ON THE NATURE OF THE FIRST EXCITED STATES OF EVEN-EVEN SPHERICAL NUCLEI

S. T. BELYAEV

Submitted to JETP editor June 21, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 1387-1400 (November, 1960)

Possible collective excitations are investigated, and their microscopic structure is analyzed. It is shown that the excitations are bound states of the particle-particle or particle-hole types, depending on the interaction. The quadrupole excitations correspond to the excitations previously found by macroscopic study. The significance of the polarizability of the core is investigated.

1. INTRODUCTION

THE spectra of the excited states of even-even undeformed nuclei are subject to definite laws.[1] The first excited state is 2^+, and its energy increases monotonically as the closed shell is approached.* The probability of the E2 transition to the ground state decreases thereby monotonically, and its magnitude exceeds considerably the single-particle value. The second excited state (4^+, 2^+, or 0^+) lies approximately twice as high, and in many nuclei a nearby doublet is found (4^+, 2^+), and in some a triplet (4^+, 2^+, 0^+). Definite laws have also been established for electromagnetic transitions from these states.

The presence of these laws suggests that the excitations are collective. The model of hydrodynamic surface oscillations is inconsistent, since the theoretical value of the mass coefficient must be increased by several times 10 to explain the experimental values of the energy levels. In addition, this model does not explain the strong dependence of the vibrational energy (of the first 2^+ level) on the filling of the upper shell.

The author has shown earlier,[2] that if the pair correlation of the nucleons and their "quadrupole" interaction are taken into account, a new form of oscillation of the spherical nucleus arises, which is essentially non-hydrodynamic in nature. These "oscillations" are connected essentially with the variation of the configurations of the internal nucleons, and can be realized even if the nuclear surface is fixed. The energies of the obtained oscillations, the mass coefficient, and their dependence on the filling of the upper shell are in qualitative agreement with experimental data.

In the present article we shall consider the microscopic structure of these excitations, so that their nature can be explained in greater detail and the connection with the single-particle description established. To ascertain whether it is legitimate to take into account only the interaction that causes pairing and the quadrupole interaction of the nucleons, we investigate the possible collective excitations for an arbitrary form of interaction. The influence of the polarizability of the core on the effective interaction between the nucleons is also established.

With a view towards explaining the principal questions on the nature of collective excitations and the conditions under which they arise, we confine ourselves to an examination of the particular case of an unfilled shell with one j level. The results obtained, however, are valid qualitatively in the general case, too.

*A similar behavior, but to a lesser extent, is observed also for certain subshells.

2. MACROSCOPIC DESCRIPTION OF THE EXCITATIONS

Let us consider briefly the macroscopic picture of quadrupole oscillations,[2] which will help to establish the connection with the microscopic description. We shall consider nucleons of one kind on the subshell $j \gg 1$. The Hamiltonian in the second quantization has the form

$$H = H_0 + H_{int} = \sum_m (\varepsilon - \lambda) a_m^+ a_m + \tfrac{1}{2} \sum_m \langle m_1 m_2 | V | m_2' m_1' \rangle a_{m_1}^+ a_{m_2}^+ a_{m_2'} a_{m_1'}, \qquad (2.1)$$

where a_m^+ and a_m are the operators of creation and annihilation of the nucleon in the state $|jm\rangle$, and the chemical potential λ has been introduced, as usual, to eliminate difficulties with the determination of the number of particles in the system.

Let us calculate, first, the nucleon interaction that describes the pairing of two particles in a state with total momentum $J = 0$:

$$H_{int}'' = -\tfrac{1}{4} G \sum_{mm'} (-1)^{j+m} a_m^+ a_{-m}^+ (-1)^{j+m'} a_{-m'} a_{m'}, \qquad (2.2)$$

where G is the interaction constant. The Hamiltonian (2.2) describes the interaction of the pair in the state J = 0. The particles do not interact in any other state. Therefore, the breaking of a pair with J = 0 leads to the same excitation energy, independently of the final state of the "broken" pair. The real interaction naturally removes this degeneracy, but qualitatively such an excitation picture remains correct.

Along with the "single-particle" excitations, connected with the breaking of one pair, there exist excitations of the collective type, in which a slight realignment of all the pairs takes place. Here, strictly speaking, the system is no longer an aggregate of independent pairs, and it must be described by certain "collective" parameters. We choose as such a parameter the total quadrupole moment Q.* Corresponding to the ground state is Q = 0, and the collective excited states of a given type correspond to a certain realignment of the single-particle states, and the appearance of a nonvanishing Q. Assuming an adiabetic variation of the collective parameter Q, we find first the energy of the system for a fixed value of Q, and then consider small oscillations in Q.

An important role is played in the indicated collective excitations by the interactions between particles of the self-consistent field type. We shall take into account the part of such an interaction, connected with the quadrupole asymmetry ("quadrupole interaction"):

$$H'_{int} = -\tfrac{1}{2}\varkappa \sum_{mm'} q_m q_{m'} a^+_m a^+_{m'} a_{m'} a_m, \qquad (2.3)$$

where $q_m = \langle jm | \hat{q} | jm \rangle$ is the matrix element of the single-particle quadrupole moment, and κ is the interaction parameter.

To find the ground state of the system for a fixed value of Q, it is convenient to take account of the additional condition Q = const by the method of Lagrange multipliers, adding to the Hamiltonian a term $-\sigma\hat{Q}$ ($\hat{Q} = \Sigma q_m a_m a^+_m a_m$ is the quadrupole-moment operator), and determine σ from the condition $\langle \hat{Q} \rangle = Q$.

Thus, let us consider the auxiliary Hamiltonian

$$\bar{H} = H - \sigma\hat{Q} = \sum_m (\varepsilon - \lambda - \sigma q_m)\, a^+_m a_m + H''_{int} + H'_{int}. \qquad (2.4)$$

Let us carry out a canonical transformation to new Fermi operators[3]

$$\alpha_m = U_m a_m - (-1)^{j+m} V_m a^+_{-m}, \qquad U^2_m + V^2_m = 1 \qquad (2.5)$$

and require that $\bar{H}$ be a minimum for the ground state $\Psi_0(Q)$, defined by the condition $\alpha_m \Psi_0(Q) = 0$ ("vacuum" in the quasi-particles). We obtain here an equation for U_m and V_m

$$(\varepsilon - \lambda - \tilde{\sigma} q_m)\, 2U_m V_m - \Delta (U^2_m - V^2_m) = 0, \qquad (2.6)$$

where

$$\Delta = \tfrac{1}{2} G \sum_m U_m V_m, \qquad Q = \sum_m q_m V^2_m, \qquad \tilde{\sigma} = \sigma + \varkappa Q. \qquad (2.7)$$

Expressing U_m and V_m in terms of Δ and using the connection between the chemical potential λ and the total number of particles ($N = \Sigma V^2_m$), we obtain a system of equations for the determination of Δ and λ:

$$\frac{G}{4}\sum_m \frac{1}{[\Delta^2 + (\varepsilon - \lambda - \tilde{\sigma} q_m)^2]^{1/2}} = 1,$$

$$\sum_m \frac{\varepsilon - \lambda - \tilde{\sigma} q_m}{[\Delta^2 + (\varepsilon - \lambda - \tilde{\sigma} q_m)^2]^{1/2}} = 2(\Omega - N), \qquad (2.8)$$

where $2\Omega = 2j + 1$ is the total number of states in the sub-shell.

The energy of the ground state of the system, W, is defined as the average value of the initial Hamiltonian $H = H_0 + H'_{int} + H''_{int}$ in the state $\Psi_0(Q)$. As a result we obtain

$$W = \langle \Psi_0(Q), H\Psi_0(Q) \rangle = (\varepsilon - \lambda) N - \tfrac{1}{2}\varkappa Q^2 - \Delta^2 / G. \qquad (2.9)$$

For the energy of the quasi-particle we obtain analogously

$$E_m = \langle \Psi_0(Q), \alpha_m H \alpha^+_m \Psi_0(Q) \rangle - W = \sqrt{\Delta^2 + (\varepsilon - \lambda - \tilde{\sigma} q_m)^2}. \qquad (2.10)$$

For small σ the solution (2.8) for Δ, U_m, and V_m is of the form

$$\Delta \approx \tfrac{1}{2}\Omega G \sin\vartheta \left\{1 - \frac{\tilde{\sigma}^2}{(\Omega G)^2}\frac{1}{\Omega}\sum_m q^2_m\right\}, \qquad (2.11)$$

$$U_m \approx U - V\frac{\tilde{\sigma}}{\Omega G} q_m \sin\vartheta, \quad V_m = V + U\frac{\tilde{\sigma}}{\Omega G} q_m \sin\vartheta, \qquad (2.12)$$

where

$$V = (N/2\Omega)^{1/2}, \quad U = (1 - N/2\Omega)^{1/2}, \quad \sin\vartheta = 2UV. \qquad (2.13)$$

(The magnitude of $\sin\vartheta$ characterizes the degree of filling the shell, and varies from unity for a half-filled shell, to 0 on the edge of the shell.) The parameter $\tilde{\sigma}$ is connected with the quadrupole moment:

$$Q = \sum_m q_m V^2_m = \frac{\tilde{\sigma}}{\Omega G}\sin^2\vartheta \sum_m q^2_m. \qquad (2.14)$$

The wave function of state with a non vanishing Q and $\Psi_0(Q)$ can be expressed in terms of the "spherical-nucleus" functions $\Psi(0)$. For this purpose, we introduce the quasi-particles for the spherical nucleus $\alpha^{(0)}_m$, defined from (2.5) with

*We have in mind the quadrupole moment of the mass, not of the charge.

$U_m = U$ and $V_m = V$. It is easy to see that the following relation holds

$$\alpha_m \approx (UU_m + VV_m)\alpha_m^{(0)} + (VU_m - UV_m)(-1)^{j+m}\alpha_{-m}^{(0)+},$$

from which it follows that

$$\Psi_0(Q) = \exp\left\{\frac{1}{2}\sum_m \frac{UV_m - VU_m}{UU_m + VV_m}(-1)^{j+m}\alpha_m^{(0)+}\alpha_{-m}^{(0)+}\right\}\Psi_0(0),$$

and for small Q

$$\Psi_0(Q) = \exp\left\{Q\left(2\sin\vartheta\sum q_m^2\right)^{-1} \times \sum_m (-1)^{j+m} q_m \alpha_m^{(0)+}\alpha_{-m}^{(0)+}\right\}\Psi_0(0). \quad (2.15)$$

For the energy in the ground state, in the same approximation of small Q, we obtain from (2.9), (2.11), and (2.14)

$$W(Q) - W(0) = \tfrac{1}{2}\Omega G(1 - \Theta_0^{-1}\sin^2\vartheta)Q^2(\sin^2\vartheta\sum q_m^2)^{-1} \equiv \tfrac{1}{2}C_Q Q^2, \quad (2.16)$$

where

$$\frac{1}{\Theta_0} = \frac{\varkappa}{\Omega G}\sum_m q_m^2.$$

(The quantity Θ_0 defines the critical filling at which the spherical shape of the nucleus becomes unstable.) Expression (2.16) determines the dependence of the energy of the system on the collective parameter Q, i.e., it is the "potential energy" of the collective motion. The total Hamiltonian of the collective motion, for small values of Q, has the form

$$H_Q = \tfrac{1}{2}B_Q\dot{Q}^2 + \tfrac{1}{2}C_Q Q^2,$$

where the "mass coefficient" and B_Q is determined by the following expression ($\hbar = 1$):

$$B_Q = 2\sum_{i\neq 0}\left|\left\langle \Psi_i, \frac{\partial}{\partial Q}\Psi_0\right\rangle\right|^2 \Big/ (W_i - W_0). \quad (2.17)$$

From (2.15) we obtain

$$\left[\frac{\partial}{\partial Q}\Psi_0(Q)\right]_{Q=0} = \left(2\sin\vartheta\sum q_m^2\right)^{-1}\sum_m(-1)^{j+m}q_m\alpha_m^{(0)+}\alpha_{-m}^{(0)+}\Psi_0(0),$$

from which it is clearly seen that the operator $\partial/\partial Q$ transfers the system to a state with 2 quasi-particles; therefore $W_i - W_0 = 2E = \Omega G$. After simple transformations we obtain

$$B_Q = (\Omega G\sin^2\vartheta\sum q_m^2)^{-1}. \quad (2.18)$$

For the oscillation energy we obtain, according to (2.6) and (2.18),

$$\omega = \sqrt{C_Q/B_Q} = 2E\sqrt{1 - \Theta_0^{-1}\sin^2\vartheta}. \quad (2.19)$$

As follows from (2.19), the energy of the resultant collective excitation is actually less than the energy required to break the pair, $2E = \Omega G$ ("single-particle" excitation), and ω decreases monotonically as the filling is increased ($\sin^2\vartheta \to 1$).

3. MICROSCOPIC ANALYSIS OF QUADRUPOLE EXCITATIONS

In this section we consider the same problem by a different method, which will enable us to establish the microscopic structure of the resultant excitations. Where possible we shall emphasize the analogy with infinite systems, for which the character of the excitations has been sufficiently well investigated.

As is well known, the lower excited states of a many-particle system can be approximately described as an aggregate of elementary excitations—quasi-particles. In an ideal degenerate Fermi gas, the quasi-particles are holes below the Fermi level and particles above it. The switching on of the interaction leads, firstly, to a change of the effective mass of the quasi-particles, and secondly to the appearance of an interaction between the quasi-particles themselves. If the latter does not lead to the appearance of bound states, but only to scattering processes, it can usually be neglected, since the scattering of quasi-particles yields only corrections proportional to Ω^{-1} (Ω is the volume of the system).

In the case when bound states become possible, new "collective" excitations appear. Thus, in the repulsion between particles, new bound states of the particle and hole at the Fermi boundary can occur in the Fermi system, and this leads to a new type of excitation — zero sound.[4] It can be stated that an excited state with two quasi-particles is unstable under the formation of a bound pair of these quasi-particles. In the case of attraction, the properties of the Fermi system change more radically; bound states become possible between particles near the Fermi surface, and this leads to instabilities even in the ground state relative to pairing between particles, and the sharp Fermi boundary becomes diffuse. The quasi-particles near the new ground state are superpositions of holes and particles.

One multiply-degenerate level ($j \gg 1$) is equivalent to an infinitesimally thin spherical layer below the diffuse Fermi level. The number of states $2\Omega = 2j + 1$ plays the role of the "volume" of the system. According to the foregoing, solving the problem of the interacting nucleons at this

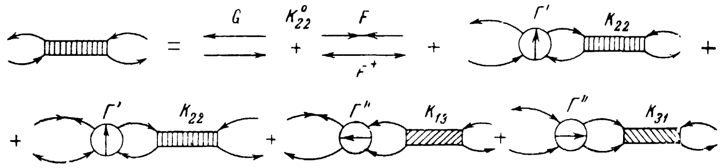

FIG. 1

level, accurate to Ω^{-1}, denotes firstly finding the quasi-particles, and secondly finding the "bound states." (In a system of finite dimensions, the concept of a bound state is rather arbitrary, and we shall use it to denote strongly correlated states.)

To investigate the excitations of a many-particle system, it is best to use the method of Green's functions.[5] Single-particle excitations (the spectrum of quasi-particles) are described by single-particle functions. In the presence of pairing in the ground state, it is necessary to use three functions (see Gor'kov[6]):

$$G(m;\ t_1 - t_2) = -i\langle T\{a_m(t_1)\overset{+}{a}_m(t_2)\}\rangle,$$

$$F(m;\ t_1 - t_2) = -ie^{i\lambda(t_1+t_2)}\langle T\{a_m(t_1)a_{-m}(t_2)\}\rangle,$$

$$F^+(m;\ t_1 - t_2) = -ie^{-i\lambda(t_1+t_2)}\langle T\{\overset{+}{a}_m(t_1)\overset{+}{a}_{-m}(t_2)\}\rangle, \qquad (3.1)$$

which in our case (spherical symmetry) have the following Fourier transforms

$$G(m;\ \varepsilon) = \frac{U^2}{\varepsilon - E + i\delta} + \frac{V^2}{\varepsilon + E - i\delta} \qquad (\delta \to +0),$$

$$F(m;\ \varepsilon) = -F^+(m;\ \varepsilon)$$

$$= -(-1)^{j+m}UV\left(\frac{1}{\varepsilon - E + i\delta} - \frac{1}{\varepsilon + E - i\delta}\right), \qquad (3.2)$$

when U and V are as defined in (2.13) while E is the energy of the quasi-particle (in the first approximation, its value is $\Omega G/2$).

To investigate the collective excitations, it is necessary to consider two-particle Green's functions. In the presence of pairing we cannot in general write down a closed equation for a single function, and we must introduce three different functions, for example

$$K_{22} = \langle T\{a_1\overset{+}{a}_2 a_3\overset{+}{a}_4\}\rangle, \qquad K_{13} = \langle T\{\overset{+}{a}_1\overset{+}{a}_2 a_3\overset{+}{a}_4\}\rangle,$$
$$K_{31} = \langle T\{a_1a_2a_3\overset{+}{a}_4\}\rangle. \qquad (3.3)$$

Thus, the equation for K_{22} has the form (see Fig. 1)

$$K_{22}(12;\ 34) = K^0_{22}(12;\ 34) + i\{G(1;\ 5)G(7;\ 2)$$
$$+ F(17)F^+(25)\}\Gamma'(56;\ 78)K_{22}(86;\ 34)$$
$$+ iF(17)G(8;\ 2)\Gamma''(56;\ 78)K_{13}(56;\ 34)$$
$$+ iG(1;\ 5)F^+(26)\Gamma''(56;\ 78)K_{31}(87;\ 34) \qquad (3.4)$$

(integration over repeated indices 5 — 8 is implied). Here Γ is a compact (irreducible) four-pole, which in the first approximation is equal to the corresponding interaction matrix element (see 2.1):

$$2\Gamma''_0(12;\ 2'1') = \Gamma'_0(12;\ 2'1') = \langle 12|V|2'1'\rangle$$
$$-\langle 12|V|1'2'\rangle, \qquad (3.5)$$

In higher approximations in Γ it is necessary to distinguish between directions of "irreducibility" (Fig. 2). The aggregate of the graphs, which is irreducible in the direction of motion of the two particles, is denoted by Γ'', and the aggregate which is irreducible in the particle — hole direction is denoted by Γ'.

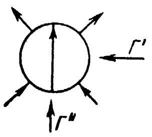

FIG. 2

In the right half of (3.4) there are contained, along with K_{22}, two other functions, so that in general it is necessary to consider a system of three equations. It will prove advantageous to uncouple the functions K. Assuming that $t_1, t_2 > t_3, t_4$, we write

$$K_{22} = \sum_s \varphi_s(12)\tilde{\varphi}_s(34); \qquad K_{31} = \sum_s \chi_s(12)\tilde{\varphi}_s(34),$$
$$K_{13} = \sum_s \bar{\chi}_s(12)\tilde{\varphi}_s(34),$$

where

$$\varphi_s(12) = \langle 0|T\{a_1\overset{+}{a}_2\}|s\rangle, \qquad \tilde{\varphi}_s(34) = \langle s|T\{a_3\overset{+}{a}_4\}|0\rangle,$$
$$\chi_s(12) = \langle 0|T\{a_1a_2\}|s\rangle, \qquad \bar{\chi}_s(12) = \langle 0|T\{\overset{+}{a}_1\overset{+}{a}_2\}|s\rangle. \qquad (3.6)$$

In the investigation of bound states we can discard the inhomogeneities in the equations for K [the term K^0_{22} in (3.4)].[7,5] Then, using the fact that $\tilde{\varphi}_S(34)$ is contained in each term, we can "divide" all the equations by $\tilde{\varphi}_S(34)$. As a result we obtain a system of homogeneous equations for φ_S, χ_S, and $\bar{\chi}_S$.

For further simplification, we shall assume that Γ describes a non-retarded interaction.* Then both times in the functions φ_S, χ_S, and $\bar{\chi}_S$ can be taken to be equal ($t_1 - t_2 \to -0$), so that the problem reduces to a determination of the three functions

*The time delay of the interaction of the nucleons in the nucleus is $\sim\varepsilon_F^{-1}$(ε_F is the Fermi boundary energy), which is much shorter than the times ω^{-1} of interest to us.

$$\varphi_{\omega\mu}(m) = \langle 0 | a_m^+ a_{m+\mu} | \omega\mu \rangle, \quad \chi_{\omega\mu}(m) = \langle 0 | a_{-m} a_{m+\mu} | \omega\mu \rangle,$$

$$\overline{\chi}_{\omega\mu}(m) = \langle 0 | a_m^+ a_{-m-\mu}^+ | \omega\mu \rangle, \qquad (3.7)$$

where the state $|s\rangle$ is characterized by an energy ω, and a momentum projection μ. The system of equations for (3.7) has the form

$$(4E^2 - \omega^2)\varphi_{\omega\mu}(m) = -2E\sin^2\vartheta \sum_{m'} \Gamma'_\mu(m; m')\varphi_{\omega\mu}(m')$$

$$-E\sin\vartheta\cos\vartheta \sum_{m'} \Gamma''_\mu(m; m')\chi^{(+)}_{\omega\mu}(m')$$

$$-\tfrac{1}{2}\omega\sin\vartheta \sum_{m'} \Gamma''_\mu(m; m')\chi^{(-)}_{\omega\mu}(m'), \qquad (3.8a)$$

$$(4E^2 - \omega^2)\chi^{(+)}_{\omega\mu}(m) = -2E\cos^2\vartheta \sum_{m'} \Gamma''_\mu(m; m')\chi^{(+)}_{\omega\mu}(m')$$

$$-\omega\cos\vartheta \sum_{m'} \Gamma''_\mu(m; m')\chi^{(-)}_{\omega\mu}(m')$$

$$-4E\sin\vartheta\cos\vartheta \sum_{m'} \Gamma'_\mu(m; m')\varphi_{\omega\mu}(m'),$$

$$(4E^2 - \omega^2)\chi^{(-)}_{\omega\mu}(m) = -2E \sum_{m'} \Gamma''_\mu(m; m')\chi^{(-)}_{\omega\mu}(m')$$

$$-\omega\cos\vartheta \sum_{m'} \Gamma''_\mu(m; m')\chi^{(+)}_{\omega\mu}(m')$$

$$-2\omega\sin\vartheta \sum_{m'} \Gamma'_\mu(m; m')\varphi_{\omega\mu}(m'), \qquad (3.8b)$$

where instead of χ and $\bar{\chi}$ we introduce their linear combinations

$$\chi^{(\pm)}_{\omega\mu}(m) = (-1)^{j+m}\{\chi_{\omega\mu}(m) \pm (-1)^\mu \overline{\chi}_{\omega\mu}(m)\} \qquad (3.9)$$

and for the sake of abbreviation we put

$$\Gamma'_\mu(m; m') = \tfrac{1}{2}\{\Gamma'(m+\mu, m'; m'+\mu, m)$$

$$+(-1)^\mu \Gamma'(-m, m'; m'+\mu, -m-\mu)\},$$

$$\Gamma''_\mu(m; m')$$

$$= (-1)^{j+m}\Gamma''(m+\mu, -m; -m', m'+\mu)(-1)^{j+m'}. \qquad (3.10)$$

So far we did not refer to any specific interaction between particles. Let us consider now the particular case of "quadrupole" interaction (2.3), for which we obtain, according to (3.5) and (3.10),

$$\Gamma'_\mu(m; m') = -\delta_{\mu 0}\varkappa q_m q_{m'}, \qquad \Gamma''_\mu = 0. \qquad (3.11)$$

Substituting (3.11) in (3.8), we get

$$(4E^2 - \omega^2)\varphi_\omega(m) = 2\varkappa E\sin^2\vartheta\, q_m \sum_{m'} q_{m'}\varphi_\omega(m'), \qquad (3.12a)$$

$$(4E^2 - \omega^2)\chi^{(+)}_\omega(m) = 4\varkappa E\cos\vartheta\sin\vartheta\, q_m \sum_{m'} q_{m'}\varphi_\omega(m'), \qquad (3.12b)$$

$$(4E^2 - \omega^2)\chi^{(-)}_\omega(m) = 2\varkappa\omega\sin\vartheta\, q_m \sum_{m'} q_{m'}\varphi_\omega(m'). \qquad (3.12c)$$

The first equation contains only φ_ω. From the condition of solvability of this equation, we obtain the energy of the bound state

$$\omega = 2E\left[1 - \frac{\varkappa}{2E}\sin^2\vartheta \sum_m q_m^2\right]^{1/2}, \qquad (3.13)$$

which coincides exactly with (2.19).

The function φ_ω(3.7) describes a particle-hole pair, and consequently the resultant excitation is a bound state of the hole and the particle. In this sense, it is an analog of the zero sound of an infinite Fermi system. However, the state obtained is not described by the function φ_ω alone. It is seen from (3.12b and c) that the functions $\chi^{(\pm)}_\omega$ are also different from zero. This can be understood by recalling that in the presence of pairing, the quasi-particles are superpositions of holes with particles. In quadrupole interaction, the "hole" component of one quasi-particle is bound with the "particle" component of its partner, but in the bound state there exist also other components. It must be emphasized that the possibility of production of a balanced state of a particle with a hole occurs only when the system contains pairs with J = 0, of the Cooper type (particle — particle). This is seen, in particular, from expression (3.13) for the energy of the bound state, where the energy of the Cooper pairing 2E is contained in explicit form.

As can be seen from (3.12), the dependence of the functions φ_ω, $\chi^{(\pm)}_\omega$ on the projection of the momentum m is determined by the quantity q_m, which is proportional to the Clebsch-Gordan coefficient $(jm\, j - m | 20)$, and consequently the state found is characterized by a momentum J = 2.

Let us consider now the pairing interaction (2.2), for which we obtain from (3.5) and (3.10)

$$\Gamma'_\mu(m; m') = 0, \qquad \Gamma''_\mu(m; m') = -\delta_{\mu 0}G/2. \qquad (3.14)$$

When $\Gamma' = 0$, the energy of the bound state is determined actually from the last two equations of (3.8), which contain in this case only $\chi^{(\pm)}$, describing pairs of particles or holes. Consequently, in this case the bound state is caused by pairing of the Cooper type (particle — particle or hole — hole). The solution of the secular equation for the interaction (3.14) yields $\omega = 0$ for the energy of the bound state. This result is obvious, since the bound pairs of the Cooper type with J = 0 [which are described by the interaction (2.2)] have already been taken into account in the ground state of the system. We note that in the solution of Eqs. (3.8) the interaction that describes the pairing, and the quadrupole interaction can be considered independently, since they lead to bound states with different momenta.

We emphasize that although formulas (2.19) and (3.13) coincide formally, there is an essential difference between them. In the derivation of (2.19) we used the adiabaticity requirement, which formally limits the region of its application to the condition $\omega \leq 2E$, whereas expression (3.13) is free of this limitation. Another advantage of the microscopic analysis is the possibility of expressing the wave functions of the "collective" state in terms of single-particle functions (the functions φ_ω, and $\chi_\omega^{(\pm)}$ give directly the contribution of the various single-particle combinations).

Thus, in the presence of an interaction that describes the pairing (2.2) and a quadrupole interaction (2.3), collective excitations are possible in the system; these excitations can be considered macroscopically as oscillations of the quadrupole moment, and microscopically as a bound state of a hole and a particle with $J = 2$. The question arises: what justifies the special choice of the Hamiltonian, and how essential is it? Will the character of the excitations change when a more real interaction is examined? These questions are analyzed in the next section.

4. COLLECTIVE EXCITATIONS IN AN ARBITRARY INTERACTION

As was established in the preceding section, the quantities Γ' and Γ'', which describe the effective interaction, lead to bound states of different nature. Γ'' binds a particle to a particle (a hole to a hole), while Γ' binds a particle to a hole. So far we have considered only pairs of the first type, with momentum $J = 0$, and pairs of the second type, with $J = 2$. In the general case Γ' and Γ'' describe an interaction in states with all possible momenta.* It is convenient to represent Γ' and Γ'', in the form of a sum of terms corresponding to pairing with a definite J.

We consider first Γ' and Γ'' of the first approximation (3.5). To determine the dependence of the matrix element on the projections of the single-particle momenta, it is convenient to separate in the interaction potential the angular dependence, writing it in terms of spherical tensor operators:[8]

$$V(\mathbf{r}_1, \mathbf{r}_2) = \sum_k v_k(r_1 r_2)(T^k(1)\,T^k(2)). \tag{4.1}$$

From this we obtain for an arbitrary matrix element between the single particle states $|1\rangle = |n_1 l_1 j_1 m_1\rangle$:[8]

$$\langle 12|V|2'1'\rangle = -\sum_{k\mu}(-1)^\mu F'_k(-1)^{j'_1-m'_1}(j_1m_1\,j'_1-m'_1|k\mu) \times(-1)^{j'_2-m'_2}(j_2m_2\,j'_2-m'_2|k-\mu), \tag{4.2}$$

where the quantities F'_k, which are independent of m, are defined as

$$F'_k = -(2k+1)^{-1}(1\|T^k\|1')(2\|T^k\|2')F^{(k)}(122'1'). \tag{4.3}$$

[Here $F^{(k)}$ are the Slater radial integrals, and $(1\|T^k\|1')$ are the reduced matrix element of the tensor operators.]

Using (4.2), we can write the interaction Hamiltonian in the form

$$H_{int} \approx -\tfrac{1}{2}\sum_{\substack{122'1'\\k\mu}} F'_k(122'1')(-1)^\mu(a_1^+ a_{1'})_{k\mu}(a_2^+ a_{2'})_{k-\mu}, \tag{4.4}$$

where the quantity

$$(a_1^+ a_{1'})_{k\mu} = \sum_m (-1)^{j'_1-m'_1}(j_1m_1\,j'_1-m'_1|k\mu)\,a_1^+ a_{1'} \tag{4.5}$$

can be considered as an operator that describes a particle and a hole in a state with momentum k. Along with (4.4), it is possible to expand H_{int} in states in which the particle—particle and hole—hole pairs have definite momenta

$$H_{int} = -\tfrac{1}{2}\sum_{\substack{122'1'\\k\mu}} f_k(a_1^+ a_2^+)_{k\mu}(a_{2'}a_{1'})_{k\mu}, \tag{4.6}$$

corresponding to a representation of the matrix element in the form

$$\langle 12|V|2'1'\rangle = -\sum_{k\mu} f_k(j_1m_1j_2m_2|k\mu)(j'_1m'_1j'_2m'_2|k\mu). \tag{4.7}$$

Using the algebra of the Clebsch-Gordan coefficients (see, for example, references 8 and 9), we can relate the coefficients of expansions (4.2) and (4.7). As a result we get

$$f_k = (-1)^{j_1+j'_2-k}\sum_l (2l+1)\,W(j_1j_2j'_1j'_2;\,kl)\,F'_l, \tag{4.8}$$

where W is the Racah coefficient. Analogously we can obtain expressions for the "exchange" matrix element in (3.5), the addition of which is equivalent to changing the quantities F'_k and f_k in (4.2) and (4.8). In the case of a single j-level the exchange corrections to f_k and F'_k are equal to f_k and $-f_k$, respectively. As a result we obtain, with account of (3.5) and (3.10) ($F_k = F'_k - f_k$):

$$\Gamma'_\mu(m;\,m') = -\sum_k F_k(-1)^{j-m}(j,m+\mu,j,-m|k\mu) \times(-1)^{j-m'}(j,m'+\mu,j,-m'|k\mu),$$

$$\Gamma''_\mu(m;\,m') = -\sum_k f_k(-1)^{j-m}(j,m+\mu,j,-m|k\mu) \times(-1)^{j-m'}(j,m'+\mu,j,-m'|k\mu). \tag{4.9}$$

*Even momenta $0 \leq J \leq 2j-1$ are possible for the j^N configuration.

We note that in the derivation of (4.9), the expansion (4.2) was used only to separate the regular dependence on the projection of the momentum, that is, the geometrical factor. Therefore expressions (4.9) are valid not only in the first approximation in the interaction potential, but also in the general case (neglecting delay).

The quantity F_k in (4.9) is the effective interaction (binding energy) between a particle and a hole in a state with momentum k, and f_k is the analogous characteristic of two particles (holes).* Relation (4.8) between the quantities f_k and F_k does not take place in general, since the four-poles Γ'' and Γ' correspond to different irreducible aggregates of diagrams, but it becomes valid for Γ which are irreducible in both direction [see text following formula (3.5)].

Substituting (4.9) in (3.8) and using the orthogonality of the Clebsch-Gordan coefficients, we can obtain a system of algebraic equations for the wave functions of the pairs with definite momentum k (and its projection μ)

$$\varphi_{\omega k} = \sum_m (-1)^{j-m}(j, m+\mu, j, -m\,|\,k\mu)\,\varphi_{\omega\mu}(m),$$

$$\chi^{(\pm)}_{\omega k} = \sum_m (-1)^{j-m}(j, m+\mu, j, -m\,|\,k\mu)\,\chi^{(\pm)}_{\omega\mu}(m); \qquad (4.10)$$

$$(4E^2-\omega^2-2F_k E\sin^2\vartheta)\,\varphi_{\omega k} - f_k E\sin\vartheta\cos\vartheta\,\chi^{(+)}_{\omega k} - \tfrac{1}{2} f_k\,\omega\sin\vartheta\,\chi^{(-)}_{\omega k} = 0, \qquad (4.11a)$$

$$(4E^2-\omega^2-2f_k E\cos^2\vartheta)\,\chi^{(+)}_{\omega k} - f_k\,\omega\cos\vartheta\,\chi^{(-)}_{\omega k} - 4F_k E\sin\vartheta\cos\vartheta\,\varphi_{\omega k} = 0, \qquad (4.11b)$$

$$(4E^2-\omega^2-2f_k E)\,\chi^{(-)}_{\omega k} - f_k\,\omega\cos\vartheta\,\chi^{(+)}_{\omega k} - 2F_k\,\omega\sin\vartheta\,\varphi_{\omega k} = 0. \qquad (4.11c)$$

The condition for the solvability of the system (4.11) has the form

$$(4E^2-\omega^2)^2\,[(2E-f_k)^2 - (2E-f_k)(F_k-f_k)\sin^2\vartheta - \omega^2] = 0, \qquad (4.12)$$

from which we obtain for the energy of the bound state with momentum k

$$\omega_k = (2E-f_k)\left[1-\frac{F_k-f_k}{2E-f_k}\sin^2\vartheta\right]^{1/2}. \qquad (4.13)$$

If there is no pairing of the Cooper type ($f_k = 0$), then (4.14) goes into (3.13) as $F_k \to F_2 = \kappa\Sigma q_m^2$. In the general case the bound state is determined by the pairing of both types (particle — particle and particle — hole). If $F_k > f_k$, then, as can be seen from (4.13), account of f_k leads only to a renormalization of the quantities E and F_k, and the dependence of ω_k on the filling of the shell remains the same.

In the case when $F_k < f_k$ this dependence changes considerably. Thus, in the limiting case $F_k = 0$, we get

$$\omega_k = 2E\left[\left(1-\frac{f_k}{2E}\right)\left(1-\frac{f_k}{2E}\cos^2\vartheta\right)\right]^{1/2}. \qquad (4.14)$$

It is impossible to establish a general relation between f_k and F_k. For orientation purposes, we consider the case when relation (4.8) is valid. When $j \gg 1$ the main contribution to the right half of (4.8) is given by terms with large l, and we can use the following approximation[9] ($j, l \gg k$)

$$W(jjjj;\,kl) \approx -(2j+1)^{-1} P_k(\cos\alpha_l),$$

$$\cos\alpha_l = [l(l+1)-2j(j+1)]/2j(j+1).$$

In this approximation (4.8) assumes the form

$$f_k \approx \tfrac{1}{2}\left(j+\tfrac{1}{2}\right)\int_{-1}^{1} P_k(\cos\alpha_l)\,F'_l\,d\cos\alpha_l. \qquad (4.15)$$

We consider the case of a δ interaction $V_{12} \sim (\mathbf{r}_1 - \mathbf{r}_2)$, for which[10]

$$F'_0 = 2G\left(j+\tfrac{1}{2}\right);\quad F'_2 = \tfrac{1}{4}F'_0, \ldots,$$

$$F'_{l\gg 1} \approx \frac{2G}{\pi}\sqrt{\left(\frac{2j+1}{l}\right)^2 - 1} \approx \frac{2G}{\pi}\tan\frac{\alpha_l}{2}, \qquad (4.16)$$

where G is a constant connected with the radial integral. Substituting (4.16) in (4.15), we get

$$f_k \approx \frac{4G}{\pi}\left(j+\tfrac{1}{2}\right)\int_0^{\pi/2} P_k(\cos 2\beta)\sin^2\beta\,d\beta. \qquad (4.17)$$

In particular, $f_0 = G(j + 1/2)$, $f_2 = 1/4\, f_0$, and from a comparison with (4.16) it follows that for a δ interaction $f_k = 1/2\, F'_k = F_k$. When the interaction radius if finite, this relation is violated, and F_k diminishes more rapidly with k than f_k.* As a result we have $F_k > f_k$ for small $k\,(k \ll 2j-1)$, and $F_k < f_k$ for large k. We emphasize that these estimates are only tentative, since the effective interaction Γ cannot be approximated in general by means of a potential, and the quantities F_k and f_k may be non-monotonic functions of k. Thus, account of polarizability of the core, as will be shown in the next section, leads to a sharp increase in the effective quadrupole interaction F_2.

5. EFFECT OF POLARIZATION OF THE CORE

Interaction between particles at the j level occurs both directly and through the nucleons of

*With the sign used in (4.3), F_k and f_k are positive in the case of attraction.

*This can be seen from the limiting case of "long-range action," when $F_k \sim \delta_{k0}$. It follows then from (4.8) that $f_k \approx F_0/(2j+1) = \mathrm{const}$.

the core. The part Γ' that corresponds to such an interaction can be represented in the following form (see Fig. 3)

$$\tilde{\Gamma}'(13;\,24) = -i\gamma(15;26)\,\tilde{K}(65;\,87)\,\gamma(73;\,84), \qquad (5.1)$$

where γ is the effective interaction of the outer nucleon with the core nucleon (the wavy line on Fig. 3), and K is the Green's function for the core.

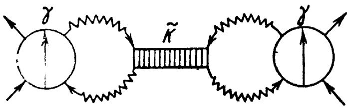

FIG. 3

Going over to a representation in which the momentum of the particle — hole pair is fixed, we obtain

$$\tilde{F}_k = i\sum_{\alpha\beta}\gamma_k(\alpha)\,\tilde{K}_k(\alpha;\,\beta)\,\gamma_k(\beta), \qquad (5.2)$$

where $\tilde{F}_k$ is the part of F_k connected with the interaction through the core, $\gamma_k(\alpha)$ describes the transition of the particle — hole pair with momentum k from the j level to the core. The two-particle function of the core is given by the equation

$$\tilde{K}_k(\alpha;\,\beta;\,t-t') = \langle T\{(a^+(t)\,a(t))_{\alpha k}\,(a^+(t')\,a(t'))_{\beta k}\}\rangle. \qquad (5.3)$$

Here $(a^+a)_{\alpha k}$ denotes a pair state characterized by momentum k and the remaining quantum numbers α. We assume that γ_k does not contain a delay, and therefore we can consider in $\tilde{K}$ the times of the ends to be pairwise equal. Breaking up $\tilde{K}$ similar to (3.6), we obtain

$$\tilde{K}_k(\alpha;\,\beta;\,\tau) = \begin{cases}\sum_s e^{-iE_{s0}\tau}\langle 0\,|\,(a^+a)_{\alpha k}\,|\,s\rangle\langle s\,|\,(a^+a)_{\beta k}\,|\,0\rangle, & \tau > 0\\ \sum_s e^{iE_{s0}\tau}\langle 0\,|\,(a^+a)_{\beta k}\,|\,s\rangle\langle s\,|\,(a^+a)_{\alpha k}\,|\,0\rangle, & \tau < 0.\end{cases} \qquad (5.4)$$

For the Fourier transform of $\tilde{K}_k$ we obtain from (5.4)

$$\tilde{K}_k(\alpha;\,\beta;\,\omega) = i\sum_s\left\{\frac{\langle 0\,|\,(a^+a)_{\alpha k}\,|\,s\rangle\langle s\,|\,(a^+a)_{\beta k}\,|\,0\rangle}{E_{s0}-\omega} + \frac{\langle 0\,|\,(a^+a)_{\beta k}\,|\,s\rangle\langle s\,|\,(a^+a)_{\alpha k}\,|\,0\rangle}{E_{s0}+\omega}\right\}. \qquad (5.5)$$

The excitation energy of the core, $E_{s0} = E_s - E_0$, is equal in order of magnitude to the energy difference between the shells, whereas the ω of interest to us correspond to the pairing energies of the outer nucleons; therefore $\omega \ll E_{s0}$. Substituting (5.5) in (5.2) and neglecting ω, we get

$$\tilde{F}_k = -\sum_{\alpha\beta}\gamma_k(\alpha)\,\gamma_k(\beta)\sum_s\frac{2}{E_{s0}}\langle 0\,|\,(a^+a)_{\alpha k}\,|\,s\rangle\langle s\,|\,(a^+a)_{\beta k}\,|\,0\rangle$$
$$\equiv\sum_{\alpha\beta}\gamma_k(\alpha)\,\gamma_k(\beta)\,h_k(\alpha\beta). \qquad (5.6)$$

Introducing $\bar{F}_k$, namely that part of F_k connected with the direct interaction ($F_k = \bar{F}_k + \tilde{F}_k$), we can write

$$\tilde{F}_k = \eta_k\bar{F}_k, \qquad (5.7)$$

where

$$\eta_k = \sum_{\alpha\beta}\frac{\gamma_k(\alpha)\,\gamma_k(\beta)}{\bar{F}_k}h\;(\alpha\beta) \equiv \bar{\gamma}_k\sum_{\alpha\beta}h\;(\alpha\beta). \qquad (5.8)$$

Let us show that the coefficient of renormalization of the direct interaction, η_k, can be connected with the polarizability of the core by the external nucleon. The interaction of the external nucleon (b, b^+) with the core can be written, in analogy with (4.4), as

$$\tilde{H} = \sum_{\alpha k\mu}\gamma_k(\alpha)\,(-1)^{\mu}\,(b^+b)_{k-\mu}\,(a^+a)_{\alpha k\mu}. \qquad (5.9)$$

In the first approximation we can neglect the reaction of the core on the nucleon, and assume the state of the latter to be specified (that is, consider the nucleon as an external field). Averaging (5.9) over the nucleon state $|\,jm_0\rangle$, we obtain the additional energy of the core in the presence of an external particle

$$\langle\tilde{H}\rangle_b = \sum_{\alpha k}\varphi_k\,\gamma_k(\alpha)\,(a^+a)_{\alpha k}, \qquad (5.10)$$

where

$$\varphi_k = \langle(b^+b)_{k0}\rangle = (-1)^{j-m_0}(jm_0\,j-m_0\,|\,k0). \qquad (5.11)$$

Considering (5.10) as a perturbation, we obtain the change in the wave function of the core

$$\Psi_0^{(1)} = -\sum_{\alpha k}\varphi_k\gamma_k(\alpha)\sum_s\frac{\langle s\,|\,(a^+a)_{\alpha k}\,|\,0\rangle}{E_{s0}}\Psi_s^0. \qquad (5.12)$$

In the presence of an external nucleon, the core loses its spherical symmetry and its polarization can be characterized by the following average values of the multipole moments

$$\hat{Q}_l = \sum_{12}\left\langle 1\left|\sqrt{\frac{2l+1}{4\pi}}\,r^lY_{l0}\right|2\right\rangle a_1^+a_2$$
$$= \sum\left(1\left\|\frac{r^l}{\sqrt{4\pi}}Y_{l0}\right\|2\right)(-1)^{j_2-m}(j_1mj_2-m\,|\,l0)\,a_1^+\,a_2$$
$$\equiv\sum_{\alpha}q_l(\alpha)\,(a^+a)_{\alpha l}. \qquad (5.13)$$

Using (5.12), we obtain the induced moment of the core

$$\langle 0\,|\,\hat{Q}_l\,|\,0\rangle = -\varphi_l\sum_{\alpha\beta}[q_l(\alpha)\,\gamma_l(\beta) + q_l(\beta)\,\gamma_l(\alpha)]\sum_s\frac{1}{E_{s0}}$$
$$\times\langle 0\,|\,(a^+a)_{\alpha l}\,|\,s\rangle\langle s\,|\,(a^+a)_{\beta l}\,|\,0\rangle. \qquad (5.14)$$

Introducing the multipole moment of the external nucleon $\langle Q_l\rangle_p = \varphi_l q_l$, we can rewrite (5.14) in the form

$$\langle 0|\hat{Q}_l|0\rangle = \eta_l' \langle Q_l\rangle_p, \qquad (5.15)$$

where η_l', the polarizability of the core, is determined from the relation

$$\eta_l' = \sum_{\alpha\beta} \frac{1}{2q_l}[q_l(\alpha)\gamma_l(\beta) + q_l(\beta)\gamma_l(\alpha)]\,h_l(\alpha\beta) \equiv \overline{\gamma}_l' \sum_{\alpha\beta} h_l(\alpha\beta). \qquad (5.16)$$

From a comparison of (5.16) and (5.8) we see that the quantities η_l and η_l' actually coincide. (The small difference is due only to a somewhat different determination of the average quantities $\bar{\gamma}$ and $\bar{\gamma}'$.) Thus, the increase in the direct interaction of the external nucleons due to the coupling with the core is determined by the polarizability of the latter.*

The quadrupole polarizability of the core (η_2) is measured experimentally from the value of the quadrupole moment for the magic nuclei plus one nucleon, and lies in the range $1 < \eta_2 < 3$.[11] The polarizability of the higher multipolarities is apparently much less, leading to a "selective" increase in the quadrupole interaction.

We note that the influence of the core on the interaction between the external nucleons is not confined to the considered effect. The remaining effects, however, particularly the renormalization of Γ'', are not connected with the presence of the surface, and exist also in infinite nuclear matter.

6. CONCLUSION

The main results obtained above can be formulated in the following manner:

Depending on the character of the interaction there can be formed in spherical nuclei "bound" (correlated) states of pairs with non-vanishing momenta of both the Cooper type and of the particle—hole type. This possibility arises only in the presence of pairing, that is, a condensate of Cooper pairs with zero momentum.

The polarizability of the core leads to a strong increase in the effective quadrupole interaction between the outer nucleons, and this contributes to the formation of a bound state of a particle and a hole with momentum $J = 2$. Macroscopically these excitations can be considered as oscillations of a quadrupole moment of external nucleons. The surface oscillations arising thereby are only a consequence of the static polarizability of the core, and have nothing in common with the hydrodynamic surface oscillations.

These general results are not connected with the specific model of a single j level used in the calculation. (The more general case leads only to a mathematical complication of non-principal nature.*)

*For the quadrupole interaction this result was obtained by the author[2] from macroscopic considerations.

Which of the possible bound states are realized in real nuclei is a question that calls for a comparison of the experimental data with the results of quantitative calculations. For quadrupole excitations, such a comparison was carried out by Kisslinger and Sorensen.[13] One can classify as this type of excitation the first level 2^+, and in most cases also the following levels (2^+, 4^+), which correspond to two bound pairs. The question of the existence of bound states with higher moments (and also with odd moments in the case of mixture of several j levels) calls for a thorough analysis of the experimental data. Apparently they lie somewhat above the "two-phonon" state of the quadrupole branch.

The author is grateful to V. M. Galitskiĭ and A. B. Migdal for numerous discussions.

*Veneroni and Arvieu have considered a case of several j levels, but without Cooper pairing.

[1] A. Bohr, and B. Mottelson, Kgl. Danske Videnskab Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).

[2] S. T. Beliaev, Kgl. Danske Videnskab Selskab, Mat.-fys. Medd. **31**, No. 11 (1959).

[3] Bogolyubov, Tolmachev, and Shirkov, Новый метод в теории сверхпроводимости (A New Method in the Theory of Superconductivity), Acad. of Sci. Press, 1959.

[4] L. D. Landau, JETP **32**, 59 (1957), Soviet Phys. JETP **5**, 101 (1957); Yu. L. Klimontovich and V. P. Silin, JETP **23**, 151 (1952).

[5] V. M. Galitskiĭ and A. B. Migdal, JETP **34**, 139 (1958), Soviet Phys. JETP **7**, 96 (1958).

[6] L. P. Gor'kov, JETP **34**, 735 (1958), Soviet Phys. JETP **7**, 505 (1958).

[7] M. Gell-Mann, and F. Low, Phys. Rev. **84**, 350 (1951).

[8] J. P. Elliot, and A. M. Lane, Handbuch d. Physik, 39, (1957).

[9] A. R. Edmonds, Angular Momentum in Quantum Mechanics, Princeton University Press, (1957).

[10] K. Helmers, Nucl. Phys. **12**, 647 (1959).

[11] Z. Ṡzymański, Nucl. Phys. **11**, 454 (1959).

[12] M. Veneroni and R. Arvieu, Compt. rend. **250**, 2155 (1960).

[13] L. S. Kisslinger and R. A. Sorensen, Preprint, 1960.

Translated by J. G. Adashko
254