FLOW OF A PLASMA INTO VACUUM IN THE PRESENCE OF A MAGNETIC FIELD

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Submitted to JETP editor March 23, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 657-661 (September, 1960)

Hydromagnetic waves excited by disintegration of the boundary between the plasma and vacuum are investigated. The boundary velocity (escape velocity) is determined. The amplitude of the electromagnetic wave radiated into the vacuum during the disintegration of the discontinuity is determined.

1. In the present article we investigate the flow of plasma into vacuum in the presence of a magnetic field. This problem has many applications in astrophysics and plasma physics. The formulation of the problem is as follows. A stationary plasma of infinite conductivity fills the half-space $x > 0$ at the initial instant of time $t = 0$. The state of the plasma is characterized by a pressure p_1, a density ρ_1, and magnetic-field components H_x and H_{1y} (to be specific, we put $H_x > 0$, $H_{1y} > 0$, $H_{1z} = 0$). A constant magnetic field with components H_x and H_{0y} and a constant electric field $E_{0z} \ll H_{0y}$ are produced in the vacuum. The necessary boundary conditions are not satisfied on such a discontinuity (when $H_x \neq 0$),* which therefore breaks up into several shock waves and self-similar waves. In the plasma there will propagate a rapid wave (shock or self-similar), followed by an Alfvén discontinuity, and finally by a slow wave. An electromagnetic wave propagates in the vacuum.† Some of the aforementioned waves may be missing. The problem is to determine the character of these waves, their amplitudes, and the velocity of the plasma on the boundary with the vacuum.

This problem was solved by Golitsyn[4] for the case when the normal magnetic-field component H_x vanishes. In the solution, Golitsyn used the fact that when $H_x = 0$ the equations of magnetohydrodynamics assume the same form as the equations or ordinary hydrodynamics,[5,6] provided the pressure p and the energy per unit mass ϵ are replaced by the total pressure $p^* = p + H^2/8\pi$ and the total energy $\epsilon^* = \epsilon + H^2/8\pi\rho$.

*If the vacuum is considered as a limiting case of a magnetohydrodynamic medium of zero density, the relativistic equations must be used.[1,2]

†The emission of an electromagnetic wave from a discontinuity with a conductivity jump was first noted by Kulikovskiĭ and Lyubimov.[3]

The presence of a longitudinal magnetic-field component H_x leads to a qualitatively different picture of the flow of plasma to the vacuum.

As already noted, three waves travel in the plasma (instead of one in the case $H_x = 0$). After the departure of these waves, the following conditions should be satisfied on the boundary between the plasma and the vacuum:

$$p = 0, \qquad \{H_y\} = 0, \qquad \{E_z\} = 0. \tag{1}$$

The first of these conditions means that the pressure vanishes behind the waves that go into the plasma, i.e., cavitation takes place. Since cavitation is impossible on a fast rarefaction wave,[7] the cavitation takes place on the slow wave, which is thus self-similar. The fast wave can be either self-similar or a shock wave, depending on the initial conditions.

The discontinuities of the electric and magnetic field in the electromagnetic wave propagating in the vacuum are connected by the relation $\Delta H_y = \Delta E_z$. If the velocity of the flowing plasma becomes nonrelativistic, then the discontinuity of the magnetic field in the electromagnetic wave will be considerably smaller than the magnitude of the magnetic field. Therefore, to solve the problem of the flow of a plasma into a vacuum it is necessary to satisfy only the first two boundary conditions in (1). The concomitant discontinuity of the electric field E_z determines the amplitude of the electromagnetic wave radiated into the vacuum.

Thus, the amplitudes of the fast and the slow waves are obtained from the equation

$$p_1 + \Delta_+ p + \Delta_- p = 0, \tag{2}$$

$$H_{1y} + \Delta_+ H_y + \Delta_A H_y + \Delta_- H_y = H_{0y}, \tag{3}$$

where Δ_+, Δ_-, and Δ_A denote the jumps in the magnetohydrodynamic quantities on the fast wave, the slow wave, and Alfvén discontinuity. Since the fast and the slow waves are plane (the magnetic field inside the wave and behind the wave are in the same plane that passes through the x axis, in our case the xy plane), the Alfvén discontinuity, if it does exist at all, can rotate the magnetic field only through 180°. Therefore the jump of the magnetic field in the Alfvén discontinuity is equal to $\Delta_A H_y = -2H_y$, where H_y is the transverse magnetic field ahead of the discontinuity. The velocity jump in the Alfvén discontinuity is

$$\Delta_A v_y = 2H_y/\sqrt{4\pi\rho}.$$

Since the transverse magnetic field does not reverse sign in shock and self-similar waves,[8] there is no Alfvén discontinuity when $H_{0y} > 0$, and a 180° Alfvén discontinuity occurs when $H_{0y} < 0$.

In order to find the amplitudes of the fast and slow waves, we must express $\Delta_+ H_y$ in terms of $\Delta_+ p$ and $\Delta_- H_y$ in terms $\Delta_- p$. Equations (2) and (3) then determine the amplitudes $\Delta_+ p$ and $\Delta_- p$.

2. To obtain the dependence of $\Delta_+ H_y$ on $\Delta_+ p$ and of $\Delta_- H_y$ on $\Delta_- p$, we use the fact that the solution of the equations of simple waves reduces to the integration of the differential equation[9]

$$dq_\pm/dr = q_\pm^2 (q_\pm - 1)/\theta (rq_\pm^2 - 1), \qquad (4)$$

where

$$r = c^2/U_x^2 \equiv 4\pi\gamma p/H_x^2, \qquad c(r) = \text{const} \cdot r^{(\gamma-1)/2\gamma},$$
$$\mathbf{U} = \mathbf{H}/\sqrt{4\pi\rho}, \qquad \theta = \gamma/(2-\gamma), \qquad q_\pm = U_\pm^2/c^2,$$
$$U_\pm = \{U^2 + c^2 \pm [(U^2+c^2)^2 - 4c^2U_x^2]^{1/2}\}^{1/2}/\sqrt{2},$$

the upper sign pertains to the fast wave and the lower one to the slow wave; γ is the Poisson-adiabat exponent (c is the velocity of sound and r is the dimensionless pressure).

The quantities q_{1+} and r_1 ahead of the fast simple wave are specified. Consequently Eq. (4) determines the function $q_+(r)$, and the jumps in the velocity are then determined from the well-known formulas[10,11]

$$\Delta_+ v_x = -\frac{1}{\gamma}\int_{r_2}^{r_1} c(r)\sqrt{q_+(r)}\,\frac{dr}{r} \qquad (5)$$

$$\Delta_+ v_y = \frac{1}{\gamma}\int_{r_2}^{r_1} \frac{c(r)}{r}\left[\frac{q_+(r)-1}{rq_+(r)-1}\right]^{1/2} dr. \qquad (6)$$

The subscript 2 refers to the region behind the fast wave and ahead of the slow wave.

The quantities $r_0 = 0$ and $q_0 = H_x^2/H_0^2$ are specified behind the slow wave. Therefore Eq. (4) determines the function $q_-(r)$, after which the jump in the velocity is determined from the formulas

$$\Delta_- v_x = -\frac{1}{\gamma}\int_0^{r_2} c(r)\sqrt{q_-(r)}\,\frac{dr}{r}, \qquad (7)$$

$$\Delta_- v_y = -\frac{1}{\gamma}\int_0^{r_2} \frac{c(r)}{r}\left[\frac{1-q_-(r)}{1-rq_-(r)}\right]^{1/2} dr. \qquad (8)$$

The magnetic field in the region contained between the two waves is defined by the relation

$$H_{2y}^2/H_x^2 = Q_+(r_2) = Q_-(r_2),$$
$$Q_\pm(r) = [q_\pm(r) - 1)][rq_\pm(r) - 1]/q_\pm(r). \qquad (9)$$

The pressure jump in the fast shock wave is connected with the jump in the magnetic field $h \equiv (H_{2y} - H_{1y})/H_1$ by the equation[12]

$$\frac{r_2 - r_1}{r_1} = \frac{\gamma}{r_1}\left\{-\frac{h^2}{2} + h\left[\frac{(\gamma h/2)\sin\theta_1 + r_1 - 1 \pm \sqrt{R(h)}}{2\sin\theta_1 - (\gamma-1)h}\right]\right\}, \qquad (10)$$

where

$$R(h) = h^2[\tfrac{1}{4}\gamma^2\sin^2\theta_1 - (\gamma-1)] + (2-\gamma)(1+r_1)h\sin\theta_1$$
$$+ [4r_1\sin^2\theta_1 + (1-r_1)^2],$$
$$\sin\theta_1 = H_{1y}/H_1.$$

The sign ahead of the root in formula (10) is chosen to satisfy the condition $r_2 > r_1$. The jumps in the velocity in the fast shock wave are determined from the relations

$$\Delta_+ v_x = U_{1x}\bar\eta(1 - \bar\eta h^{-1}\sin\theta_1)^{-1/2}, \qquad (11)$$

$$\Delta_+ v_y = U_{1x}h(1 - \bar\eta h^{-1}\sin\theta_1)^{1/2}/\cos\theta_1, \qquad (12)$$

where

$$\bar\eta = h\left[\frac{-(\gamma h/2)\sin\theta_1 + r_1 - 1 \pm \sqrt{R(h)}}{2r_1\sin\theta_1 - (\gamma-1)h}\right]. \qquad (13)$$

The sign ahead of the root in (13) is chosen in the same way as in (10).

If $H_{1y}^2/H_x^2 \equiv Q_+(r_1) > Q_-(r_1)$, then the equation $Q_+(r) = Q_-(r)$ determines the root $r_2 < r_1$, after which the values of $\Delta_\pm v$ are determined. In this case the fast wave will be self-similar.

If $H_{1y}^2/H_x^2 < Q_-(r_1)$, then the fast wave will be a shock wave, and the value of $r_2 > r_1$ is determined from (13) and from $H_{2y}^2/H_x^2 = Q_-(r_2)$.

The escape velocities v_x and v_y in the absence of an Alfvén discontinuity are given by

$$v_x = \Delta_+ v_x + \Delta_- v_x, \qquad v_y = \Delta_+ v_y + \Delta_- v_y, \qquad (14)$$

where the quantities $\Delta_+ v_x$ and $\Delta_+ v_y$ are determined from (5) and (6) if the fast wave is self-similar, and from (11) and (12) if the fast wave is a shock wave. The quantities $\Delta_- v_x$ and $\Delta_- v_y$ are given by (7) and (8).

The escape velocity in the presence of an Alfvén discontinuity is given by

$$v_x = \Delta_+ v_x + \Delta_- v_x, \qquad v_y = \Delta_+ v_y + 2U_{2y} - \Delta_- v_y, \qquad (15)$$

where $\Delta_\pm v$ have the same meaning as in (14).

The electric field on the boundary between the plasma and the vacuum is determined by

$$E_z = v_y H_x - v_x H_y \qquad (16)$$

(the velocity of light is set equal to unity). The amplitude of the electromagnetic wave ΔE_z radiated into the vacuum is $\Delta E_z = E_z - E_{0z}$, where E_z is given by (16).

3. The foregoing relations become much simpler when the Alfvén velocity $U_1 = H_1/\sqrt{4\pi\rho_1}$ is considerably smaller than the velocity of sound. In this case the fast shock wave will be the same as in the absence of the magnetic field, and the jump in the magnetohydrodynamic quantities in the self-similar waves can be obtained in explicit form[10]. In this approximation, the character of the waves that travel into the plasma is determined by the quantity $\mu \equiv H_{0y}/\sqrt{8\pi\rho_1}$. The Alfvén discontinuity will exist only when $\mu < 0$. The fast wave will be a shock wave if $|\mu| > 1$ and self-similar if $|\mu| < 1$.

The escape velocities v_x and v_y and the electric field E_z on the plasma—vacuum boundary are determined by the following formulas (see the figure)

1) In the case when

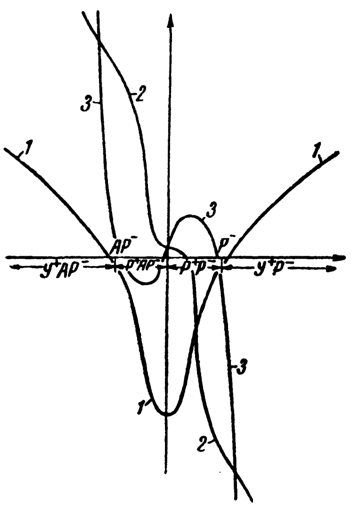
$$|\mu| \ll (U_{1x}/c_1)\,\xi_m^{(\gamma-1)/\gamma}, \quad \xi_m = \xi_1[\xi_1/(\gamma-1)r_1]^{1/(\theta-1)}, \quad \xi_1 = q_{1+} - 1$$

(the subscript 1 refers to the unperturbed region in the plasma), we have

$$v_x/c_1 = -2/(\gamma-1),$$

$$v_y/c_1 = (U_{1y}/c_1)(U_{1x}/c_1)^{1/\gamma} h(\gamma) - \sqrt{2/\gamma}\,\xi_m^{1/\gamma} r_1^{1/2\gamma}\mu,$$

$$E_z/H_x c_1 = (U_{1y}/c_1)(U_{1x}/c_1)^{1/\gamma} h(\gamma) + 2\sqrt{2}\,c_1\mu/(\gamma-1)\sqrt{\gamma}\,U_{1x}.$$



1 – longitudinal escape velocity v_x/c_1, 2 – transverse escape velocity v_y/c_1, 3 – electric field $E_z/H_x c_1$ on the plasma–vacuum boundary. The abscissas represent the quantity $\mu \equiv H_{0y}\sqrt{8\pi\rho_1}$. The x axis is directed into the plasma. The letters Y^+, P^+, P^-, and A denote the presence of a fast shock wave, of a fast and slow rarefaction wave, and of an Alfvén discontinuity.

Here

$$h(\gamma) = \Gamma[(\gamma-1)/2\gamma]\,\Gamma(1/\gamma)/\gamma\Gamma[(\gamma+1)/2\gamma], \qquad h(5/3) = 3.52.$$

2) In the case when $U_{1x}/c_1 \ll |\mu| \le 1$, we have

$$v_x/c_1 = -2/(\gamma-1) + 2|\mu|^{(\gamma-1)/\gamma}/(\gamma-1) - U_{1x} f(\gamma)/c_1|\mu|^{1/\gamma},$$

$$v_y/c_1 = -\mu g(\gamma)|\mu|^{-1/\gamma},$$

$$E_z/H_x c_1 = -(g - \sqrt{2/\gamma}\,f)\mu|\mu|^{-1/\gamma}$$

$$+ 2\sqrt{2}\,c_1\mu(1 - |\mu|^{(\gamma-1)/\gamma})/(\gamma-1)\sqrt{\gamma}\,U_{1x}.$$

Here

$$f(\gamma) = \frac{1}{\sqrt{2\gamma}}\int_0^1 \frac{\sigma^{-(\gamma+1)/2\gamma}}{\sqrt{1-\sigma(\theta+1)^{-1}}}\,d\sigma,$$

$$g(\gamma) = \frac{1}{\gamma}\int_0^1 \sqrt{\frac{1-\sigma(\theta+1)^{-1}}{1-\sigma}}\,\sigma^{-(\gamma+1)/2\gamma}\,d\sigma,$$

$$f(5/3) = 2{,}78, \qquad g(5/3) = 3.67.$$

3) In the case when $|\mu| \gg 1$, we have

$$v_x/c_1 = |\mu|\sqrt{2/\gamma(\gamma+1)},$$

$$v_y/c_1 = -\sqrt{(\gamma-1)/(\gamma+1)}\,g(\gamma)\mu,$$

$$E_z/H_x c_1 = -2c_1\mu|\mu|/\gamma\sqrt{\gamma+1}\,U_{1x}.$$

The author thanks L. I. Sedov for a general formulation of the problem, and A. I. Akhiezer and G. Ya. Lyubarskiĭ for variable discussions.

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Translated by J. G. Adashko
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