RECURRENT CONSTRUCTION OF ANGULAR OPERATORS. II

Introduction of an Integral Spin and an Arbitrary Orbital Momentum

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The method previously proposed by the authors[1] for practical construction of angular operators, in which one uses only differential operations, is applied to studying the change in the form of the angular operators resulting from the adding of a new orbital angular momentum or an integer spin to the process. Formulas for practical computations and examples are given.

INTRODUCTION

In studying processes in which more than four particles participate, the computation of the angular operators (their definition and properties can be found in references 1 — 3) becomes very complicated and long. The practical question therefore arises of obtaining a complete set of angular operators for a given process on the assumption that one knows the angular operators for a simpler process. In its simplest form* this question can be expressed in the form of the relation

$$\Omega_0 \mathcal{Y}(a_1 + a_2 \to \sum_1^n a_i') = \mathcal{Y}(a_1 + a_2 \to \sum_1^n a_i' + s), \quad (1)$$

or

$$\Omega_0 \mathcal{Y}(a \to \sum_1^n a_i') = \mathcal{Y}(a + s \to \sum_1^n a_i').$$

In other words, the operator Ω_0 "adds" a new scalar particle to the process. However, it is not difficult to show that this question can be formulated as a special case of the problem of finding an operator Ω, which changes the quantum number of one of the angular momenta in the reaction from l to l', without changing the number of particles participating in the process. This enables us to find a complete set of angular operators for the process if we know just one term of the set. Therefore we shall first find the explicit form of the operator Ω, and then in Sec. 3 express Ω_0 in terms of Ω.

It will also be shown that the proposed method can be used also for finding angular operators for a process in which, compared with the initial process, one scalar particle has been replaced by a vector or a tensor particle.

1. CHANGE IN QUANTUM NUMBER OF AN ORBITAL ANGULAR MOMENTUM

Let us consider reactions of the type

$$a_1 + a_2 \to a_1' + a_2' + \ldots + a_k' \quad (2a)$$

and decays of the type

$$a \to a_1' + a_2' + \ldots + a_k'. \quad (2b)$$

In processes of type (2a) there is one initial and $k - 1$ final unknown orbital angular momenta in the system of the center of mass. [In the decays (2b) there are no initial orbital angular momenta.] If we also include the spins of all the particles in the processes (2a) and (2b), then there are altogether M initial and N final independent angular momenta, where

$$0 \leqslant M \leqslant 3, \quad k-1 \leqslant N \leqslant 2k-1,$$

depending on how many particles with non-zero spin are present. If M and N are greater than unity, these angular momenta can combine with one another into total angular momenta; in this process the number of initial angular momenta increases up to $2M - 1$ and the number of final angular momenta to $2N - 1$. All these angular momenta and the corresponding quantum numbers will be denoted (without distinguishing orbital,

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‡The symbols a, a_i, a_i' denote any particles or nuclei, s is a particle with spin zero, $\mathcal{Y}$ is the angular operator for the process written inside the parentheses.

spin, and total angular momenta) by the symbol l with different subscripts.

Let us consider any process of type (2a) or (2b). Let us assume that its initial and final states are characterized by complete sets of quantum numbers for the angular momenta. Let us consider three angular momenta $\mathbf{l}_a, \mathbf{l}_b, \mathbf{l}_c$ of the set for the final state, and assume that they are connected by the relation

$$\mathbf{l}_a + \mathbf{l}_b = \mathbf{l}_c. \tag{3}$$

As yet we make no assumptions concerning the nature of the angular momenta $\mathbf{l}_a$ and $\mathbf{l}_b$. The angular operator $\mathcal{Y}$ with the quantum numbers l_a, l_b, and l_c can be written in the form

$$\mathcal{Y}(\ldots l_a l_b l_c \ldots) = \sum_{M=-l_c}^{l_c} X_{l_c M} \Psi_{l_c M}^{l_a l_b}, \tag{4}$$

where

$$\Psi_{l_c M}^{l_a l_b} = \sum_{m=-l_a}^{l_a} C_{l_c M}^{l_a m l_b M-m} \psi_{l_a m} \varphi_{l_b M-m}$$

and where the $X_{l_c M}$ are functions of the other variables in the process, and do not depend on l_a and l_b. If on the right side of (4) we substitute in place of the function Ψ the functions Ψ' with the same values of l_c and M, but with different l_a' and l_b', then the expression

$$\sum_M X_{l_c M} \Psi_{l_c M}^{l_a' l_b'} = \mathcal{Y}(\ldots l_a' l_b' l_c \ldots)$$

is another angular operator for the same process. Therefore the transition from $\mathcal{Y}(\ldots l_a l_b l_c \ldots)$ to $\mathcal{Y}(\ldots l_a' l_b' l_c \ldots)$ is equivalent to a transition on the right-hand side of (4) from Ψ to Ψ', and can be described formally as follows:

$$\Omega \mathcal{Y}(\ldots l_a l_b l_c \ldots) = \mathcal{Y}(\ldots l_a' l_b' l_c \ldots), \tag{5}$$

i.e., we have

$$\Omega \Psi_{l_c M}^{l_a l_b} = \Psi_{l_c M}^{l_a' l_b'}, \tag{5'}$$

where the operator Ω depends on l_a and l_b and is a scalar with respect to simultaneous rotations of the spaces characterizing $\mathbf{l}_a$ and $\mathbf{l}_b$.*

The fundamental problem is to find the explicit form of Ω^{rs} for the case $l_a' = l_a + r$, $l_b' = l_b + s$, where r, s = 1, 0, −1. All the other cases are solved by repeated application of the Ω^{rs} (cf. Sec. 3).

Let us use Johnson's theorem:[4] If there are vector operators **A**, **B** with the commutation relations

$$[l_{ai}, A_j] = i \sum_k \varepsilon_{ijk} A_k, \qquad [l_{bi}, A_j] = 0, \qquad [l_{ai}, B_j] = 0,$$
$$[l_{bi}, B_j] = i \sum_k \varepsilon_{ijk} B_k \qquad (ijk = xyz,\ \varepsilon_{xyz} = 1), \tag{6}$$

then the matrix elements of their scalar product are equal [cf., for example, reference 5, formula (3.101)] to

$$\langle l_a + r, l_b + s, l_c', M' | \mathbf{AB} | l_a, l_b, l_c, M\rangle = \langle l_a + r \vdots A \vdots l_a\rangle \langle l_b + s \vdots B \vdots l_b\rangle \chi^{rs} \delta_{l_c' l_c} \delta_{M'M}, \tag{7}$$

where the χ^{rs} are numerical coefficients having the values

$$\chi^{11} = -\tfrac{1}{2} [(l_a + l_b + l_c + 3)(l_a + l_b + l_c + 2) \times (l_a + l_b + 2 - l_c)(l_a + l_b + 1 - l_c)]^{1/2},$$
$$\chi^{10} = -\tfrac{1}{2} [(l_c + l_a - l_b + 1)(l_c + l_b - l_a) \times (l_c + l_a + l_b + 2)(l_a + l_b - l_c + 1)]^{1/2},$$
$$\chi^{1-1} = \tfrac{1}{2} [(l_c + l_a - l_b + 1)(l_c + l_a - l_b + 2) \times (l_c - l_a + l_b - 1)(l_c - l_a + l_b)]^{1/2},$$
$$\chi^{0-1} = \tfrac{1}{2} [(l_c - l_a + l_b)(l_c + l_a - l_b + 1) \times (l_c + l_a + l_b + 1)(l_a + l_b - l_c)]^{1/2},$$
$$\chi^{-1-1} = -\tfrac{1}{2} [(l_a + l_b + l_c + 1)(l_a + l_b + l_c) \times (l_a + l_b - l_c)(l_a + l_b - l_c - 1)]^{1/2},$$

with the symmetry properties $\chi^{rs}(l_a, l_b, l_c) = (-1)^{r+s} \chi^{sr}(l_b, l_a, l_c)$. The symbol $\langle l_a + r \vdots A \vdots l_a\rangle$ is defined by the relations

$$\langle l_a + 1 \vdots A \vdots l_a\rangle = [(l_a + 1)^2 - m_a^2]^{-1/2} \langle l_a + 1, m_a | A_z | l_a, m_a\rangle,$$
$$\langle l_a \vdots A \vdots l_a\rangle = m_a^{-1} \langle l_a, m_a | A_z | l_a, m_a\rangle,$$
$$\langle l_a - 1 \vdots A \vdots l_a\rangle = (l_a^2 - m_a^2)^{-1/2} \langle l_a - 1, m_a | A_z | l_a, m_a\rangle. \tag{8}$$

Analogous relations hold for $\langle l_b + s \vdots B \vdots l_b\rangle$.

To raise or lower l_a and l_b we need operators Ω^{rs} whose matrix elements are all equal to zero, except for the one where we have on the left $\langle l_a + r, l_b + s, l_c, m|$ and on the right $|l_a, l_b, l_c, m\rangle$. We shall write them as scalar products of two vectors $\mathbf{A}^{(r)}, \mathbf{B}^{(s)}$ of the type of Eq. (6) which, according to Johnson's theorem, should have the form

$$\langle l_a + r' \vdots A^{(r)} \vdots l_a\rangle = \langle l_a + r \vdots A^{(r)} \vdots l_a\rangle \delta_{rr'},$$
$$\langle l_b + s' \vdots B^{(s)} \vdots l_b\rangle = \langle l_b + s \vdots B^{(s)} \vdots l_b\rangle \delta_{ss'}. \tag{9}$$

Then, according to Eq. (7), the matrix elements of $\mathbf{A}^{(r)} . \mathbf{B}^{(s)}$ will be

$$\langle l_a + r', l_b + s', l_c', M' | \mathbf{A}^{(r)} \mathbf{B}^{(s)} | l_a l_b l_c M\rangle = \langle l_a + r \vdots A^{(r)} \vdots l_a\rangle \langle l_b + s \vdots B^{(s)} \vdots l_b\rangle \chi^{rs} \delta_{rr'} \delta_{ss'} \delta_{l_c' l_c} \delta_{MM'}. \tag{10}$$

*The angular operators $\mathcal{Y}$ are scalar with respect to simultaneous rotations of all the orbital and spin dynamical variables of the process.

Consequently, the operators

$$\Omega^{rs}(l_a l_b l_c) = \mathbf{A}^{(r)}\mathbf{B}^{(s)} / \chi^{rs}\langle l_a + r \vdots A^{(r)} \vdots l_a\rangle \langle l_b + s \vdots B^{(s)} \vdots l_b\rangle \quad (11)$$

have the required properties [cf. Eqs. (5) and (5′)]:

$$\Omega^{rs}(l_a l_b l_c)\Psi^{l_a l_b}_{l_c, M} = \Psi^{l_a+r,\, l_b+s}_{l_c, M}$$

In other words,

$$\Omega^{rs}(l_a l_b l_c)\mathcal{Y}(\ldots l_a, l_b, l_c \ldots) = \mathcal{Y}(\ldots l_a + r, l_b + s, l_c \ldots), \quad (12)$$

where r, s = 1, 0, −1.

In order for formulas (12) and (11) to be practical, one must find the explicit form of the vectors $\mathbf{A}^{(r)}$ and $\mathbf{B}^{(s)}$ having the properties (6) and (9). The next section is devoted to this problem.

2. EXPLICIT FORM OF THE VECTORS $\mathbf{A}^{(r)}$

The explicit form of the vectors $\mathbf{A}^{(r)}$ depends on whether the angular momentum $\mathbf{l}_a$ is the sum of two (or more) angular momenta or not.*

1. Let us first assume that the eigenfunction $\psi_{l_a m_a}$ is a spherical harmonic:

$$\psi_{l_a m_a} = Y_{l_a m_a}(\mathbf{p}), \quad (13)$$

where $\mathbf{l}_a = -i\,[\mathbf{p} \times \partial/\partial\mathbf{p}]$.

If $\mathbf{l}_a$ is an angular momentum, then $\mathbf{p}$ will be the unit momentum vector $\mathbf{p} = \mathbf{P}/P$; if on the other hand $\mathbf{l}_a$ denotes an integer spin, then $\mathbf{p}$ will be the spin variable α.

Omitting the mathematical derivations (cf. reference 8), we give only the final results†

$$\mathbf{A}^{(1)} = \nabla^{+1} \equiv P^{l+2}\frac{\partial}{\partial\mathbf{P}}P^{-(l+1)}, \quad \mathbf{A}^{(0)} = \nabla^{0} \equiv \mathbf{l} = -i\left[\mathbf{P}\times\frac{\partial}{\partial\mathbf{P}}\right],$$

$$\mathbf{A}^{(-1)} = \nabla^{-1} \equiv P^{-(l-1)}\frac{\partial}{\partial\mathbf{P}}P^{l}. \quad (14)$$

It can be shown by direct calculation that the vectors ∇^{+1}, ∇^{0}, ∇^{-1} satisfy the commutation relations (6) and have the property (9). From (8) and (13) it also follows that

$$\langle l+1 \vdots \nabla^{+1} \vdots l\rangle = -\sqrt{\frac{2l+1}{2l+3}}, \qquad \langle l \vdots \nabla^{0} \vdots l\rangle = 1,$$

$$\langle l-1 \vdots \nabla^{-1} \vdots l\rangle = \sqrt{\frac{2l+1}{2l-1}}. \quad (15)$$

2. Now let us assume that the function ψ_{lm} is a combination of two or more spherical harmonics:

$$\psi_{lm} = \sum C^{l_1\mu l_2 m-\mu}_{lm}\chi_{l_1\mu}\varkappa_{l_2 m-\mu}. \quad (13')$$

*All the formulas of this section are obtained only for vectors $\mathbf{A}^{(r)}$. The corresponding formulas for $\mathbf{B}^{(s)}$ are obtained by the trivial replacements

$$\mathbf{A} \to \mathbf{B}, \quad r \to s, \quad l_a \to l_b, \quad m_a \to m_b, \quad \psi \to \varphi.$$

*Here and in what follows we drop the subscript a on l_a, m_a, etc.

This corresponds to the addition of angular momenta $\mathbf{l} = \mathbf{l}_1 + \mathbf{l}_2$. If now we apply to ψ_{lm} a vector operator l_{1i} (i = x, y, z), then, according to the general theory (cf. reference 5), we obtain a mixture of states ψ_{l+1}, ψ_l and ψ_{l-1}. It is then sufficient to apply the projection operators $Z_0 Z_-$, $Z_- Z_+$ and $Z_+ Z_0$ where

$$Z_0 = \mathbf{l}^2 - l(l+1), \qquad Z_- = \mathbf{l}^2 - l(l-1),$$

$$Z_+ = \mathbf{l}^2 - (l+1)(l+2),$$

in order to obtain the pure states ψ_{l+1}, ψ_l, and ψ_{l-1} respectively. Thus it is easy to see that the operators $\mathbf{A}^{\pm 1}$ will have the form:

$$\mathbf{A}^{(\pm 1)} = \mathbf{V}^{\pm 1} \equiv Z_{\mp} Z_0 \mathbf{l}_1. \quad (16)$$

The computation of $\mathbf{V}^{\pm 1}$ from formula (16) (cf. reference 8) gives the result

$$\mathbf{V}^{\pm 1} = 2\mathbf{l}_1\{l(l+1) - l_1(l_1+1) + l_2(l_2+1)\}$$
$$- 2\mathbf{l}_2\{l(l+1) + l_1(l_1+1)$$
$$- l_2(l_2+1)\} + 4i\left(\pm l + \tfrac{1}{2} \pm \tfrac{1}{2}\right)[\mathbf{l}_1\mathbf{l}_2]. \quad (17)$$

The operator $\mathbf{A}^0$ is obviously the total angular momentum:

$$\mathbf{A}^0 = \mathbf{V}^0 \equiv \mathbf{l} = \mathbf{l}_1 + \mathbf{l}_2 \quad (17')$$

[cf. Eq. (14)].

The matrix elements of the vectors $\mathbf{V}^{+1}$, $\mathbf{V}^0$ and $\mathbf{V}^{-1}$ are equal to

$$\langle l_1 l_2 l+1 \vdots V^1 \vdots l_1 l_2 l\rangle$$
$$= 2\sqrt{\frac{2l+1}{2l+3}}[(l+1-l_1+l_2)(l+1+l_1-l_2)$$
$$\times (l_1+l_2+l+2)(l_1+l_2-l)]^{1/2},$$

$$\langle l_1 l_2 l \vdots V^0 \vdots l_1 l_2 l\rangle = 1,$$

$$\langle l_1 l_2 l-1 \vdots V^{-1} \vdots l_1 l_2 l\rangle = 2\sqrt{\frac{2l+1}{2l-1}}[(l-l_1+l_2)(l+l_1-l_2)$$
$$\times (l_1+l_2+1)(l_1+l_2+1-l)]^{1/2}. \quad (18)$$

All the other matrix elements are equal to zero, as they should be according to formula (9).

3. ANALYSIS OF RESULTS

1. Formula (11) determines the operator Ω^{rs} by means of which we can, according to (12), change the quantum numbers characterizing the angular operators $\mathcal{Y}$ of a given process by one unit. The vectors $\mathbf{A}^{(r)}$ have the form either of the vectors $\mathbf{V}^r$ of formula (17) or the vectors ∇^r of formula (14), depending on whether the corresponding angular momentum $\mathbf{l}_a$ (whose quantum number is to be changed) can be written as a sum of two other angular momenta $\mathbf{l}_{a1}$ and $\mathbf{l}_{a2}$ or not.

2. Increasing or decreasing l_a, l_b by any integer > 1 is achieved by repeated application of the operator Ω^{rs}. Thus, if we know one angular operator for the given process, we can by means of Ω^{rs} obtain all the others.

3. As was already stated in the introduction, the operator Ω can also be used for going over to a more complicated process containing one more scalar particle than participates in the original process [cf. formula (1)]. This is related to the fact that the spherical harmonic Y_{00} is equal to the constant $1/\sqrt{4\pi}$ so that the introduction in the process of a new orbital angular momentum l with quantum number l equal to zero reduces to multiplying the whole set of angular operators by $1/\sqrt{4\pi}$. For example

$$\frac{1}{\sqrt{4\pi}}\cdot\mathcal{Y}\left(a_1+a_2\to\sum_1^n a_i'\right)=\mathcal{Y}\left(a_1+a_2\to\sum_1^n a_i'+s'\right),$$

$$l_{s'}=0.$$

Having determined in this way the angular operators for the new process for $l_{s'}=0$, we can by using Ω^{1i} $(0, l, l)$ increase the value of $l_{s'}$ to unity. Similarly, the operator $\Omega^{1i}(1, l', l)$ increases $l_{s'}$ from 1 to 2, etc.

Let us take, for example, the process $s_1+s_2 \to s_1'+s_2'$, characterized by the following angular operators, ($\mathcal{P}_l$ is a Legendre polynomial):

$$\mathcal{Y}_l(s_1+s_2\to s_1'+s_2')=\frac{2l+1}{4\pi}\mathcal{P}_l(\mathbf{pr}),\quad l=0,1,2,\ldots \tag{19}$$

Multiplying by $1/\sqrt{4\pi}$ we obviously obtain $\mathcal{Y}_{l_{s_1'}, l, l}$ $(s_1+s_2\to s_1'+s_2'+s_3')$, where $l_{s_3'}=0$ and $l=0, 1, 2\ldots$ According to (11), (15), the operator

$$\Omega^{1r}(0,l,l)=\nabla_R^{+1}\nabla_Q^r/\chi^{1r}\langle 1:\nabla_R^{+1}:0\rangle\langle l+r:\nabla_Q^r:l\rangle$$

increases $l_{s_3'}$ from zero to unity:

$$\Omega^{1r}(0,l,l)\mathcal{Y}_{0,l,l}=\mathcal{Y}_{1,l+r,l}.$$

For convenience we give the following general formulae which are useful in particular computations:

$$\nabla_R^0(l)f(\mathbf{pr})=-i\left[\mathbf{R}\times\frac{\partial}{\partial\mathbf{R}}\right]f(\mathbf{pr})=i[\mathbf{p}\times\mathbf{r}]f',$$

$$\nabla_R^{+1}(l)f(\mathbf{pr})=R^{l+2}\frac{\partial}{\partial\mathbf{R}}R^{-(l+1)}f(\mathbf{pr})$$

$$=-(l+1)\mathbf{r}f+(\mathbf{p}-\mathbf{r}(\mathbf{pr}))f',$$

$$\nabla_R^{-1}(l)f(\mathbf{pr})=R^{-(l-1)}\frac{\partial}{\partial\mathbf{R}}R^l f(\mathbf{pr})=l\mathbf{r}f+(\mathbf{p}-\mathbf{r}(\mathbf{pr}))f', \tag{20}$$

and note that

$$\nabla_R^{+1}(l=0)\cdot 1=-\mathbf{r}. \tag{20'}$$

By means of these formulas we obtain, for example, for $\mathbf{r}=0$,

$$\mathcal{Y}_{1,l,l}=-\frac{1}{(4\pi)^{3/2}}\frac{i(2l+1)\sqrt{3}}{\sqrt{l(l+1)}}\mathbf{r}[\mathbf{p}\times\mathbf{q}]\mathcal{P}_l' \tag{21}$$

and similarly for $l+\mathbf{r}=l+1$ ($\mathcal{P}_l$ depends on $\mathbf{p}\cdot\mathbf{q}$). Continuing further, we can obtain $\mathcal{Y}_{l_{s_3'}, l, l}$ for $l_{s_3'}=2, 3$, etc. For example,

$$\mathcal{Y}_{2ll}=-\frac{\sqrt{5}}{(4\pi)^{3/2}}\frac{2l+1}{[l(l+1)(2l-1)(2l+3)]^{1/2}}\{l(l+1)\mathcal{P}_l$$

$$-3(\mathbf{pq}-(\mathbf{pr})(\mathbf{qr}))\mathcal{P}_l'-3(\mathbf{r}[\mathbf{p}\times\mathbf{q}])^2\mathcal{P}_l''\}. \tag{22}$$

4. The introduction of an integral spin can be done in similar fashion. One need only make the formal substitution

$$r_i\to\mathbf{e}_i\sqrt{4\pi/3},\quad i=x,y,z. \tag{23}$$

In this way, we obtain angular operators for processes of the type

$$s_1+s_2\to s_1'+v_2',$$

where v_2' is a particle with spin 1, 2, 3 etc. For example, in the case of a particle with spin 2, we obtain from (22) by using (23)

$$\mathcal{Y}=-\frac{\sqrt{5}}{3\sqrt{4\pi}}\frac{2l+1}{[l(l+1)(2l-1)(2l+3)]^{1/2}}\{\delta_{\alpha\beta}l(l+1)\mathcal{P}_l$$

$$-3(\mathbf{pq}\,\delta_{\alpha\beta}-p_\alpha q_\beta)\mathcal{P}_l'-3[\mathbf{p}\times\mathbf{q}]_\alpha[\mathbf{p}\times\mathbf{q}]_\beta\mathcal{P}_l''\}\mathbf{e}_\alpha\mathbf{e}_\beta.$$

We note that for spin 1 the form of the operators Ω simplifies considerably if we use (20′) and (23):

$$\Omega^1=\frac{1}{\sqrt{(l+1)(2l+1)}}\nabla^{+1},\qquad \Omega^0=\frac{1}{\sqrt{l(l+1)}}\nabla^0,$$

$$\Omega^{-1}=\frac{1}{\sqrt{l(2l+1)}}\nabla^{-1}.$$

It is known that these operators transform the spherical harmonics Y_l^m into spherical vectors.

5. Let us show on a few examples how one can, by this method, obtain from the operators for the reaction

$$s+N\to s'+N' \tag{24}$$

the angular operators for the reaction

$$s+N\to s_1+s_2+N',$$

which were given in Tables I, II and III of our earlier paper.[3] It is known that the angular operators for the process (24) have the form*

$$\{(l'+1)\mathcal{P}_{l'}+i\sigma\mathbf{P}_y\mathcal{P}_{l'}'\}/4\pi \text{ for } j'=l'+1/2=l+1/2=j, \tag{25.1}$$

$$\{-(l'+1)\sigma\mathbf{p}\mathcal{P}_{l'}+\sigma\mathbf{P}_x\mathcal{P}_l'\}/4\pi \text{ for } j'=l'+1/2=l-1/2=j, \tag{25.2}$$

$$\{-l'\sigma\mathbf{p}\mathcal{P}_{l'}-\sigma\mathbf{P}_x\mathcal{P}_{l'}'\}/4\pi \text{ for } j'=l'-1/2=l+1/2=j, \tag{25.3}$$

*In the following we shall use the notation introduced in reference 3 and also used in Sec. 5 of reference 1. In particular we recall that $\mathbf{P}_y=[\mathbf{p}\times\mathbf{r}]$, $\mathbf{P}_x=\mathbf{r}-\mathbf{p}(\mathbf{p}\cdot\mathbf{r})$.

$\{l'\mathcal{P}_{l'} - i\boldsymbol{\sigma}\mathbf{P}_y\mathcal{P}'_{l'}\}\,/\,4\pi$ for $j' = l' - {}^1/_2 = l - {}^1/_2 = j$, (25.4)

where l' is the orbital angular momentum of the final state and the Legendre polynomials $\mathcal{P}_{l'}$ depend on $\mathbf{p}\cdot\mathbf{r} = \cos\theta$. We note that s, s′, s_1 and s_2 correspond to particles with spin zero and without definite parity; if the parities of s and s′ are the same, only the first and fourth channels are allowed.

a) The four angular operators of Table I in reference 3, corresponding to the quantum numbers $l_1 = 0$, $l_2 = l'$, can be obtained from (25.1) – (25.4) by multiplying by $1/\sqrt{4\pi}$: (I 1) = $(4\pi)^{-1/2}$ × (25.1), (I 2) = $(4\pi)^{-1/2}$ × (25.2), etc.

b) The first four angular operators (III 1 – 4) of Table III in reference 3 can be obtained from (I 1 – 4) in the following fashion:

$$\Omega^{10}(l_1 = 0,\ \tfrac{1}{2},\ j = \tfrac{1}{2})(\text{I}\ 1;2;3;4) = (\text{III}\ 1;2;3;4), \quad (26)$$

where*

$$\Omega^{10} = \nabla^1_Q\nabla^0_\sigma/\chi^{10}\langle 1 \vdots \nabla^1 \vdots 0\rangle\langle \tfrac{1}{2} \vdots \nabla^0 \vdots \tfrac{1}{2}\rangle = -\boldsymbol{\sigma}\mathbf{q}.$$

Let us, for example, verify the second of the relations (26). We have

$$-\boldsymbol{\sigma}\mathbf{q}(\text{I}\ 2) = (4\pi)^{-3/2}\{l_2 + 1)(\boldsymbol{\sigma}\mathbf{q})(\boldsymbol{\sigma}\mathbf{p})\mathcal{P}_{l_2}$$
$$-(\boldsymbol{\sigma}\mathbf{q})(\boldsymbol{\sigma}\mathbf{P}_x)\mathcal{P}'_{l_2}\} = (\text{III. } 2).$$

We note that the angular operators (I 3) and (II 7) can be associated with diagrams which clearly show the connection of the orbital angular momenta (cf. Fig. 1). The angular momenta of the initial state are on the right, those of the final state on the left. To each angular momentum there corresponds one line, where a spin ½ is indicated by an arrow. From the diagram it is also clear how the operators Ω act.

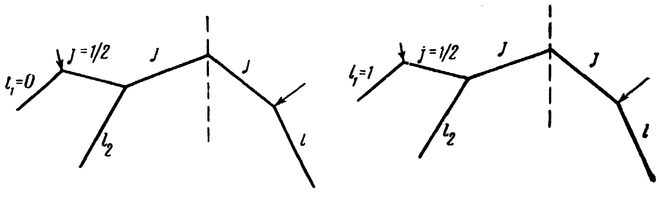

FIG. 1

c) Let us calculate the angular operator (III 7) by means of (III 1) (cf. Fig. 2). We have

$$\Omega^{10} = \mathbf{V}^1_j\nabla^0_{l_2}/\chi^{10}\langle \tfrac{3}{2} \vdots V^1 \vdots \tfrac{1}{2}\rangle\langle l_2 \vdots \nabla^0 \vdots l_2\rangle$$
$$= -\mathbf{V}^1_j\nabla^0_{l_2}/4\sqrt{l_2(2l_2+3)},$$

where $\mathbf{V}^1_j$ and $\nabla^0_{l_2}$ are equal, according to (14) and (17), to the following expressions:

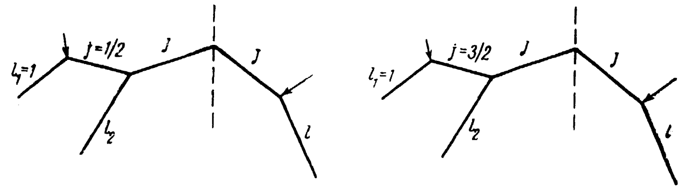

FIG. 2

$$\mathbf{V}^1_j = -\mathbf{l}_1 - 2\boldsymbol{\sigma} - 3i[\boldsymbol{\sigma}\mathbf{l}_1], \qquad \mathbf{l}_1 = -i[\mathbf{Q}\times\partial/\partial\mathbf{Q}];$$
$$\nabla^0_{l_2} = \mathbf{l}_2, \qquad \mathbf{l}_2 = -i[\mathbf{R}\times\partial/\partial\mathbf{R}].$$

The computation gives the result

$$\Omega^{10}(\text{III}\ 1) = -(4\pi)^{-3/2}[l_2(2l_2+3)]^{-1/2}\{(i(2l_2+3)\mathbf{P}_y\mathbf{q}$$
$$-(l_2+3)(\boldsymbol{\sigma}\mathbf{r})(\mathbf{pq}) + l_2(\boldsymbol{\sigma}\mathbf{p})(\mathbf{rq}) + (\boldsymbol{\sigma}\mathbf{q})(\mathbf{pr}))\mathcal{P}'_{l_2}$$
$$-(3(\boldsymbol{\sigma}\mathbf{P}_y)(\mathbf{P}_y\mathbf{q}) + \boldsymbol{\sigma}\mathbf{q}((\mathbf{pr})^2 - 1))\mathcal{P}''_{l_2}\}. \quad (27)$$

We note that in this way we can obtain the angular operators in a much simpler form than they were given in reference 3.

CONCLUSION

From the analysis given we see that the proposed method for computing angular operators is simpler than the algebraic methods which are usually used. In particular, it is simpler than the method given by the authors in reference 3, and besides it has a more general applicability. As is known, the algebraic methods (which make use of the Clebsch-Gordan and Racah coefficients) lead to difficult computations which restrict their applicability in angular and polarization analysis to only the simplest processes. On the other hand, the differential methods retain their simplicity even in the case of complex processes. Besides, if one knows one or more angular operators, the computation is simplified, since the proposed method is a recursion method.

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*The results of Sec. 2 are valid for an orbital angular momentum and an integer spin, but the relation $\nabla^0 = \sigma/2$ analogous to the second of formulas (14), is also valid.