ANOMALOUS MAGNETIC MOMENTS OF THE MUON AND OF THE ELECTRON

V. B. BERESTETSKIĬ

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The dispersion relations and unitarity conditions yield a simple method for calculating the radiative corrections to quantum electrodynamics. A correction to the magnetic moment is calculated by taking into account the "cutoff" at high momenta.

1. Krokhin, Khlebnikov, and the author[1] calculated the anomalous moment of the muon, $\delta\mu$, with allowance for the possible inapplicability of quantum electrodynamics in the region of high momenta. By introducing the Feynman cutoff factor with end-point momentum λ_0, an expression was obtained for $\delta\mu/\mu = (\alpha/2\pi)(1 - \delta F)$. When $m_\mu^2/\lambda_0^2 \ll 1$ (m_μ is the muon mass), the deviation from the Schwinger correction amounts to

$$\delta F = 2m_\mu^2/3\lambda_0^2. \tag{1}$$

Recently de Tollis[2] called attention to the fact that the introduction of cutoff multipliers by other methods leads to somewhat different values for δF. The purpose of the present note is to derive an expression $\delta\mu$ with the most illustrative introduction of the end-point momentum, so that the numerical estimate of the value of the expected effect is facilitated.

2. The amplitude of interaction between a muon and an electromagnetic wave, A_σ is characterized by two invariant form factors a and b, according to the general expression

$$A_\sigma = e\bar{u}(p_2)\left\{a(t)\gamma_\sigma + \tfrac{1}{4}b(t)m_\mu^{-1}(\gamma_\sigma\hat{q} - \hat{q}\gamma_\sigma)\right\}u(p_1)$$
$$= e\bar{u}(p_2)\left\{[a(t) + b(t)]\gamma_\sigma - \tfrac{1}{2}b(t)m_\mu^{-1}p_\sigma\right\}u(p_1). \tag{2}$$

Here p_1 and p_2 are the momenta of the muon, $u(p_i)$ are the spinor amplitudes,

$$(\hat{p}_i - m_\mu)u(p_i) = 0, \qquad q = p_1 - p_2, \quad p = p_1 + p_2, \quad t = q^2.$$

The region $t < 0$ corresponds to the radiation or absorption of a wave accompanied by a muon momentum change $p_1 \to p_2$, while the region $t > 4m_\mu^2$ corresponds to the production or annihilation of a pair (with momenta p_1 and $-p_2$). The anomalous magnetic moment is defined by the quantity

$$b(0) = \delta\mu/\mu.$$

To find $b(t)$, we use the dispersion relation

$$b(t) = \frac{1}{\pi}\int\frac{\operatorname{Im} b(t')}{t' - t}dt', \tag{3}$$

and to calculate $\operatorname{Im} b(t)$, we can use the unitarity relation

$$2\operatorname{Im} A_\sigma = \sum_n \langle n|q\rangle^* \langle n|p_1, p_2\rangle, \tag{4}$$

where the first bracket in the right half of the equation denotes the amplitude of the transformation of an electromagnetic wave into a certain aggregate of particles n, while the second denotes the amplitude of the conversion of the pair into these particles.

We assume that for values $t < \lambda_0^2$ these amplitudes can be calculated by using the formulas of quantum electrodynamics. If the laws of quantum electrodynamics are violated when $t > \lambda_0^2$, then for lack of something better we must cut off the integral (3) at the value $t' = \lambda_0^2$. Here λ_0 is the limiting value of the pair energy, which is annihilated according to the laws of quantum electrodynamics.

We can now estimate the quantity λ_0 by comparing it with the average electromagnetic radius $(\overline{r^2})^{1/2}$ of the nucleon $\lambda_0^2 \sim 6/\overline{r^2}$. Using the data of Hofstadter, $\overline{r^2} \approx {}^1/_3 m_\pi^2$, where m_π, is the pion mass, we obtain

$$\lambda_0^2 \sim 18m_\pi^2 \approx 36m_\mu^2. \tag{5}$$

3. If we confine ourselves to first-order approximation in $e^2 = \alpha$, the states n can only be two-particle states (an electron or muon pair). Then

$$\langle n|q\rangle^* = e\bar{u}(k_2)\gamma_\sigma u(k_1), \tag{6}$$

where k_1 and $-k_2$ are the momenta of the pair components, and $\langle n|p_1, p_2\rangle \equiv \langle k_1, k_2|p_1, p_2\rangle$ is the amplitude of the conversion of an electron pair into a muon pair, or the amplitude of scattering of a muon by an antimuon, calculated in the first approximation (in α). The summation in (4) reduces in this case to summation over the polarizations of the particles k_1 and k_2, and integration over the values of k_1 and k_2, as permitted by the conservation laws.

The amplitude of the scattering of a muon by an antimuon has the form

$$\langle k_1, k_2 | p_1, p_2\rangle = -M + M^{(a)}$$

$$M = \alpha[\bar{u}(p_2)\gamma_\rho u(k_2)][\bar{u}(k_1)\gamma_\rho u(p_1)]/(k_1 - p_1)^2, \quad (7)$$

$$M^{(a)} = \alpha[\bar{u}(p_2)\gamma_\rho u(p_1)][\bar{u}(k_1)\gamma_\rho u(k_2)]/t. \quad (8)$$

The exchange (annihilation) part of this amplitude $M^{(a)}$ is independent of p_1 and p_2. Therefore the result of its substitution in (4) will not contain p and will make no contribution to Im b. The amplitude of the conversion of an electron pair into a muon pair is also of the form (8), and this process likewise does not contribute to Im b. We note that the corresponding terms in the amplitude A_σ are usually not related to the vertex part Γ_σ, but are described as an effect of vacuum polarization (Π_σ). Thus, if we represent the amplitude A_σ in the form

$$A_\sigma = e\bar{u}(p_2)\{\Gamma_\sigma + \Pi_\sigma\}u(p_1),$$

then the form factor b will be contained only in Γ_σ, and for its calculation it is enough to take into account in (3) only the amplitude M.

Substituting in (4) the expressions (6) and (8), we obtain

$$\operatorname{Im}\Gamma_\sigma = -\frac{\alpha}{2\pi}\int \gamma_\rho(\hat{k}_2 + m_\mu)\gamma_\sigma(\hat{k}_1 + m_\mu)\gamma_\rho \delta(k_1^2 - m_\mu^2)\,\delta(k_2^2 - m_\mu^2)\frac{d^4k_1 d^4k_2}{(k_1 - p_1)^2} = -\frac{\alpha}{8}\frac{1}{\sqrt{t(t-4m_\mu^2)}}\int_{-(t-4m_\mu^2)}^{t-4m_\mu^2} \gamma_\rho(\hat{k}_2 + m)\gamma_\sigma(\hat{k}_1 + m)\gamma_\rho \frac{d\nu}{\nu + t - 4m_\mu^2}, \quad (9)$$

where $\nu = (k_1 + k_2)(p_1 + p_2)$. We note that this expression can be obtained from the Feynman diagram for the vertex part, by replacing the product of the Green's function of the muons by double the product of their imaginary parts.[3]

From (9) we obtain

$$\operatorname{Im} b(t) = \frac{\alpha}{4}\frac{4m_\mu^2}{\sqrt{t(t-4m_\mu^2)}}. \quad (10)$$

This gives, after substitution in (3),

$$b(0) = \frac{1}{\pi}\int_{4m_\mu^2}^{\lambda_0^2} \frac{\operatorname{Im} b(t)}{t}dt = \frac{\alpha}{2\pi}\sqrt{1 - \frac{4m_\mu^2}{\lambda_0^2}}, \quad (11)$$

that is,

$$1 - \delta F = \sqrt{1 - 4m_\mu^2/\lambda_0^2}. \quad (12)$$

Using estimate (5), we obtain $\delta F \sim 0.06$.

4. The anomalous magnetic moment of the electron, with allowance for the finiteness of λ_0 (reference 4) can be calculated by the same method. This yields

$$\delta F = 2m_e^2/\lambda_0^2, \quad (13)$$

where m_e is the electron mass.

In view of the smallness of m_e, this correction can be taken into account only with the radiative corrections on the order of α^2 and α^3. At the present time only the correction of order α^2 has been calculated. It corresponds to taking into account also three-particle states (a pair plus a photon). Here

$$2\operatorname{Im} A_\sigma = \Sigma(\Gamma_\sigma M + CB),$$

where Γ_σ is the vertex part, M is the amplitude of scattering of an electron by an electron, calculated with allowance for the first radiative corrections, C is the amplitude of the Compton effect, and B is the amplitude of the bremsstrahlung on the electron (the last two in the first approximation).

For calculation of order α^3, it is necessary to take into account four-particle states (a pair plus two photons and two pairs). We have

$$2\operatorname{Im} A_\sigma = \Sigma(\Gamma_\sigma M + CB + C_{II}B_{II} + BP + C_{II}D),$$

where C_{II} is the amplitude of the double Compton effect, B_{II} is the amplitude of the bremsstrahlung of two photons, P is the amplitude of pair production upon collision of an electron with a positron, and D is the amplitude of scattering of a photon by a photon, taken in first approximation. In amplitudes C and B account should be taken of the first radiative corrections, while second correction should be taken into account in Γ_σ and M.

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[1] Berestetskiĭ, Krokhin, and Khlebnikov, JETP **30**, 788 (1959), Soviet Phys. JETP **3**, 761 (1956).
[2] B. de Tollis, Nuovo cimento **16**, 203 (1960).
[3] Mandelstam, Phys. Rev. **115**, 1741 (1959).

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