ON THE THEORY OF RUNAWAY ELECTRONS

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The influence of a relatively weak electric field on the velocity distribution of plasma electrons in the range of high velocities is considered. An expression for the stationary electron distribution function is obtained and analyzed. The magnitude of the flux of runaway electrons in a completely ionized plasma is determined. The effect on this flux of neutral plasma particles is taken into account. It is shown that under certain conditions instabilities in the plasma may occur during the development of the discharge which are due to the runaway electron flux. The results obtained are in qualitative agreement with experiment.

1. INTRODUCTION

As is well known, the frequency of collisions of an electron with ions, and also with other electrons in the plasma, falls off sharply as its velocity increases. Therefore, friction is always negligibly small for electrons possessing a sufficiently high energy. If a constant electric field is present in the plasma the velocity of such electrons increases continuously with time; they are usually called "runaway" electrons.

Clearly, in a very strong electric field (or in a plasma of sufficiently low density) all the electrons are accelerated by the field, i.e., become "runaway" electrons. In a weak field only very fast electrons will run away, i.e., those whose velocity v exceeds a certain critical value v_c. The velocity v_c depends in an essential manner on the magnitude of the field; in a weak field it is, naturally, much larger than the average thermal velocity of the electrons in the plasma and, therefore, the number of runaway electrons is not very great in this case. In order to determine it we must know the way in which the density of electrons having a velocity $v \sim v_c$ varies, i.e., we must know the velocity distribution for the electrons for $v \sim v_c$. The corresponding problem must, naturally, be solved taking collisions into account (since they determine the critical velocity v_c, and, consequently also the number of runaway electrons), and is therefore very complicated in general. In previous papers[1] only some numerical calculations for completely ionized plasma have been carried out (Dreicer, Bernstein and Rabinowitz), and also a solution has been obtained for a very strong electrical field when collisions may be neglected in the first approximation (Kovrizhnykh).

The investigation of the phenomenon of runaway electrons in the case of a weak electrical field in a completely ionized plasma is the aim of Sec. 2 of the present paper. The effect on the runaway electrons of neutral particles in the plasma is taken into account in Sec. 3.

2. THE DISTRIBUTION FUNCTION AND THE FLUX OF RUNAWAY ELECTRONS IN A COMPLETELY IONIZED PLASMA

The equation for the electron distribution function $f(v, \theta, t)$ in the domain of high velocities $(v \gg \sqrt{kT_e/m})$ in a completely singly ionized plasma situated in a constant uniform electric field **E** has the following form*

$$\frac{\partial f}{\partial t} + \frac{eE}{m}\left(\cos\theta\frac{\partial f}{\partial v} - \frac{\sin\theta}{v}\frac{\partial f}{\partial \theta}\right) - \frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^2\nu_e(v)\left[\frac{kT_e}{m}\frac{\partial f}{\partial v} + vf\right]\right\}$$

$$- \frac{\nu(v)}{2\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial f}{\partial\theta}\right\} = 0. \qquad (1)$$

Here, as usual, e, m and T_e are the charge, mass and temperature of the electrons; k is the Boltzmann constant; θ is the angle between **E** and **v**; $\nu_e(v)$ is the collision frequency of an electron of velocity v with other electrons in the plasma:

*In the velocity range $v \lesssim \sqrt{kT_e/m}$ the terms in Eq. (1) which describe collisions between electrons should be written in a more complicated form (cf., for example, references 2 and 3). However, this circumstance is not essential in our case in view of the fact that in a weak field ($E \ll E_{ci}$), which is the only one considered in the present article, any significant deviations of the distribution function from the equilibrium one appear only at high velocities, $v \gg \sqrt{kT_e/m}$ (cf. below).

$$\nu_e(v) = (4\pi e^4 N_e/m^2 v^3)\ln(mv^2 D/e^2), \qquad (2)$$

where N_e is the electron density, $D = \sqrt{kT_e/4\pi e^2 N_e}$ is the Debye radius. We have further

$$\nu(v) = \nu_i(v) + \nu_e(v)(1 - kT_e/2mv^2)$$

giving the frequency of collisions of an electron with ions and with electrons [$\nu_i(v)$ is also given by formula (2), but with N_e replaced by N_i]. We shall in future, as usual, neglect the variation of the logarithmic term in (2) and set $mv^2 = mv_c^2$ in the argument of the logarithm, where v_c is the characteristic velocity of the electrons under consideration: $mv_c^2 \approx kT_e E_{ci}/E$ (cf. below).

For subsequent developments it is convenient to go over to the dimensionless variables

$$\mu = \cos\theta, \qquad u = v/\sqrt{kT_e/m},$$
$$\tau = \nu_e(\sqrt{kT_e/m})\,t = 4\pi e^4 N_e m^{-1/2}(kT_e)^{-3/2}\ln(mv_c^2 D/e^2)\,t.$$

In terms of these variables, Eq. (1) assumes the following form

$$\frac{\partial f}{\partial \tau} + \frac{E}{E_{ci}}\left(\mu\frac{\partial f}{\partial u} + \frac{1-\mu^2}{u}\frac{\partial f}{\partial \mu}\right) - \frac{1}{u^2}\frac{\partial}{\partial u}\left\{\frac{1}{u}\frac{\partial f}{\partial u} + f\right\}$$
$$- \frac{1-1/4u^2}{u^3}\frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial f}{\partial \mu}\right\} = 0, \qquad \text{(1a)}$$

where E_{ci} is the characteristic (critical) field in a completely ionized plasma:

$$E_{ci} = \frac{4\pi e^3 N_e}{kT_e}\ln\left(\frac{mv_c^2 D}{e^2}\right)$$
$$\approx 2.6\cdot 10^{-13}\frac{N_e}{kT_e}\ln\left\{4\cdot 10^9\frac{(kT_e)^{1/2} mv_c^2}{N_e^{1/2}}\right\} \qquad (3)$$

(here in the last equation E_{ci} is in v/cm; kT_e and mv_c^2 are in ev). In what follows we shall consider only the case of a weak electric field $E \ll E_{ci}$. We note that the field E_c utilized in Dreicer's papers[1,4] is related to E_{ci} by the expression $E_c = \frac{1}{2}E_{ci}T_e/T_0$.

A. STATIONARY SOLUTION

Under stationary conditions ($\partial f/\partial\tau = 0$) it is natural to seek the solution of equation (1a) in the form

$$f = C\exp\{\varphi(u,\mu)\}, \qquad (4)$$

where C is a certain constant determined by the normalization conditions. The function $\varphi(u,\mu)$ must then satisfy the following nonlinear equation:

$$u^2\frac{E}{E_{ci}}\left[\mu u\frac{\partial\varphi}{\partial u} + (1-\mu^2)\frac{\partial\varphi}{\partial\mu}\right] - \left(\frac{\partial\varphi}{\partial u}\right)^2 - \frac{\partial^2\varphi}{\partial u^2}$$
$$- u\left(1-\frac{1}{u^2}\right)\frac{\partial\varphi}{\partial u} + 2\left(1-\frac{1}{4u^2}\right)\mu\frac{\partial\varphi}{\partial\mu} - \left(1-\frac{1}{4u^2}\right)$$
$$\times(1-\mu^2)\frac{\partial^2\varphi}{\partial\mu^2} - \left(1-\frac{1}{4u^2}\right)(1-\mu^2)\left(\frac{\partial\varphi}{\partial\mu}\right)^2 = 0. \qquad (5)$$

Before undertaking the solution of this equation we shall point out one essential characteristic feature of the electron velocity distribution in the domain of high velocities in the presence of a constant electric field. Under the action of the electric field the electrons acquire an additional velocity (with respect to the thermal velocity) directed parallel to the field. It is redistributed among other directions as a result of collisions of electrons with each other and with ions. The change in the distribution function due to the collisions is described by the last two terms in (1). The first of them describes the change in the absolute value of the velocity or of the energy of a fast electron colliding with another electron (as is well known, the change in electron energy in a collision with an ion is very small). The second term describes the change of only the direction of the momentum as a result of the collision. Collisions with electrons as well as collisions with ions contribute equally to such changes.

In a weak field ($E \ll E_{ci}$) in the domain of thermal velocities ($u \sim 1$) the distribution function is, naturally, close to Maxwellian. Under these conditions collisions between electrons are of little significance; in particular, the Maxwellian distribution function results in the vanishing of that term in (1) which describes the change in the absolute value of the electron velocity as a result of a collision. Under these conditions the principal role is played by collisions accompanied by a change of only the direction of the velocity, as a result of which, at thermal velocities, the additional velocity communicated to the electrons as a result of the action of the field turns out to be uniformly distributed among all the directions. Under these conditions the distribution function depends primarily only on the absolute value of the velocity, as is well known (cf., for example, reference 3).

At high velocities, when the collisions are infrequent and when as a result of this the electric field can produce considerable deviations of the distribution function from Maxwellian, the situation is significantly different. Under these conditions [for $u > (E_{ci}/E)^{1/4} \gg 1$] the most important collisions turn out to be those between electrons accompanied by a change in the absolute value of the velocity. The reason for this is that the gradient of a function of the Maxwellian type increases rapidly (in absolute value) as the velocity increases: $|df/du| \sim uf$; therefore for large u the change in the distribution function resulting from a change in the absolute value of u is very large [this can be clearly seen from Eq. (1)]. The last term in (1), describing the change in the direction of the velocity, is comparatively small; consequently, the redistribution of the electron energy over the different directions of the velocity is hindered. Therefore, for large values of u, when the collisions are infrequent and the electric field can produce a large increase in the component of the velocity u_z over a mean free path (the z axis is parallel to the field), this direction of the velocity must be strongly favored. The velocity distribution of the electrons for high velocities must acquire a directional character.

Taking into account the above peculiarity of the distribution function in the domain of high velocities it is natural to seek the solution of Eq. (5) in the form of a series in powers of μ in the neighborhood of $\mu = 1$ (i.e., near the z axis, since $\mu = \cos\theta$):

$$\varphi(u,\mu) = \varphi(u,1) + (\mu-1)\left(\frac{\partial\varphi}{\partial\mu}\right)_{\mu=1} + \frac{(\mu-1)^2}{2!}\left(\frac{\partial^2\varphi}{\partial\mu^2}\right)_{\mu=1} + \dots = \varphi_0(u) + (\mu-1)\varphi_1(u) + (\mu-1)^2\varphi_2(u) + \dots.$$

On substituting this expansion into (5), and on equating terms containing various powers of $\mu - 1$, we obtain the following chain of equations for the functions $\varphi_0, \varphi_1, \varphi_2, \dots$:

$$u^3\frac{E}{E_{ci}}\frac{d\varphi_0}{du} - \left(\frac{d\varphi_0}{du}\right)^2 - \frac{d^2\varphi_0}{du^2} - u\left(1-\frac{1}{u^2}\right)\frac{d\varphi_0}{du} + 2\left(1-\frac{1}{4u^2}\right)\varphi_1 = 0, \tag{5a}$$

$$u^3\frac{E}{E_{ci}}\left(\frac{d\varphi_0}{du}+\frac{d\varphi_1}{du}\right) - 2u^2\varphi_1\frac{E}{E_{ci}} - 2\frac{d\varphi_0}{du}\frac{d\varphi_1}{du} - \frac{d^2\varphi_1}{du^2} - u\left(1-\frac{1}{u^2}\right)\frac{d\varphi_1}{du} + 2\left(1-\frac{1}{4u^2}\right)(\varphi_1+\varphi_1^2) + 8\left(1-\frac{1}{4u^2}\right)\varphi_2 = 0, \tag{5b}$$

$$u^3\frac{E}{E_{ci}}\left(\frac{d\varphi_1}{du}+\frac{d\varphi_2}{du}\right) - u^2\frac{E}{E_{ci}}(\varphi_1+4\varphi_2) - 2\frac{d\varphi_0}{du}\frac{d\varphi_2}{du} - \left(\frac{d\varphi_1}{du}\right)^2 - \frac{d^2\varphi_1}{du^2} - u\left(1-\frac{1}{u^2}\right)\frac{d\varphi_2}{du} + \left(1-\frac{1}{4u^2}\right)(6\varphi_2 + 8\varphi_2\varphi_1 + \varphi_1^2) + 18\left(1-\frac{1}{4u^2}\right)\varphi_3 = 0, \dots. \tag{5c}$$

Equations (5) can be easily solved by successively terminating the chain of equations. By setting as a first approximation $\varphi_1 = 0$ and by omitting for simplicity small terms (of order $1/u^2$) we obtain

$$\varphi_0^{(1)} = -\int_0^u u\left(1-u^2\frac{E}{E_{ci}}\right)du = -\frac{u^2}{2} + \frac{u^4}{4}\frac{E}{E_{ci}}. \tag{6}$$

In the next approximation, on setting $\varphi_2 = 0$ we obtain $\varphi_0 = \varphi_0^{(1)} + \varphi_0^{(2)}$, $\varphi_1 = \varphi_1^{(1)}$, where

$$\varphi_0^{(2)} = -\left(\frac{2E_{ci}}{E}\right)^{1/2}\left[1-\left(1-\frac{E}{2E_{ci}}u^2\right)^{1/2}\right],$$

$$\varphi_1^{(1)} = u^2\left[\frac{E}{2E_{ci}}\left(1-\frac{E}{E_{ci}}u^2\right)\right]^{1/2} \tag{7}$$

etc.* Naturally, the stationary solution under discussion has sense only up to the runaway-electron limit, i.e., for $1 - u^2E/E_{ci} > 0$.

In order to determine the convergence of the sequence of successive approximations we rewrite expressions (6) and (7) for φ_0 and φ_1, by introducing the new variable $z = u^2E/E_{ci}$:

$$\varphi_0^{(1)} = -(E_{ci}/E)\left(\tfrac{1}{2}z - \tfrac{1}{4}z^2\right),$$

$$\varphi_0^{(2)} = -2^{1/2}(E_{ci}/E)^{1/2}[1-(1-z)^{1/2}],$$

$$\varphi_0^{(3)} = \tfrac{1}{4}(1-z)^{-1} + \tfrac{1}{4}\ln[z^{3/2}(1-z)^{-1}], \dots;$$

$$\varphi_1^{(1)} = 2^{-1/2}(E_{ci}/E)^{1/2}z(1-z)^{1/2},$$

$$\varphi_1^{(2)} = (-3 + 10z - 5z^2)/4(1-z), \dots.$$

From this it is clear that the expansion is in fact made in powers of the parameter $(E/E_{ci})^{1/2}$, which in the case of a weak field $E \ll E_{ci}$ is always small.

Thus, in the case $E \ll E_{ci}$ the stationary electron distribution function has the following form at high velocities ($u^2 = mv^2/kT_e \gtrsim \sqrt{E_{ci}/E}$)

$$f(v,\theta) = C\exp\left\{-\frac{mv^2}{2kT_e} + \frac{E}{4E_{ci}}\left(\frac{mv^2}{kT_e}\right)^2 - \left(\frac{2E_{ci}}{E}\right)^{1/2}\left[1-\left(1-\frac{E}{E_{ci}}\frac{mv^2}{kT_e}\right)^{1/2}\right] - \left(\frac{E}{2E_{ci}}\right)^{1/2}\frac{mv^2}{kT_e}\left(1-\frac{E}{E_{ci}}\frac{mv^2}{kT_e}\right)^{1/2}(1-\cos\theta)\right\}, \tag{8}$$

***We can also indicate a somewhat different, but a more consistent, method of solving the system of equations (5). In particular, with the aid of Eq. (5a) we can easily express the function φ_1 in terms of φ_0; with the aid of Eq. (5b) we can express φ_2 in terms of φ_1 and φ_0, etc., i.e., in the final result we can easily obtain a solution of the system of equations (5) in which all the functions $\varphi_1, \varphi_2, \varphi_3 \dots$ will be expressed in terms of φ_0. In order to determine the function φ_0 it is necessary to use an additional condition. Indeed, a stationary electron distribution is established both when there exist no sources of particles, and also when there are δ-like sources of particles; naturally, the distribution in the two cases will be different, although Eq. (5) is valid in all cases. The condition imposed on the sources (or on the flux) of electrons is the required additional condition for the determination of φ_0 (cf., for example, reference 3). In particular, in the case under consideration at present there are no sources of particles; consequently, the total flux of particles over any closed surface must be equal to zero. This condition leads to a certain complicated integral equation for the function φ_0. A solution of this equation obtained by the method of successive approximations leads to expressions agreeing with (6) and (7).**

where the normalization constant is $C = (m/2\pi kT_e)^{3/2} N_e$ (N_e is the electron density). For not very high velocities $v^2 \sim kT_e/m$ (or more accurately, for $v^2 \ll (kT_e/m)\sqrt{E_{ci}/E}$) the distribution function (8) is close to Maxwellian.* For large velocities $v^2 \gg (kT_e/m)\sqrt{E_{ci}/E}$ the distribution function (8) differs from it considerably: as the velocity increases it falls off significantly slower than the Maxwellian one does. Moreover, in the high velocity domain the distribution function (8) acquires a directional character, which is in complete agreement with the qualitative analysis of the solution of Eq. (1) carried out earlier. The directional nature of the distribution (8) is most clearly manifested for $v^2 = 2kT_eE_{ci}/3mE$. The average angular spread

$$\bar{\theta} = \int \theta f(v, \theta)\, d\Omega \Big/ \int f(v, \theta)\, d\Omega$$

has in this case a minimum value:

$$\bar{\theta} = \bar{\theta}_{min} = (27\pi^2/8)^{1/4}(E/E_{ci})^{1/4}.$$

It is of interest that not only at speeds $v^2 < 2kT_eE_{ci}/3mE$, but also at high speeds $v^2 > 2kT_eE_{ci}/3mE$ (i.e., near the runaway-electron limit), does the distribution function become less directional.†

B. THE FLUX OF RUNAWAY ELECTRONS

The average velocity of an electron parallel to the direction of the field $v_z = u_z(kT_e/m)^{1/2}$ is much greater in the high velocity domain than is its velocity in the direction orthogonal to the field $v_r = u_r(kT_e/m)^{1/2}$. Because of this, it is the distribution with respect to the velocity u_z which is particularly significant for runaway electrons:

$$F(u_z, \tau) = 2\pi \int_0^\infty f(u, \theta, \tau)\, u_r\, du_r.$$

On multiplying Eq. (1) by $2\pi u_r$ and on integrating it over du_r (neglecting small terms of order u_r^2/u_z^2) we obtain the following equation for $F(u_z, \tau)$:

$$\frac{\partial F}{\partial \tau} - \frac{\partial}{\partial u_z}\left\{\frac{1}{u_z^2}\left(3 - \frac{E}{E_{ci}} u_z^2\right) F + \frac{1 + \overline{u_r^2}}{u_z^3}\frac{\partial F}{\partial u_z}\right\} = 0. \quad (9)$$

Here

$$\overline{u_z^2} = \overline{u_r^2}(u_z) = \int_0^\infty u_r^3 f\, du_r \Big/ \int_0^\infty u_r f\, du_r$$

is the mean square velocity of the electron in the direction orthogonal to the field (for a given u_z). For the determination of $\overline{u_r^2}$ we can utilize the stationary distribution function (8) obtained earlier. We then find that (for $u_z^2 < E_{ci}/E$)

$$\overline{u_r^2} = 2\left[1 - \frac{E}{E_{ci}} u_z^2 + \frac{(E/E_{ci})^{1/2}(3 - u_z^2 E/E_{ci})}{2^{1/2}(1 - u_z^2 E/E_{ci})^{1/2}}\right]^{-1}. \quad (10)$$

The problem of finding the flux of runaway electrons has thus been reduced to finding the nonstationary solution of Eq. (9); we have obtained it earlier.[5] By utilizing this solution we obtain for the flux of the runaway electrons S the following expression:

$$S = \frac{N_e \nu_{e0}}{\sqrt{2\pi}}\left[\int_0^\infty \frac{u_z^3\, du_z}{1 + \overline{u_r^2}} \exp\left\{\int_0^{u_z} u_z\, du_z \frac{3 - u_z^2 E/E_{ci}}{1 + \overline{u_r^2}}\right\}\right]^{-1}, \quad (11)$$

where $N_e = N_e(t)$ is the number of electrons in the principal stationary domain of velocities, $\nu_{e0} = \nu_e(\sqrt{kT_e/m})$ is the mean collision frequency for an electron. On taking it into account that $E_{ci}/E \gg 1$, we can carry out the integration in the last expression. We then obtain

$$S = \frac{2}{\sqrt{\pi}} N_e \nu_{e0} \left(\frac{E}{E_{ci}}\right)^{1/2} \exp\left\{-\frac{E_{ci}}{4E} - \sqrt{2}\left(\frac{E_{ci}}{E}\right)^{1/2}\right\}. \quad (12)$$

It is shown in reference 5 that the flux of runaway particles determined by formulas (11) and (12) is established within a time $\Delta t_y \gtrsim (E_{ci}/E)^{3/2}\nu_{e0}^{-1}$. Therefore, the weakly nonstationary solution, (11) and (12), is applicable only if the parameters N_e, T_e, E do not change appreciably during this time. In particular, since the density of electrons in the principal velocity domain N_e is decreased as a result of the running away process $dN_e/dt = -S$, then in order to be able to utilize expression (12) it is necessary to have $S\Delta t_y \ll N_e$, i.e., it is necessary that the following inequality hold

$$\frac{E_{ci}}{E} \exp\left\{-\frac{E_{ci}}{4E} - \left(\frac{2E_{ci}}{E}\right)^{1/2}\right\} \ll 1,$$

*It has been pointed out earlier that both the initial equation (1) and the method of solving it utilized in this paper are valid only at high velocities. However, only at such velocities can the electron distribution function differ appreciably from Maxwellian. For low (thermal) velocities the electron distribution is Maxwellian, and the solution of (8) which we have obtained coincides with it. Consequently, the distribution function (8) represents well the nature of electron distribution over the whole stationary domain of velocities: $E_{ci}kT_e/Em \geqslant v^2 \geqslant 0$.

†An analogous problem was recently investigated by Dreicer.[4] However, he assumed that the distribution function could be represented in the form $f(v) = f_0(v) + f_1(v)\cos\theta$, where $|f_1| \ll f_0$. He obtained the result that the distribution function is Maxwellian right up to the runaway-electron limit. As is clear from (8), the above assumption, and, consequently, the result obtained, are erroneous in the domain of high velocities $v^2 \gg (kT_e/m)\sqrt{E_{ci}/E}$, and even more so at the runaway-electron limit $v_c^2 = kT_eE_{ci}/mE$.

which, naturally, is always satisfied in the case of a sufficiently weak field $E \lesssim 0.1\, E_{ci}$.

If we estimate the flux of runaway electrons in the simplest possible manner, assuming that the distribution function is Maxwellian right up to the runaway-electron limit (cf., for example, Harrison's paper[1]), we obtain* $S \sim \nu_e N_e \exp\{-E_{ci}/2E\}$. This expression for the number of runaway electrons qualitatively agrees well with the exact formula (12), although the quantitative discrepancy between them is quite large for large values of E_{ci}/E. It should be emphasized that the agreement which we have mentioned is to a certain extent accidental. Indeed, as is clear from the exact calculation, the friction that determines the flux of runaway electrons is primarily determined by collisions between electrons which are accompanied by an appreciable change in electron energy. This constitutes only a part of the total friction experienced by electrons in a plasma, but this part differs from the total friction only by a numerical factor. As a result qualitative agreement is obtained between an elementary estimate of the flux taking the total friction into account and the result of an exact calculation.†

It should be emphasized that the assumptions made above in the calculation of the flux (neglect of terms of order $\overline{u_r^2}/\overline{u_z^2}$, the utilization for the determination of $\overline{u_r^2}$ of the stationary distribution function, in the evaluation of which, moreover, only the first terms of the expansion in powers of the parameter E/E_{ci} have been taken into account) have enabled us to pick out only the principal terms in the exponential factor in formula (12). If the next approximations are taken into account this should lead to corrections of order unity in the exponential term.

3. INCLUSION OF THE EFFECT OF NEUTRAL PARTICLES

a) Weakly ionized plasma. The possibility of electrons being accelerated in a completely ionized plasma is a consequence of the fact that the frequency of collisions of an electron with ions and with other electrons falls off rapidly as its velocity increases. As a result friction is always negligibly small for electrons possessing a sufficiently high energy, and they are accelerated by even a very weak electric field. If the plasma contains a considerable number of neutral particles the situation is, in general, completely analogous, since the frequency of collisions of an electron with the neutral particles (atoms, molecules) also decreases with increasing velocity in the case of electrons possessing a sufficiently high energy $\epsilon \gtrsim (3-5)\epsilon_i$, where ϵ_i is the ionization energy. However, for $\epsilon \lesssim \epsilon_i$ the frequency of collisions usually increases with increasing ϵ. Therefore, in weakly ionized plasma, when the principal role is played by the collisions of an electron with neutral particles, only the fast electrons may run away in a relatively weak electric field; their energy must, in any case, exceed $(3 \text{ to } 5)\epsilon_i$.

Because of this, it is possible to simplify considerably the kinetic equation for the electron distribution function in a weakly ionized plasma $f(v, \theta, t)$ by utilizing the differential representation for the integral due to the inelastic collisions of an electron with neutral particles (cf. reference 3). In this case the kinetic equation assumes the following form:

$$\frac{\partial f}{\partial t} + \frac{eE}{m}\left(\cos\theta\,\frac{\partial f}{\partial v} - \frac{\sin\theta}{v}\frac{\partial f}{\partial \theta}\right) - \frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^2\left[D(v)\frac{\partial f}{\partial v} + \frac{F(v)}{m}f\right]\right\} - \frac{\nu_n(v)}{2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial f}{\partial\theta}\right) = 0. \tag{13}$$

Here

$$F(v) = N_n\left\{(\varepsilon_i + \Delta\varepsilon)\,Q_i(v) + \sum_k \hbar\omega_k Q_k(v)\right\}$$

is the effective retarding force on an electron moving with a velocity v in a gas of density N_n. Here $\Delta\epsilon$ is the energy transferred to the knock-on electron, $Q_i(v)$ is the total ionization cross section, $Q_k(v)$ is the total cross section for the excitation of the level $\hbar\omega_k$. As is well known, the retarding force F may be approximately written in the form

$$F(v) = (4\pi e^4 N_n Z / mv^2)\ln(mv^2/\bar{\varepsilon}), \tag{14}$$

where $\bar{\epsilon}$ is some average excitation energy (the calculation in reference 6 yields $\bar{\epsilon} \approx Z \times 13.5$ ev, in particular $\bar{\epsilon} = 15$ ev in hydrogen, $\bar{\epsilon} = 30$ ev in helium). For not very high electron energies the force F increases with increasing v, while at high energies it diminishes. It reaches the maximum value of $F_{max} = 4\pi e^4 N_n Z/2.72\bar{\epsilon}$ for $\epsilon_{max} = 1.36$.*

*A similar expression for the main exponential term was also obtained in Dreicer's paper.[4]

†Naturally, other cases are also possible when there is no such agreement. For example, in the case of statistical acceleration mechanisms estimates of the magnitude of the flux of runaway particles made with the aid of simple formulas of the type given above turned out to be completely useless.[5]

*It should be noted that if we use the calculation of F_{max} experimental values of the cross sections which, as is well known, are considerably lower in the neighborhood of the maximum than the theoretical ones,[7] then the value of the force

Further,

$$D(v) = \frac{N_n}{2m^2 v}\left\{\varepsilon_i(\varepsilon_i + \Delta\varepsilon) Q_i(v) + \sum_k (\hbar\omega_k)^2 Q_k(v)\right\}, \quad (15)$$

is the coefficient characterizing the velocity diffusion of an electron. For subsequent discussion it is convenient to write this expression in the form

$$D(v) = \varepsilon_i F(v) d(v) / 2m^2 v. \quad (15a)$$

Here $d(v)$ turns out to be a very slowly varying dimensionless function of v; its values are close to unity. In the case of hydrogen, for example, $d(v) \approx \bar{d} = 0.94$, in the case of helium $\bar{d} = 0.99$.

Finally, $\nu_n(v)$ is the frequency of collisions of electrons with neutral particles:

$$\nu_n(v) = N_n \int Q(v, \theta)(1 - \cos\theta)\, d\Omega,$$

where $Q(v, \theta)$ is the differential cross section for the scattering of an electron by a molecule, including elastic scattering, excitation of all levels, and ionization.

The stationary electron distribution function in a weakly ionized plasma determined by Eq. (13) turns out qualitatively to be completely analogous to the distribution function in a strongly ionized plasma: with increasing velocity it falls off more and more slowly; at sufficiently high velocities it acquires a directed character. In the first approximation, just as in the case of the strongly ionized plasma, the distribution function is symmetric; it has the following form

$$f = f(v) = f(v_0)\exp\left\{-\frac{2m}{\varepsilon_i}\int_{v_0}^{v}\frac{v\,dv}{d(v)}\left[1 - \frac{eE}{F(v)}\right]\right\}, \quad (16)$$

where v_0 is the lower limit starting at which Eq. (13) becomes valid: $v_0 \sim (5\epsilon_i/m)^{1/2}$; $f(v_0)$ is the distribution function at the boundary v_0. If, for example, the distribution for $v \leq v_0$ is Maxwellian in the temperature T_e, then we have*

$$f(v_0) = (m / 2\pi k T_e)^{3/2} N_e \exp\{-mv_0^2 / 2kT_e\}.$$

For sufficiently high velocities $v > v_0$, the account of $f(v_0)$ naturally introduces only a small correction in expression (16) for the function $f(v)$.

As is clear from (16), the limit for the stationary solution $v = v_c$ is determined by the condition

$$F(v_c) = eE. \quad (17)$$

From this it may be seen that the stationary solution exists only under the condition

$$E < E_{cn} = F_{max} / e = 4\pi e^3 N_n Z / 2.72\,\bar{\varepsilon} \approx 7 \cdot 10^{-15} N_n \text{ v/cm} \quad (18)$$

Consequently the field E_{cn} is the critical field in a weakly ionized plasma.

When condition (18) is fulfilled only electrons of velocity greater than v_c are accelerated. Naturally, the flux of runaway electrons is in this case determined by the expression

$$S = -\frac{dN_e}{dt} = \frac{eE}{mv_c} v_T^3 f(v_0)\exp\left\{-\frac{2m}{\varepsilon}\int_{v_0}^{v}\frac{v\,dv}{d(v)}\left[1 - \frac{eE}{F(v)}\right]\right\}$$
$$\approx \frac{eE}{mv_c} v_T^3 f(v_0)\exp\left\{\frac{mv_0^2}{\bar{d}\varepsilon_i}\left(1 - \frac{E}{E_{cn}}\right)\right\}$$
$$\times\exp\left\{-\frac{2\pi e^3 N_n Z \ln(\overline{mv^2}/\bar{\varepsilon})}{E\varepsilon_i \bar{d}}\right\}, \quad (19)$$

where $\overline{mv^2} \approx (0.3 \text{ to } 0.4)\, mv_c^2$, $v_T^2 = 2kT_e/m$. In the last expression we have taken into account the fact that the function d is practically independent of v and may be replaced by its average value $\bar{d}$. We note that the last exponential factor in (19) may also be represented in the form

$$\exp\left\{-\frac{E_{cn}}{4E}A\right\},$$
$$A = 8\pi e^4 N_n Z \ln\frac{\overline{mv^2}}{\bar{\varepsilon}} \Big/ \varepsilon_i \bar{d} F_{max} \approx 5{,}4\,\bar{\varepsilon}\ln\frac{\overline{mv^2}}{\bar{\varepsilon}} \Big/ \varepsilon_i \bar{d}.$$

Here the constant A turns out to be a very large quantity, $A \gtrsim 10$. This means that the flux of runaway particles for $E < E_{cn}$ is not large; it falls off very sharply with increasing ratio E_{cn}/E, considerably faster than in the case of a fully ionized plasma [cf. (12)]. Qualitatively the nature of the dependence of the flux of runaway electrons on the intensity of the field E in a weakly ionized plasma is almost the same as in a strongly ionized plasma, which is as it should be.

F_{max} is considerably diminished, while ε_{max} increases (for example, in helium such a calculation yields $\varepsilon_{max} = 120$ ev and $F_{max} = 6 \times 10^{-26} N_n$ dynes, while formula (14) gives $F_{max} = 10 \times 10^{-26} N_n$ dynes and $\varepsilon_{max} = 35$ ev). Moreover, the differential representation of the integral due to the inelastic collisions of an electron with atoms utilized in Eq. (13) is valid only at sufficiently high electron energies $\varepsilon \gg \varepsilon_i$. This condition is not sufficiently well satisfied even in the region of maximum retardation force, and more so when $\varepsilon < \varepsilon_{max}$. As a result of this the values of the force F_{max}, and, consequently, also of the field E_{cn} (cf. below) are determined here only very approximately. We also note that the expressions given here for the retarding force F and for the diffusion coefficient D are valid only at not too high plasma temperatures, when $\exp\{-\hbar\omega_k/kT\} \ll 1$; for $kT \gtrsim \hbar\omega_k \approx \varepsilon_i$ it is necessary to take into account that some of the molecules are in an excited state (cf. reference 3).

*For $v < v_0$ the distribution function $f(v)$ can, of course, differ considerably from Maxwellian. For example, in inert gases the Druyvesteyn distribution holds up to $\varepsilon \sim \varepsilon_i$, while above ε_i there applies a function $f(v)$ which falls off even more sharply with increasing velocity (cf., for example, reference 3).

b) Arbitrary degree of ionization. The equation for the electron distribution function in the domain of high velocities for an arbitrary degree of plasma ionization is completely analogous to Eqs. (1) and (13) discussed previously; only we must now take into account in this equation collisions both with neutral particles and with free electrons and ions. The stationary solution of this equation has at high velocities the same character as before; in the first approximation the following expression holds for f(v)

$$f(v) = f(v_0)\exp\left\{-\int_{v_0}^{v} mv\,dv\,\frac{F(v) + mv\nu_e(v) - eE}{\varepsilon_i F(v)\,d(v)/2 + kT_e mv\nu_e(v)}\right\}, \quad (20)$$

where all the quantities have the same meaning as before.*

The boundary v_c of the domain in which the stationary solution is valid is determined by the condition

$$F(v_c) + mv_c\nu_e(v_c) - eE = 0, \quad (21)$$

and, correspondingly, in the first approximation the flux of runaway electrons is given by

$$S = R(v_c)\,v_T^3 f(v_0)\exp\left\{-\int_{v_0}^{v_c} mv\,dv\,\frac{F(v) + mv\nu_e(v) - eE}{\varepsilon_i F(v)\,d(v)/2 + kT_e mv\nu_e(v)}\right\}, \quad (22)$$

where $R(v_c)$ is some effective frequency which varies from eE/mv_c to $\nu_{e0}(E/E_c)^{1/2}$ depending on the degree of plasma ionization. For sufficiently high degrees of plasma ionization [when the electron distribution in the principal domain of velocities is Maxwellian (cf. reference 3)] the exponential term in formula (22) can be written in the form $\exp\{-E_c/4E\}$; if the next approximation is also taken into account (cf. Sec. 2) the expression for the flux assumes the form

$$S = N_e\nu_{e0}(E/E_c)^{1/2}\exp\{-E_c/4E - \sqrt{2}(E_c/E)^{1/2}\}, \quad (23)$$

where E_c is the critical field:

$$E_c = 4\pi e^3\left(N_e \ln\frac{mv_c^2 D}{e^2} + N_n Z\ln\frac{\overline{mv^2}}{\bar{\varepsilon}}\right)^2 \Bigg/ \left(kT_e N_e \ln\frac{mv_c^2 D}{e^2} + \frac{\varepsilon_i \bar{d}}{2} N_n Z \ln\frac{\overline{mv^2}}{\bar{\varepsilon}}\right). \quad (24)$$

At high degrees of plasma ionization formula (23) agrees with (12), while at low degrees of ionization (in the first approximation) it agrees with (19), as it should.* At low temperatures $kT_e < \epsilon_i$ the field E_c increases monotonically with increasing degree of plasma ionization.

c) Instability of a spatially homogeneous plasma. It was shown above that the flux of runaway electrons in a plasma in the case of a relatively weak electric field increases sharply with increasing electron temperature and usually falls off with increasing degree of plasma ionization. Both these quantities vary appreciably in the course of the development of a discharge in a gas in a constant field. The flux of runaway electrons varies correspondingly.

It is obvious that the magnitude of the flux S is uniquely related to the value of the electron distribution function with respect to the velocity v_z (the z axis is parallel to the field E) in the domain of very high velocities, beyond the limit for runaway electrons $v_z^2 \gg v_c^2 = E_c kT_e/Em$:

$$F(v_z) = F\left(\frac{eE}{m}(t - t_0)\right) = \frac{m}{eE}S(t_0).$$

From this it follows that if the flux $S(t_0)$ decreases with time then the distribution function $F(v_z)$ acquires a region in which $\partial F/\partial v_z > 0$, since

$$\frac{\partial F}{\partial v_z} = -\frac{m}{eE}\frac{\partial F}{\partial t_0} = -\frac{m^2}{e^2E^2}\frac{dS}{dt_0}.$$

However, it is well known that a spatially homogeneous plasma may become unstable with respect to longitudinal excitations[8] (plasma waves) if the distribution function has a domain in which $\partial F/\partial v_z > 0$. The criterion for instability

$$\pi(\partial F/\partial v_z)\,v_z^2\omega_0/2N_e\nu \gg 1$$

can in our case be rewritten in the following form

$$(-dS/dt_0)(\Delta t)^2\omega_0/N_e\nu \gg 1. \quad (25)$$

Here $\Delta t = t - t_0$ is the time which will have elapsed from the instant t_0 when the flux begins to decrease to the instant t when an instability will arise in the plasma; $\omega_0 = (4\pi e^2N/m)^{1/2}$ is the plasma frequency. The characteristic size of the excited inhomogeneities is $\lambdabar \sim eE\Delta t/m\omega_0$ [the time Δt is determined by relation (25)]; their characteristic frequency is $\omega \approx \omega_0$.

An experimental investigation of runaway electrons in a helium plasma placed in a constant

*If the electrons in the principal stationary domain of velocities do not have a Maxwellian distribution, then in formula (20) we have to replace kT_e by $mv_T^2/3$, where $mv_T^2/2$ is the average electron energy.

*For low degrees of plasma ionization formulas (23) and (24) given here, which take the second approximation [i.e., terms of order $(E_c/E)^{1/2}$] into account, are valid only for hydrogen (Z = 1). For other gases the coefficient $\sqrt{2}$ in front of this term must be replaced by $\sqrt{Z+1}$; the index of the power of the ratio E/E_c in the factor preceding the exponential is also changed.

electric field was carried out in a stellarator.[9] In this case the electron temperature at first increased sharply, but later if the field was lower than $E_p \approx 1.5 \times 10^2 p_0$ v/cm it stayed for a certain length of time at an almost stationary level $kT_e \sim (0.5 \text{ to } 0.9)\epsilon_i$ (cf. references 10 and 11). During this period (the period of the temperature or the current "plateau") the degree of plasma ionization q_i continued to increase slowly. Therefore, the flux of runaway electrons diminished, since at a constant temperature the flux decreases with increasing degree of plasma ionization. Consequently it was maximum at a time close to the instant of formation of the plateau; in this case the field is given by $E_c = E_{c\,min} \approx (50 \text{ to } 35)E$. Consequently, the total number of runaway (accelerated) electrons is given by*

$$\Delta N_e = \int S dt \approx S\left(\frac{E_{c\,min}}{E}\right) \Big/ \frac{dq_i}{dt} \sim (10^{-7} - 10^{-5})\, N_e.$$

Further, in virtue of the fact that during the current "plateau" the flux of runaway electrons decreases with time, an instability of a spatially homogeneous plasma should arise. The criterion (25) for instability to occur will be satisfied after a time

$$\Delta t \sim \left(\frac{1}{\omega_0 dq_i/dt}\,\frac{N_e}{\Delta N_e}\right)^{1/2} \sim 10^{-4} - 10^{-5} \text{ sec}$$

elapsed since the beginning of a decrease in the flux. Consequently, such an instability must arise during the current "plateau". Experimentally,[9] bursts of hard x rays were observed just at this time, due to the impact of a large number of accelerated electrons against the walls of the vessel; they are generally accompanied by powerful microwave radiation from the discharge at frequencies $\omega \sim (1 \text{ to } 3) \times 10^{11}$ (under these conditions the plasma frequency is of the same order of magnitude). Apparently, it can be concluded that these bursts of radiation during the current "plateau" (according to the terminology of reference 9 — bursts of "type A") are a result of the mechanism for the instability of a homogeneous plasma indicated above.

*The first of the numbers quoted here corresponds to the case $E = 0.6\, E_p$, the second corresponds to $E = 0.9\, E_p$ [in making these estimates the values of $T_e(q_i, E/E_p)$ calculated in reference 11 were utilized]. It should also be emphasized that the total number of runaway electrons is very sensitive to the parameters. For example, if the temperature or electron density is changed by 10% then ΔN_e changes by almost an order of magnitude. In virtue of this, an investigation of the flux of runaway particles (with respect to number and to the energy distribution of the fast electrons) could serve as a good method for the measurement of electron temperature.

We should also point out another type of instability produced by runaway electrons in an equilibrium plasma pinch confined by a strong longitudinal magnetic field.* This instability is associated with the inhomogeneity of the electron density in the pinch, and arises only in the case of a high degree of plasma ionization. Indeed, the electron density in an equilibrium pinch falls off as it approaches the boundary: $N_e \to 0$ as $r \to a$. Correspondingly the field $E_c(r)$ also decreases: $E_c(r) \to E_{cn}$ as $r \to a$, with E_{cn} being very small at sufficiently high degrees of plasma ionization. Therefore, even if in the principal region of the pinch the electric field E is weak ($E \ll E_c$), yet in a layer of thickness

$$\Delta r = EkT_e \Big/ 4\pi e^3 \left(\frac{\partial N_e}{\partial r}\right)_{r=a} \ln \frac{kT_e D}{e^2}$$

near the boundary of the pinch it is always strong ($E \gtrsim E_c$). In this layer, therefore, all the electrons run away at once. Therefore, in this region the plasma consists of a stream of electrons moving among the ions. As is well known,[8] such a system is unstable with respect to excitation of plasma waves as soon as the velocity of the electron stream v_0 exceeds their mean thermal velocity. If an instability arises the amplitude of those waves builds up most rapidly whose wavelength is given by $\lambda = v_0/\omega_0 = Dv_0/(kT_e/m)^{1/2}$. Their frequency ω is determined in this case by the following expression:

$$\omega \approx \left(EkT_e \Big/ 2^{7/3} e m^{1/3} M^{2/3} \ln \frac{kT_e D}{e^2}\right)^{1/2}$$

$$\approx 1.2 \cdot 10^9 \left(\frac{M_p}{M}\right)^{1/3} (kT_e)^{1/2} E^{1/2}. \qquad (26)$$

Here M is the ion mass, M_p is the proton mass; numerically kT_e is expressed in ev and E in v/cm. The velocity of the stream required for the excitation of the above instability, $v_0 \approx (2 \text{ to } 3)\ (kT_e/m)^{1/2}$, is attained after a time $\Delta t = mv_0/eE$ has elapsed since the electric field has been switched on. The excitation of such an instability must be accompanied by a burst of radiation at a frequency of order ω determined by formula (26), and also by a sharp decrease in the current associated with the runaway electrons, i.e., by a decrease of the total current I_0 by a quantity of order $I_0 \Delta r/a$. At the same time x rays may be completely absent since the instability is excited by electrons of low energy $\epsilon = mv_0^2/2 \sim (2 \text{ to } 5)kT_e$.

*The whole investigation is carried out here without taking into account the magnetic field H_I due to the current itself. This is only valid in the presence of a strong external longitudinal magnetic field $H_0 \gg H_I$.

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