SOVIET PHYSICS

JETP

A translation of the Zhurnal Éksperimental'noĭ i Teoreticheskoĭ Fiziki.

Vol. 12, No. 5, pp. 819-1033 (Russ. orig. Vol. 39, No. 5, pp. 1177-1487, Nov., 1960) May, 1961

MULTIPLE PARTICLE PRODUCTION IN THE INTERACTION BETWEEN $> 10^{11}$-ev *NUCLEONS AND EMULSION NUCLEI*

A. P. ZHDANOV, I. M. KUKS, N. V. SKIRDA, and R. M. YAKOVLEV

Radium Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 6, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) 39, 1177-1185 (November, 1960)

We analyze 80 events of multiple meson production detected in an emulsion stack, using methods of mathematical statistics. The experimental material is compared with the hydrodynamical theory of particle production. Meson production due to central collisions between the nucleons and heavy nuclei is considered in particular detail.

1. EXPERIMENTAL DATA

A stack consisting of 180 layers of NIKFI-R emulsion, each 10 cm × 10 cm and 400 μ thick, was irradiated for 9 hours at 24 km altitude. A total of 120 interactions with more than five relativistic particles produced by singly-charged particles or neutral particles was found in the systematic scanning of the stack. In the scanning, the number of thin (N_s), grey (N_g), and black (N_h) tracks was counted for each of these stars, and the angle $\theta_{1/2}$ containing one half of the relativistic particles was approximately measured with a goniometer. 64 events were selected for subsequent measurements according to two criteria: the value of the half-angle $\theta_{1/2} < 10°$, or (independently of $\theta_{1/2}$) the multiplicity $N_s > 25$. These data were supplemented by 16 events from another stack of 0.25-liter volume irradiated for 15 hours at 20 km altitude.

The grain count and accurate angle measurements were made with MBI-8, MBI-8M, and Cooke 4005 microscopes. Tracks with a grain density 1.3 to 4 times greater than minimum were regarded as grey. The method of measuring the angles of the shower particles with respect to the direction of motion of the primary particle has been described earlier.[1] In order to ensure a constant relative error in the determination of both small and large angles, sections of the tracks were usually measured in two or three cross sections of the shower. The error in the determination of the smallest angle in each cross section was not greater than 20%. As a rule, the measurements were carried out in one emulsion layer. The shrinkage of the emulsion was determined immediately after the measurements, by determining the thickness of the emulsion layer at five points.

The main characteristics of the analyzed showers are presented in the table. (The notation used in the table is explained in the text.)

2. MULTIPLICITY OF PARTICLE PRODUCTION AND THE NUMBER OF NUCLEONS TAKING PART IN MESON PRODUCTION

To determine the number l of target-nucleus nucleons that take part in the meson production, the following relation was used:

$$N_s = (2l)^{1/4}(l+1)\gamma_c^{1/2}, \qquad \gamma_c = [1-(v_c/c)^2]^{-1/2}, \tag{1}$$

which is correct within the framework of the hydrodynamical theory of Landau (v_s is the center-of-mass velocity of the system containing the primary nucleon and the l nucleons of the nucleus). The values of l calculated in this manner (see table) are often found to be greater than those expected from the model of a cylindrical tunnel. (Not more than 7 nucleons can be found in the emulsion nuclei along the line of motion of the primary particle.)

Moreover, the variation of $\log N_s$ with $\log \gamma_c$ yields a correlation coefficient $r = -0.33 \pm 0.18$, which indicates a monotonically decreasing function. Showers with $\gamma_c \geq 6$ were used in the calculation, since the selection of such showers was carried out independently of their multiplicity (according to the criterion $\theta_{1/2} \geq 10°$.).

The results obtained can be explained in the following manner: In the $10^{11} - 10^{12}$ ev energy range, the factor γ_c is small, and a considerable part of the primary-nucleon energy can be transferred to a part of the nucleus lying outside the

Serial No.	Z	N_s	N_g	N_h	$\gamma_c(\bar{x})$	$\gamma_c(\theta_{1/2})$	l	E_0^{****}, 10^{10} ev	s	σ	$\bar{\gamma}$	w_k	$P(w < w_k)$
1	2	3**	4	5	6***	7	8	9	10	11	12	13	14
1	1	18	2	9	2.5	2.9	6	7.2	0.29	0.62			
2	1	38(5)	12	22	2.6	2.6	11	13.2	0.54	0.50		0.063	0.60
3	1	13	6	6	2.8	2.8	4	5.6	0.45	0.68			
4	1	39(6)	7	18	2.9	2.4	10	16.0	0.69	0.58	2.3	0.123	0.92
5	1	20(2)	11	15	2.9	2.9	6	10.8	0.51	0.66	1.5		
6	1	18	4	18	3.0	3.2	5	8.5	0.53	0.67	2.5		
7	1	26	6	10	3.02	3.0	7	12.6	0.31	0.64			
8	1	19	8	15	3.1	4.1	6	10.8	0.54	0.67			
9	1	17	3	6	3.4	3.6	5	9.5	0.44	0.74			
10	1	9	3	21	3.6	4.0	3	6.9	0.56	0.74			
11	0	15	5	20	3.7	4.5	4	10.5	0.36	0.72			
12	1	30	9	19	3.7	3.7	8	11.2	0.41	0.60		0.027	0.13
13	1	14(1)	4	6	3.8	3.6	3	5.1	0.29	0.72			
14*	1	36	10	17	3.8	3.2	9	26.0	0.61	0.64		0.146	0.96
15	1	21(1)	3	2	4.2	4.5	5	16.0	0.40	0.72			
16	1	8	5	13	4.2	6.3	2	6.6	0.32	0.88			
17	1	44(1)	10	18	4.24	3.24	10	32.0	0.51	0.74		0.093	0.84
18	1	31(1)	2	11	4.3	5.3	8	28.0	0.53	0.67		0.032	0.14
19*	1	11	9	10	8.1	4.3	2	20.0	1.02	0.94			
20	0	17	4	13	4.6	5.7	4	14.0	0.26	0.75			
21	1	8	2	6	4.4	4.4	2	7.4	0.15	0.75			
22	1	16(3)	6	10	4.7	5.0	4	16.4	0.46	0.75			
23	0	23	0	13	4.8	4.0	5	21.5	0.44	0.74			
24	1	15	9	12	4.9	4.5	4	17.6	0.56	0.76			
25	1	25	7	12	4.9	4.4	6	26.4	0.50	0.72			
26	1	18	1	10	5.5	8.5	4	34.0	0.49	0.80	1.5		
27	1	41	17	21	5.1	5.3	9	47.7	0.43	0.69		0.046	0.40
28	0	15	1	7	5.3	5.1	4	21.0	0.39	0.78			
29*	1	38(1)	4	11	5.3	5.4	8	40.0	0.58	0.70		0.030	0.14
30	1	7	8	17	5.4	7.6	2	11.0	0.30	0.82			
31	1	14	0	6	6.2	6.3	3	16.6	0.45	0.79			
32	1	16	8	17	5.6	6.8	4	23.2	0.39	0.78			
33	1	17	4	11	5.6	6.0	4	23.2	0.71	0.78			
34	1	10	3	6	6.0	7.1	2	13.5	0.40	0.81			
35	0	14	2	5	6.2	7.3	2	14.0	0.35	0.84			
36*	1	22	2	15	6.3	7.6	5	37.5	0.30	0.78			
37	1	8	3	14	6.4	5.9	1	7.5	0.28	0.88			
38	1	14	7	11	6.5	8.2	3	23.4	0.23	0.82			
39	1	9	5	6	7.2	7.6	2	15.6	0.42	0.86			
40	1	23	4	8	6.5	7.8	5	39.0	0.53	0.775			
41*	1	28	5	9	6.6	6.8	6	48.5	0.51	0.71			
42	1	9	8	18	6.7	6.3	2	16.5	0.30	0.85			
43	0	19	1	4	7.2	8.2	4	33.0	0.54	0.81			
44*	1	13	3	9	6.7	7.1	3	25.0	0.61	0.88	1.5		
45	1	9	7	3	7.3	10.9	2	20.0	0.35	0.84			
46	1	21	4	4	7.9	7.8	4	44.0	0.52	0.83			
47	1	33	3	3	7.9	7.7	6	66.0	0.51	0.79		0.065	0.61
48	1	24	0	6	8.2	9.9	5	65.0	0.62	0.82			
49	1	8	1	8	8.4	11.8	1	13.0	0.36	0.90			
50	1	14	4	12	8.5	13.3	3	39.0	0.65	0.86	1.5		
51	1	11	3	12	8.7	11.0	2	28.0	0.53	0.88			
52	1	18	2	10	8.7	8.8	3	42.0	0.56	0.86			
53	1	8	6	4	8.9	8.9	1	15.0	0.50	0.91			
54	1	51	3	10	8.9	9.6	8	120.0	0.49	0.79		0.063	0.65
55*	1	25	6	2	8.9	8.3	5	75.0	0.43	0.83			
56	1	6	3	6	9.5	8.3	1	17.0	0.37	0.92			
57	1	6	3	9	9.5	9.6	1	17.0	0.51	0.92			
58	1	17	6	11	10.5	10.5	3	60.0	0.50	0.88			
59	1	12	2	13	10.6	12.4	2	42.0	0.34	0.91			
60	1	18	6	9	11.7	12.5	3	66.0	0.57	0.89			
61	0	8	2	7	10.9	14.0	1	22.0	0.31	0.93			
62	1	8	2	7	11.0	18.8	1	22.0	0.33	0.93	1.5		
63	1	12	2	7	11.8	14.0	2	52.0	0.38	0.90			
64	1	8	1	7	11.8	10.9	1	26.0	0.40	0.95			
65*	1	68(1)	13	17	12.6	11.2	9	261.0	0.54	0.81		0.103	0.89

1	2	3**	4	5	6***	7	8	9	10	11	12	13	14
66	1	18	3	7	13.3	28.3	3	105,0	0.65	0.91	2,3		
67	0	10	1	10	13.8	18.2	1	37.0	0.59	0.97			
68*	1	10	2	15	15	19.0	2	82.0	0.34	0.95			
69	0	8	3	9	16.3	14.0	1	50,0	0,25	0.98			
70*	0	69(2)	13	16	17.5	15.4	8	456,0	0,63	0.87		0,046	0,30
71	1	11	1	1	18,0	28.9	1	65,0	0,52	1.00			
72	1	17	1	5	28,1	28.2	2	240.0	0,72	1.01			
73*	0	27	4	5	37,6	29.6	3	780.0	0.51	1.03			
74*	1	7	3	10	43,0	32,0	1	340.0	0.86	1.09			
75	0	10	2	3	31,5	32.5	1	180.0	0.86	1.09			
76	1	5	0	5	24.5	32,0	1	270.0	0.63	1.08			
77	1	5	2	2	58.6	54.7	1	360.0	0.38	1.10			
78*	0	46	5	6	49.1	60.3	4	1800.0	0.46	1.04			
79*	1	14	1	8	72,4	85.4	1	960,0	0.48	1.13			
80*	1	14	2	2	56.9	107.4	1	700,0	0,62	1.11			

*Jets detected in the small chamber.

**The numbers in the parentheses denote the number of shower particles in the backward hemisphere.

***The value γ_c was estimated (assuming a symmetrical angular distribution of shower particles about the angle $\pi/2$ in the c.m.s.) in two ways: $\gamma_c(\bar{x}) = -(1/N_s)\times \Sigma \log\tan\theta_i$ and $\gamma_c = \cot\theta_{1/2}$ (column 7).

****Energy of the primary nucleon E_0 was calculated according to the formula $E_0 = Mc^2 2\gamma_c^2 l$ (where M is the nucleon mass).

cylindrical tunnel. Within the framework of the hydrodynamical model, we can approximately estimate the dilatation of the tunnel during the process of its "drilling" through the nucleus. We shall consider this process at any moment of time in the laboratory system (l.s.). A part of the matter in the future tunnel behind the front of the shock wave moves in the direction of the primary particle (along the x axis), and undergoes a Lorentz contraction in the ratio γ, which varies from $\gamma_0 = E_0/Mc^2$ to γ_c as the added mass of the matter joins the motion. In the coordinate system of the moving matter, the walls of the tunnel expand with the velocity of propagation of the shock wave $v'_\perp = c/3$. Using the Lorentz transformation formula, we obtain $v_\perp = c/3\gamma$ for the velocity of dilatation of the tunnel wall in the l.s. We find γ by using the momentum conservation law: $\gamma = (M_0/M_X)\gamma_0 \lesssim \gamma_0/X$, where M_X is the mass of the matter which is already moving along the path X measured in nucleon diameters. Finally, an account of the dilatation of the tunnel leads to the following increase of the number l of nucleons that can take part in the meson production, as compared to the number of nucleons in the cylindrical tunnel l_{geom}:

$$l/l_{geom} \gtrsim 1 + 0.68\, l_{geom}/\gamma_0^{1/2} + 0.175\, l_{geom}^3/\gamma_0. \qquad (2)$$

If, instead of the values of l calculated by means of Eq. (1), we now take those found from Eq. (2), then the contradiction with the geometrical dimensions of the emulsion nuclei disappears. In addition, it is also possible to explain the anomalous correlation between N_s and γ_c. In fact, with increasing energy of the primary nucleon, two competing effects in the meson production appear: the mean multiplicity per nucleon increases, and the number of excited nucleons of the target nucleus decreases. In the energy range up to 10^{12} ev, the second effect is more pronounced.*

The tunnel dilatation can also be useful in studying the excitation energy of the residual nucleus. Thus, according to our data, the mean number of grey and black tracks in stars at $E_0 < 5\times 10^{11}$ ev equal to $\overline{N}_g + \overline{N}_h = 4.7 + 11.2$ is greater than at $E_0 \geq 5\times 10^{11}$ ev, where $\overline{N}_g + \overline{N}_h = 3.7 + 7.2$.

3. ANISOTROPY OF THE ANGULAR DISTRIBUTION OF SHOWER PARTICLES

In the analysis of the angular distribution of shower particles, it is convenient to consider the quantity $x_i = \log\tan\theta_i$. In the variables x_i, the mean square deviation σ of the distribution serves as a measure of the shower anisotropy. The esti-

*Equation (2) gives the volume contained between the radial front of the shock wave and two planes with separation l_{geom}. The influence of the spherical form of the target nucleus for $\gamma_0 > 100$ and for heavy target nuclei is negligible, especially since the shock-wave model completely loses its meaning for meson-production events at $\gamma_0 < 100$. However, an effect analogous to the tunnel expansion can, at low energies, be represented by a branching off of the meson-producing cascade, independently of the hydrodynamical model.

mates of s found for each shower* are given in column 10 of the table. In column 11, the values of σ^2 calculated from the hydrodynamical model according to the following formula of Milekhin[2] are shown for comparison:

$$L = \sigma^2 = 0.25 \log \gamma_0 + 0.7 \log \frac{2}{l+1} + 0.3. \qquad (3)$$

The values of σ obtained from Eq. (3) are greatly overestimated, as can be seen from the table. At the same time, Eq. (3) apparently reflects the increase of the anisotropy with increasing primary-particle energy correctly. Thus, for the variation of $\sigma^2 (\log\gamma_0)$, the slope of the straight line as found by the least-squares method from the data for all 80 showers is $+0.32 \pm 0.07$ for a correlation coefficient $r = +0.61^{-0.06}_{+0.08}$.

For a constant primary-particle energy, one should observe a marked difference in the anisotropy for nucleon-nucleon showers and for those showers that result from central collisions between the primary nucleon and a heavy nucleus. According to Eqs. (2) and (3), we have calculated the mean value of s for two groups of showers with the same mean energy of about 10^{12} ev. The first group, which, for simplicity's sake, will henceforth be called the "high-multiplicity" group, includes twelve showers with maximum tunnel length $l \geq 7$. The second group (nucleon-nucleon collisions) contains data sent from the laboratories in Moscow, Alma-Ata, Warsaw, Prague, and Berlin, and selected according to the following criteria: $N_g + N_h \leq 3$; $20 \leq \gamma \leq 100$; $N_s \leq 18$ (in all, 32 jets with a total number of shower particles equal to 398). For nucleon-nucleon showers, we obtained the mean value s = 0.55 for showers of high-multiplicity 0.54.† If Eq. (2) were correct and our results due only to statistical fluctuations, the probability of such an event would be less than 0.01%.

In a narrow range of γ_c ($4 \leq \gamma \leq 8$), the disagreement between predictions of the hydrodynamical theory‡ and our data is even more marked, since the dispersion σ^2 is found to be an increasing function of $\log l$: the slope of the straight line obtained by the least-squares method is 0.39 for a correlation coefficient $r = 0.43^{+0.11}_{-0.12}$. Analogous results have also been obtained by other authors.[3-5] Since the conclusion about decreasing anisotropy with an increasing length of the nuclear tunnel follows from the hydrodynamical approach (which is independent of additional postulates, say the equation of state), serious doubts arise as to the validity of applying such an approach to the description of multiple-meson production.

4. TEST OF THE UNIFORMITY OF THE ANGULAR DISTRIBUTION OF SHOWER PARTICLES

The fact that the anisotropy of the angular distribution does not decrease with increasing number of excited nucleons is understandable if we use those meson-production models in which there is no assumption about the existence of a common excited state of the system (the two-center model,[5] the "accompanying-showers" model of Chernavskiĭ,[7] the scheme of plural-multiple processes). In connection with the above, it would be very interesting to obtain some information on the general properties, not related to anisotropy, of the angular distribution.

In theories with one center of emission, the angular distributions of produced particles are given for the variables $x = \log\tan\theta$ by the probability of approximately Gaussian form $f(x; a, \sigma^2)$, where the mathematical expectation and the dispersion σ^2 of the variable x determine all the individual features of the showers. A two- or a multi-center theory leads to a superposition of functions of the same type $G(x) = \Sigma\alpha_i f(x; a_i, \sigma_i^2)$, and the mathematical expectation and the dispersion of the value x no longer determine the function G(x) fully. These distributions will be nonuniform in the Gaussian sense.

Thus, the choice of one or the other approach to the description of the multiple particle production depends on the uniformity of the experimentally obtained angular distributions. Since it is assumed that the hydrodynamical theory is most applicable to those interactions in which a large number of secondary particles are produced,[6] it becomes especially important to verify the uniformity in the group of high-multiplicity stars. In this section, we shall describe only the results of the investigation, the details being given in the Appendix.

The integral angular distributions of the shower particles of a group of nucleon-nucleon collisions and of a group of high-multiplicity stars

*Here and in the following discussion we differentiate between the mathematical expectation E(x), the mean-square deviation σ of the quantity x in the general set, and the estimates of these parameters from direct measurements, $\bar{x}$ and $\bar{s}$ respectively.

†It is interesting to note that the spreads of individual values in the group of nucleon-nucleon collisions (mean-square value 0.17) and in the high-multiplicity group (0.08) are strictly proportional to their statistical estimates based on the different mean multiplicity of the jets in these groups.

‡We can, in the present case, compare the experimental results with the hydrodynamical theory because of the high average multiplicity of the showers.

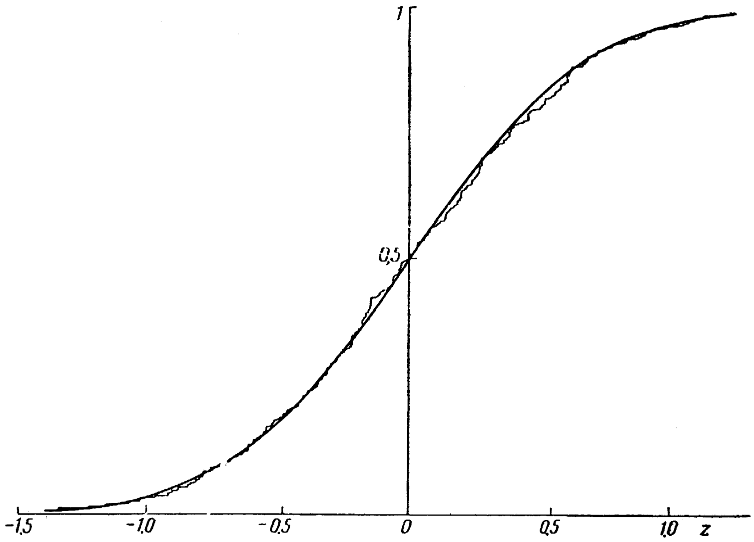

FIG. 1. Integral angular distribution for a group of nucleon-nucleon collisions in normalized variables $z = (x - \bar{x})/s$.

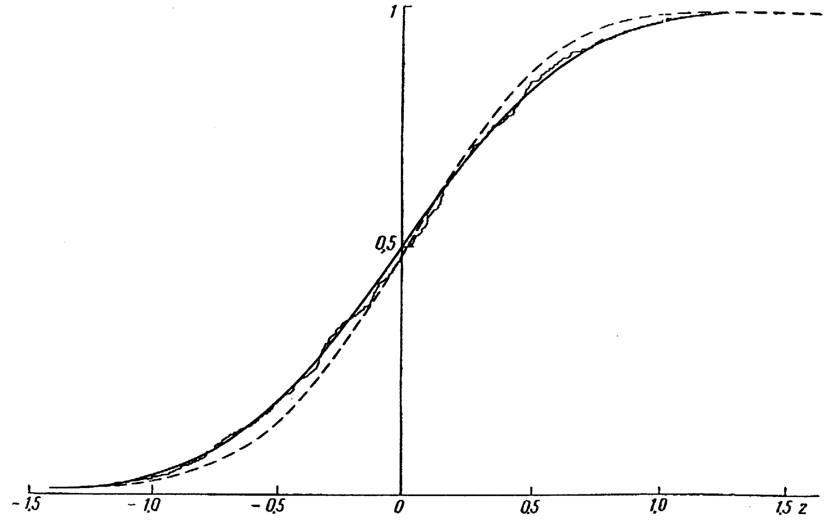

FIG. 2. Angular distribution of particles for a group of high-multiplicity interactions in normalized variables.

are shown in Figs. 1 and 2 in normalized variables $z = (x - \bar{x})/s$, where s^2 is the measure of dispersion obtained from experimental data. The solid curves represent the Gaussian function with the parameter equal to 0 and 1. It can be seen that, in both cases, the empirical distribution functions can be considered as normal ones. The w-test does not reveal marked deviations even at a significance level of 20%. The corresponding values of w_k for separate jets are shown in column 13 of the table. The arithmetical mean in the group of high-multiplicity jets $\overline{w} = 0.07$ is in good agreement with the assumption of the uniformity of the distribution. With a fit of 94%, the mean difference between the angular distribution of these jets and the normal distribution is less than 3×10^{-4}.

Chernavskiĭ[7] proposed a hypothesis suggesting the existence of a special type of non-uniformities in nucleon-nucleus collisions connected with the production of an accompanying shower by virtual π mesons, independent of the main shower. Stars of such a type should differ by possessing a large number of grey and black prongs, i.e., the same features as are characteristic of high-multiplicity stars. One can therefore separate showers of such type in the last group only by the form of the angular distribution of shower particles.

The dotted curve in Fig. 2 represents, in normalized variables, the calculated total angular distribution of the main shower and of the accompanying one. For constructing the curves, we used Eq. (2) with $E_0 = 10^{12}$ ev, $l = 5$, and under the following assumptions: the angular distribution for the accompanying shower is shifted from the main one by the quantity $\log\sqrt{M/\mu}$, the dispersion is decreased by $0.25 \log(M/\mu)$, and the number of particles in it amounts to 0.4 of the total. The quantity characterizing the difference between the solid and the dotted line $\Delta = 9 \times 10^{-4}$ (see Appendix). The probability of an agreement between the calculated distribution and the total distribution of all groups of high-multiplicity showers is less than 0.01%.

For all the 12 showers of this group, the comparison with the dotted curve produced a value of Δ less than with the solid curve only for No. 14 (see table). If we assume that the agreement is, in that case, not due to purely accidental factors, and if we take into account the fact that the detection probability of the accompanying showers in the high-multiplicity group should be greater than the corresponding total probability, we obtain the upper limit of probability of the appearance of an accompanying shower as equal to 0.04.*

It follows from the above that the angular distributions of shower particles are sufficiently uniform so that the fraction of showers of the two-center type cannot be great. In fact, in the variables $x = \log\tan\theta$, $y = \log[F/(1-F)]$, only eight out of 80 showers are represented by curves with a well-marked split into two branches, characteristic of two-center stars. The value of $\bar{\gamma}$ (the Lorentz factor of the emission centers in the general center-of-mass system) for these showers is shown in column 12 of the table. Thus, the two-center mechanisms of particle production cannot be responsible for the anomalous behavior of the distribution anisotropy, as is maintained by Bartke et al.[5]

CONCLUSIONS

1. In the analysis of the interactions of high-energy nucleons (up to 10^{12} ev) with heavy nuclei, it is necessary to take the widening of the tunnel in the course of penetrating through the nucleus into account.

2. The anisotropy of the angular distribution of nucleon-nucleus showers does not decrease with an increase in the number of excited nucleons, at least not up to energies of 5×10^{12} ev.

3. Within the same energy range, the relative probability for the appearance of accompanying showers in the form predicted by Chernavskiĭ is smaller than 0.04. Apparently, an "accompanying" tunnel cannot be considered independently of the main one.

4. The angular distributions of relativistic particles in the showers are uniform and, expressed in terms of the variables $x = \log\tan\theta$, are well described by Gaussian functions.

The authors are greatly indebted to A. A. Blyudzin, who was very helpful in collecting the experimental material. The authors would like to thank D. M. Samoĭlovich for the processing of the emulsion stack, A. N. Charakhch'yan and V. P. Grigor'ev for help in organizing the flight, and E. L. Feĭnberg and G. A. Milekhin for their interest in the experiment and for their comments.

APPENDIX

In the article, we have made use of the w test to check the agreement between the experimental results and a hypothetic curve for a small amount of data. The test is based on the use of the quantity w_n which relates the integral empirical distribution function $\tilde{F}(x; \bar{x}, s)$, for a chosen volume n out of the total Gaussian set, to the Gaussian function

$$F(x;\ a,\ \sigma) = (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{x} e^{-(x-a)^2/2\sigma^2} dx,$$

where

$$w_n = \int [\tilde{F}(x) - F(x \mid a = \bar{x},\ \sigma = s)]^2\, dF(x \mid \bar{x}, s)$$

$$= 1/12n + \sum_{\nu=1}^{n} [F(x_\nu \mid \bar{x}, s) - (2\nu - 1)/2n]^2,$$

where $\bar{x}$, s are the estimates of the parameters a and σ corresponding to the given choice, and the quantities x_ν are ordered according to increasing value. It has been shown[8] that the distribution of w is independent of the values $\bar{x}$ and s and, for increasing n, soon becomes independent of the chosen volume. In the same reference, tables for the asymptotic form of the distribution w and the distribution w_n for n = 25 and n = 100 are given.

Let us consider r showers with multiplicity n_k each with a corresponding empirical distribution function $\tilde{G}_k(x)$. We wish to test whether the observed $\tilde{G}_k(x)$ is better described by a Gaussian function (assumption I) or by a superposition of such functions $G'_k(x) = \sum_i \alpha_{ik} F(x; a_{ik}, \sigma_{ik})$ (assumption II).

---

*Let us denote the number of the ordinary showers and of the Chernavskiĭ-type showers by N_0 and N_{ch} respectively. The high-multiplicity group consists of events with $l \geq 7$. The values of l calculated for the Chernavskiĭ-type showers using Eq. (1) take the nucleons both in the main and in the accompanying tunnel into account. The total value $l \geq 7$ is obtained if the main tunnel contains five nucleons. The relative probability in the high-multiplicity group is therefore

$$N_{ch}(l_{main} \geqslant 5) / [N_{ch}(l_{main} \geqslant 5) + N_0(l \geqslant 7)],$$

and the total probability from geometrical considerations is

$$N_{ch}(l_{main} \geqslant 5) / [N_0(l \geqslant 5) + N_{ch}(l_{main} \geqslant 5)] \approx N_{ch}(l_{main} \geqslant 5) / N(l \geqslant 5).$$

The number of showers with $N(l \geq 5)$ is taken from the table.

For each shower, we shall pass from the variable x to the normalized variable $z_{\nu k} = (x_{\nu k} - \bar{x}_k)/s_k$, where

$$\bar{x}_k = \frac{1}{n_k}\sum_{\nu}^{n_k} x_{\nu k}, \qquad s_k = \frac{1}{n_k - 1}\sum_{\nu}^{n_k}(x_{\nu k} - \bar{x}_k)^2,$$

and we shall construct the total empirical distribution function

$$\tilde{G}(z) = (1/r)\sum_{k}^{r}\tilde{G}_k(z).$$

If the assumption II is correct, then the following two cases can occur:

a) The parameters α_{ik}, a_{ik}, and σ_{ik} are such that all the distributions $G_k(z)$, or a considerable part of them, are identical. The function $\tilde{G}(z)$, for a sufficient amount of data, is then markedly different from the function $F(z; 0, 1)$, which can easily be established, e.g., by means of the w test.

b) Let us consider the case where α_{ik}, a_{ik}, and σ_{ik} are not connected by any relation in individual showers, and all $G_k(z)$ values are different. The function $\tilde{G}(z)$ can then coincide with $F(z)$, but $\tilde{G}_k(z)$ differs for different showers by more than can be ascribed to purely statistical fluctuations.

We define the variance of the functions $G_k(z)$ and $F(z)$ as

$$\Delta_k = \int [G_k(z) - F(z)]^2\, dF(z).$$

For each of the r showers, let us find the quantity

$$w'_k = n_k \int (F - \tilde{G}_k)^2\, dF(z).$$

If F and G_k are different, then

$$w'_k = n_k \int [(F - G_k) + (G_k - \tilde{G}_k)]^2 dF(z)$$
$$= n_k \int (F - G_k)^2\, dF(z) + n_k \int (G_k - \tilde{G}_k)^2\, dF(z).$$

The first term of the sum equals $n_k\Delta_k$, and the second term is distributed approximately according to the w-distribution law. Averaging w'_k over all showers, we find

$$\frac{1}{r}\sum_{k}^{r} w'_k = \bar{w} + \overline{n_k\Delta_k} \approx \bar{w} + \bar{n}_k\bar{\Delta}_k.$$

For a large r, we can find such a value of w_P that, with probability $1 - P$, we have

$$\bar{w} \leqslant w_P = E(w) + \lambda_r(P), \qquad \lambda_r(P) \to 0 \text{ as } r \to \infty$$

where $E(w)$ denotes the mathematical expectation of w. In order to find $\lambda_r P$ for a finite r, we can use the limiting theorem of Lyapunov.

In principle, if we desire a given accuracy of $1 - P$, we can always select a sufficiently large r in order to determine the accuracy of the standard deviation $(\bar{\Delta}_k)$ for the angular distribution functions of single showers and of the total distribution.

Finally, if for the selected level of significance P, we simultaneously find that

$$\frac{1}{r}\sum w'_k < E(w) + \lambda_r(P),$$
$$\Big(\sum_{k}^{r} n_k\Big)\int [\tilde{G}(z) - F(z)]^2\, dF(z) < E(w) + \lambda_r(P),$$

then we can consider the assumption I as correct, and the functions $\tilde{G}_k(z)$ as homogeneous within the limits of the variance

$$\Big[E(w) - \lambda_r(P) - \frac{1}{r}\sum w'_k\Big](1/\bar{n}_k).$$

[1] Zhdanov, Berkovich, Lepekhin, Skirda, and Khokhlova, Приборы техника и эксперимента (Instruments and Exptl. Techniques) 4, 32 (1957).

[2] G. A. Milekhin, JETP 35, 1185 (1958), Soviet Phys. JETP 8, 829 (1959).

[3] Gurevich, Mishakova, Nikol'skiĭ, and Surkova, JETP 34, 265 (1958), Soviet Phys. JETP 7, 185 (1958).

[4] G. Bozoki and E. Gombozi, Nucl. Phys. 9, 400 (1958).

[5] Bartke, Ciok, Gierula, Holyński, Męsowicz, and Saniewska, Proceedings of the Moscow International Conference on Cosmic Rays 1, 1960.

[6] E. L. Feĭnberg, Usp. Fiz. Nauk 70, 333 (1960), Soviet Phys.-Uspekhi 3, 147 (1960).

[7] D. S. Černavskij, Suppl. Nuovo cimento 8, 775 (1958).

[8] Kac, Kiefer, and Wolfowitz, Ann. Math. Stat. 26, 189 (1955).

Translated by H. Kasha
223