HIGHER BORN APPROXIMATIONS IN PAIR CONVERSION*

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Submitted to JETP editor March 1, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 591-599 (September, 1960)

Higher Born approximations with respect to the nuclear Coulomb field are considered in pair creation processes. The integrals of the first Born approximation can be computed exactly and lead to a simple analytic result.

In a number of quantum-mechanical problems one neglects the nuclear Coulomb field in the so-called Born approximation [we shall call this the zeroth Born approximation (z. B. a.)]. The computation of the higher approximations with respect to the Coulomb field is connected with considerable mathematical difficulties (see, for example, references 1 and 2). These difficulties are particularly great in the discussion of nuclear conversion with formation of electron-positron pairs,[2] so that up to now expressions for the probability of this process have only been obtained in the z. B. a.[3] At the same time, the calculations with exact wave functions in the field of the nucleus for the pair conversion are exceedingly cumbersome, so that it is very desirable to develop approximate methods.

In the present paper we consider the higher Born approximations. As the perturbing potential we choose a potential of the Yukawa type. In the final formulas the limit of a pure Coulomb potential is taken. In Sec. 1 we give the general expressions which can be applied to other problems (e.g., the photoeffect) besides the pair conversion. Section 2 is devoted to the discussion of the higher Born approximations for pair conversion. In Sec. 3 and the Appendices the calculations for the pair conversion are carried out to the end in the first Born approximation.

1. GENERAL RELATIONS

The matrix element describing quantum transitions of the electron (positron) due to the action of an electromagnetic field with frequency ω has the form (reference 4)†

$$S_{1\to 2} = -2\pi ie W\delta(E_2 - E_1 - \omega),$$
$$W = \int \overline{\Psi}_2(\mathbf{r}) B(\mathbf{r}) \Psi_1(\mathbf{r})\, d^3r,$$
$$B = \gamma\mathbf{B} + \gamma_4 B_4, \qquad \gamma_j = -i\beta\alpha_j, \qquad \gamma_4 = \beta, \qquad \overline{\Psi} = \Psi^+\gamma_4, \tag{1}$$

where α_j, β are the Dirac matrices. $B(\mathbf{r})$ corresponds to the electromagnetic field, as for example, the photon (photoeffect, bremsstrahlung) or the potential of the nuclear current (conversion).

The functions Ψ_i in (1) are the wave functions of the free or bound electron (positron) in the static field of the nucleus. If we regard the latter as a perturbation for one or both of the functions Ψ_i and proceeding as in reference 4 (see Sec. 29), we obtain

$$\Psi_i(\mathbf{r}) = \sum_{k=0}^{\infty} \psi_i^{(k)}(\mathbf{r}),$$
$$\psi_i^{(k)}(\mathbf{r}) = -\frac{ie}{(2\pi)^3}\int d^3f e^{i\mathbf{f}\mathbf{r}} \frac{i\hat{f} - m}{f^2 + m^2 - i\varepsilon}\int d^3r' \, e^{-i\mathbf{f}\mathbf{r}'} A^{(e)}(\mathbf{r}')\psi_i^{(k-1)}(\mathbf{r}'),$$
$$\psi_i^{(0)}(\mathbf{r}) = u(\mathbf{p}_i) e^{i\mathbf{p}_i\mathbf{r}}. \tag{2}$$

For the field of the nucleus we take

$$A^{(e)}(\mathbf{r}) = A_4^{(e)}(\mathbf{r}) = -i\frac{eZe^{-\lambda r}}{4\pi r}. \tag{3}$$

In the momentum representation these formulas take the form

$$\Psi(\mathbf{r}) = \int \varphi(\mathbf{f}) e^{i\mathbf{f}\mathbf{r}} d^3f, \tag{4a}$$
$$\varphi(\mathbf{f}) = \left\{\sum_{n=0}^{\infty} \beta^n \varphi^{(n)}(\mathbf{f})\right\} u(\mathbf{p}), \qquad \beta = \frac{\alpha Z}{2\pi^2}, \tag{4b}$$
$$\varphi^{(n)}(\mathbf{f}) = \frac{m - i\hat{f}}{f^2 + m^2 - i\varepsilon}\int \frac{\gamma_4}{(\mathbf{f}-\mathbf{s})^2 + \lambda^2}\varphi^{(n-1)}(\mathbf{s})\, d^3s,$$
$$f^0 = E, \qquad \varphi^0(\mathbf{f}) = \delta(\mathbf{f} - \mathbf{p}). \tag{4c}$$

The matrix element (1) is in the momentum representation

$$W = i\int d^3f_1 d^3f_2 \overline{\varphi}_2(\mathbf{f}_2)\, \hat{b}(\mathbf{f}_2 - \mathbf{f}_1)\, \varphi_1(\mathbf{f}_1), \tag{5}$$

*This paper was presented at the Ninth All-Union Conference on Nuclear Spectroscopy (Khar'kov, January 1959).

†If not noted otherwise, the notation is the same as in the book of Akhiezer and Berestetskii.[4] Heaviside units are used throughout.

where

$$b(\mathbf{k}) = -i\int e^{-i\mathbf{kr}} B(\mathbf{r})\, d^3r.$$

Using (4c), we can write W in the form

$$W = i\bar{u}(p_2)\, wu(\mathbf{p}_1), \tag{6a}$$

$$w = w(0\,|\,0) + \beta\,[w(0\,|\,1) + w(1\,|\,0)] + \beta^2\,[w(0\,|\,2) + w(1\,|\,1) + w(2\,|\,0)] + \ldots, \tag{6b}$$

$$w(k\,|\,n) = \int d^3f_1 d^3f_2\, \bar{\varphi}_2^{(k)}(\mathbf{f}_2)\, \hat{b}(\mathbf{f}_2 - \mathbf{f}_1)\, \varphi_1^{(n)}(\mathbf{f}_1). \tag{6c}$$

2. PAIR CONVERSION

For the description of nuclear conversion processes with formation of pairs we must make the following changes in the expressions of Sec. 1:* $\mathbf{p}_1 \to -\mathbf{p}_1$, $\mathbf{f}_1 \to -\mathbf{f}_1$, $E_1 \to -E_1$, and $\hat{B}$ must be replaced by the singular multipole potentials $\hat{B}_{lm}^{(\lambda)}$. Using the explicit expressions for the $\hat{B}_{lm}^{(\lambda)}$ of reference 4, we obtain in the momentum representation

$$b_{lm}^{(0)}(\mathbf{k}) = J_l(k)\, \mathbf{Y}_{l,l,m}(\mathbf{k}/k)\, \gamma, \tag{7a}$$

$$\hat{b}_{lm}^{(1)}(\mathbf{k}) = \frac{1}{\sqrt{l+1}}\,[\sqrt{2l+1}\, J_{l-1}(k)\, \mathbf{Y}_{l,l-1,m}(\mathbf{k}/k)\,\gamma - i\sqrt{l}\, J_l(k)\, Y_{lm}(\mathbf{k}/k)\,\gamma_4], \tag{7b}$$

$$J_l(k) = -i\int G_l(\omega r)\, g_l^*(kr)\, r^2\, dr. \tag{8}$$

The spherical vectors $\mathbf{Y}_{l,l+\lambda,m}$ and the functions G_l and g_l, which are proportional to the spherical Hankel and Bessel functions, respectively, are defined in reference 4 (pp. 33 and 426).

The integral $J_l(k)$ will be discussed in Appendix B.

The expression for the differential conversion coefficient for pair creation is, of course, of the same form as in the z. B. a.:

$$d\beta_{l\lambda} = \frac{\alpha\omega}{4}\frac{d^3p_1 d^3p_2}{(2\pi)^6}\, \sigma\delta(E_2 + E_1 - \omega),$$

$$\sigma = \sum_{\mu_1\mu_2} |\, W_{lm}^{(\lambda)}\,|^2, \tag{9}$$

where the summation goes over the spin states of the electron and the positron.

Using (6), we obtain

$$\sigma = \frac{1}{E_1E_2}\,\frac{1}{4}\,\mathrm{Sp}\,[w\,(i\hat{p}_1 + m)\,\bar{w}\,(i\hat{p}_2 - m)] = \frac{1}{E_1E_2}\sum_{s=0}^{\infty}\beta^s\sigma_s, \tag{10a}$$

$$\sigma_s = \sum_{k+n+k'+n'=s} \sigma\begin{pmatrix} k & n \\ k' & n' \end{pmatrix}, \qquad \bar{w} = \gamma_4 w^+ \gamma_4, \tag{10b}$$

$$\sigma\begin{pmatrix} k & n \\ k' & n' \end{pmatrix} = \frac{1}{4}\,\mathrm{Sp}\,[w\,(k\,|\,n)\,(i\hat{p}_1 + m)\,\bar{w}\,(k'\,|\,n')\,(i\hat{p}_2 - m)]. \tag{10c}$$

*Below the index 1 will thus refer to the positron and the index 2 to the electron.

Using the properties of traces and the relation of charge conjugation

$$\varphi_1(\mathbf{f}) = \{C\bar{\varphi}_2^T(\mathbf{f})\}_{(2\rightleftarrows 1)}$$

[$(2 \rightleftarrows 1)$ signifies the interchange of 1 and 2, the index T denotes the transpose, and C is the matrix of charge conjugation], we easily derive the following properties of the symbols (10c):

$$\sigma\begin{pmatrix} k & n \\ k' & n' \end{pmatrix} = \left[\sigma\begin{pmatrix} k' & n' \\ k & n \end{pmatrix}\right]^*, \quad \sigma\begin{pmatrix} k & n \\ k' & n' \end{pmatrix} = (-1)^s\sigma\begin{pmatrix} n & k \\ n' & k' \end{pmatrix}_{(2\rightleftarrows 1)},$$

$$s = n + n' + k + k'. \tag{11}$$

3. CALCULATION OF THE PROBABILITY OF CONVERSION WITH FORMATION OF PAIRS IN THE FIRST APPROXIMATION

In the present paper we restrict ourselves to the calculation of σ_1.* With the help of (11) we can write

$$\sigma_1 = 2\,\mathrm{Re}\,\sigma\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - (2\rightleftarrows 1),$$

$$\sigma\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{4}\,\mathrm{Sp}\,[w\,(1\,|0)\,(i\hat{p}_1 + m)\,\bar{w}\,(0\,|\,0)\,(i\hat{p}_2 - m)],$$

$$w\,(1\,|\,0) = -\int d^3f_2\,\gamma_4\,(i\hat{f}_2 - m)\, b_{lm}^{(\lambda)}(\mathbf{f}_2 + \mathbf{p}_1)\,/\,[(\mathbf{f}_2 - \mathbf{p}_2)^2 + \lambda^2]\,(\mathbf{f}_2^2 - \mathbf{p}_2^2 - i\varepsilon),$$

$$\bar{w}\,(0\,|\,0) = \bar{\hat{b}}_{lm}^{(\lambda)}(\mathbf{p}_1 + \mathbf{p}_2) = -\,\hat{b}_{lm}^{(\lambda)*}(\mathbf{p}_1 + \mathbf{p}_2). \tag{12}$$

Here we have used the identity

$$f_2^2 + m^2 = \mathbf{f}_2^2 + m^2 - E_2^2 = \mathbf{f}_2^2 - \mathbf{p}_2^2.$$

To simplify the subsequent calculations we introduce the following notation:

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{v}, \qquad \mathbf{f}_2 + \mathbf{p}_1 = \mathbf{k}, \qquad \mathbf{f}_2 - \mathbf{p}_2 = \mathbf{q} = \mathbf{k} - \mathbf{v},$$

$$\hat{b}_1 = \hat{b}_{lm}^{(\lambda)}(1\,|\,0) = \int d^3k\hat{b}_{lm}^{(\lambda)}(\mathbf{k})\,/\,(\mathbf{q}^2 + \lambda^2)\,(\mathbf{f}_2^2 - \mathbf{p}_2^2 - i\varepsilon),$$

$$\hat{b}_0 = \hat{b}_{lm}^{(\lambda)}(0\,|\,0) = \hat{b}_{lm}^{(\lambda)}(\mathbf{v}) \tag{13}$$

Then the expression for σ_1 has the form

$$\sigma_1 = 2\mathrm{Re}\,[\tfrac{1}{4}\,\mathrm{Sp}\,\{\gamma_4\,(i\hat{f}_2 - m)\,\hat{b}_1\,(i\hat{p}_1 + m)\,\hat{b}_0^*\,(i\hat{p}_2 - m)\}] - (2\rightleftarrows 1). \tag{14}$$

It should be recalled that b_1 is an integral operator which acts on functions of $\mathbf{k}$ both to the left and to the right.

First we calculate the traces and then the integrals, because in this case the expressions for the integrals assume a simpler form.

Let us rewrite (14) in the form

*The calculation of the correction $\sim(\alpha Z)^2$ will be published in a later paper.

$$\sigma_1 = 2\mathrm{Re}\,[\tfrac{1}{4}\,\mathrm{Sp}\,\{\hat{b}_1(i\hat{p}_1 + m)\,\hat{b}_0^*(i\hat{p}_2 - m)\,\gamma_4\,(i\hat{f}_2 - m)\}] - (2 \rightleftarrows 1). \tag{15}$$

Using the notation (13), we obtain

$$(i\hat{p}_2 - m)\,\gamma_4\,(i\hat{f}_2 - m) = -2E_2\,(i\hat{p}_2 - m) + (i\hat{p}_2 - m)\,\gamma_4\,i\hat{q}. \tag{16}$$

We now split σ_1 into two parts, one containing $\mathbf{q}$, the other not containing $\mathbf{q}$:

$$\sigma_1 = \sigma_1(0) + \sigma_1(\mathbf{q}),$$

$$\sigma_1(0) = \{-4E_2\,\mathrm{Re}\,\tfrac{1}{4}\,\mathrm{Sp}\,[\hat{b}_1(i\hat{p}_1 + m)\,\hat{b}_0^*\,(i\hat{p}_2 - m)]\} - (2 \rightleftarrows 1),$$

$$\sigma_1(\mathbf{q}) = 2\mathrm{Re}\,\tfrac{1}{4}\,\mathrm{Sp}\,\{\hat{b}_1\,(i\hat{p}_1 + m)\,\hat{b}_0^*\,(i\hat{p}_2 - m)\,\gamma_4\,i\hat{q}\} - (2 \rightleftarrows 1).$$

After evaluating the traces we find

$$\sigma_1(0) = 4E_2\,\mathrm{Re}\,\{(m^2 + E_1E_2 - \mathbf{p}_1\mathbf{p}_2)\,(b_1b_0^*) + (b_1p_1)(b_0^*p_2) + (b_1p_2)(b_0^*p_1)\} - (2 \rightleftarrows 1),$$

$$\begin{aligned}\sigma_1(\mathbf{q}) = 2\mathrm{Re}\,\{&(\mathbf{b}_1\mathbf{q})[E_1\,(b_0^*\,p_2) + E_2\,(b_0^*p_1) - (m^2 + E_1E_2 - \mathbf{p}_1\mathbf{p}_2)b_0^*] \\ &+ (b_1b_0^*)[E_1\,(\mathbf{p}_2\mathbf{q}) - E_2\,(\mathbf{p}_1\mathbf{q})] + (b_1p_1)\,[E_2\,(\mathbf{b}_0^*\mathbf{q}) - b_0^*\,(\mathbf{p}_2\mathbf{q})] + (b_1p_2)[b_0^*\,(\mathbf{p}_1\mathbf{q}) \\ &- E_1\,(\mathbf{b}_0^*\mathbf{q})] - b_1[(p_1b_0^*)(\mathbf{p}_2\mathbf{q}) + (\mathbf{p}_1\mathbf{q})(p_2b_0^*) + (m^2 + E_1E_2 - \mathbf{p}_1\mathbf{p}_2)(\mathbf{b}_0^*\mathbf{q})]\} - (2 \rightleftarrows 1).\end{aligned} \tag{17}$$

Using the definitions of b_0 and b_1 [formulas (13) and (7)], we average the scalar products of the spherical vectors and the products of the spherical harmonics in the numerators of the integrands over the magnetic quantum numbers. The averaged expressions must, of course, be invariant under spatial rotations, i.e., they must be expressed in terms of functions of scalar products of ordinary vectors. These invariants are calculated in Appendix A for all occurring combinations of spherical vectors of the electric or magnetic type. It follows from the formulas of Appendix A and (13) that the numerators of all integrals are proportional to functions of the form

$$(1, \mathbf{k})\,P_l\left(\frac{\mathbf{kv}}{kv}\right), \qquad (1, \mathbf{k})\,\frac{k_\alpha}{k}\,P_l'\left(\frac{\mathbf{kv}}{kv}\right), \qquad (1, \mathbf{k})\,\frac{k_\alpha k_\beta}{k^2}\,P_l''\left(\frac{\mathbf{kv}}{kv}\right),$$

where P_l is the Legendre polynomial. The required real parts of these integrals are calculated in Appendix C.

It is seen from the final expressions (C.6) to (C.9) that the infinities arising in the limit of the pure Coulomb field drop out and that the result of the integration can be formally obtained in the following fashion: replace the integral sign by $\pi^3/2$, change the denominator of the integral (13) to p_2, and replace $\mathbf{k}$ by $\mathbf{v}$ everywhere in the remaining part of the function under the integral.

After the general proof of the necessity of performing these operations we can change the order of application and first perform the alterations in the expressions which formally correspond to the integration and then average over the magnetic quantum numbers.

Making the above-mentioned changes, we find that $\sigma_1(\mathbf{q}) = 0$.

For $\sigma_1(0)$ we obtain

$$\sigma_1(0) = 2\pi^2\cdot\pi\,(E_2/p_2 - E_1/p_1)\,\sigma_0,$$

$$\sigma_0 = \sigma\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = (m^2 + E_1E_2 - \mathbf{p}_1\mathbf{p}_2)(b_0b_0^*) + (b_0p_1)(b_0^*p_2) + (b_0p_2)(b_0^*p_1). \tag{18}$$

Using (18), (10a), and (9), we have finally

$$d\beta_{l\lambda} = d\beta_{l\lambda}^0\,M\,(Z;\,E_1,\,E_2),$$

$$M\,(Z;\,E_1,\,E_2) = 1 - \pi\alpha\,Z\,(E_1/p_1 - E_2/p_2),$$

$$d\beta_{l\lambda}^0 = \frac{\alpha\omega}{4}\,\frac{d^3p_1d^3p_2}{(2\pi)^6}\,\sigma_0\delta\,(E_2 + E_1 - \omega), \tag{19}$$

i.e., $d\beta_{l\lambda}^0$ is the differential conversion coefficient for pair formation in the z. B. a.*

The first Born approximation result is therefore obtained by multiplying the differential conversion coefficient for pair formation in the z. B. a. by the factor $M(Z;\,E_1,\,E_2)$.

Since $M(Z;\,E_1,\,E_2)$ is antisymmetric in the indices 1 and 2, the term proportional to αZ drops out in the integration over E_1 or E_2, i.e., it does not appear in the expressions for the angular distribution and the total conversion coefficient for pair formation. This is a consequence of the charge symmetry of the theory which requires that only even powers appear in the expansions of the total conversion coefficient for pair formation and of the angular distribution in powers of αZ.

The authors are grateful to B. S. Dzhelepov and L. A. Sliv for a discussion of the results of this paper and for valuable comments.

APPENDIX

A. FORMULAS FOR THE SCALARS FORMED OUT OF THE SPHERICAL VECTORS

To derive the formula for the probability we must write down the explicit covariant form of the

*Formula (19) is, of course, not valid near the limits of the energy spectrum, where the parameter $\alpha ZE/p$ is not small.

expressions*

$$S^{(1)}_{l\lambda}=\sum_{m=-l}^{l}(\mathbf{Y}_{l,l+\lambda,m}(\mathbf{k}/k)\,\mathbf{Y}^*_{l,l+\lambda,m}(\mathbf{v}/v)),$$

$$S^{(2)}_{l\lambda}=\sum_{m=-l}^{l}(\mathbf{p}\mathbf{Y}_{l,l+\lambda,m}(\mathbf{k}/k)(\mathbf{q}\mathbf{Y}_{l,l+\lambda,m}(\mathbf{v}/v))^*,$$

$$S^{(3)}_{l}=\sum_{m=-l}^{l}Y_{lm}(\mathbf{k}/k)(\mathbf{q}\mathbf{Y}_{l,l-1,m}(\mathbf{v}/v))^*.$$

We are interested in the form of these expressions for $\lambda = 0$ and $\lambda = -1$.

Choosing the z axis in the direction of $\mathbf{v}$ and using the addition theorem for spherical functions, we obtain

$$S^{(1)}_{l,0}=\frac{2l+1}{4\pi}P_l\left(\frac{\mathbf{kv}}{kv}\right), \tag{A.1}$$

$$S^{(1)}_{l,-1}=\frac{2l+1}{2l-1}\frac{2l+1}{4\pi}P_l\left(\frac{\mathbf{kv}}{kv}\right). \tag{A.2}$$

To derive the remaining formulas we make use of the covariant differential representation of the spherical vectors (see, for example, reference 4, p. 34):

$$\mathbf{Y}_{l,l,m}\left(\frac{\mathbf{k}}{k}\right)=\mathbf{Y}^{(0)}_{l,m}\left(\frac{\mathbf{k}}{k}\right)=\frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}\left(\frac{\mathbf{k}}{k}\right)$$
$$=\frac{-i}{\sqrt{l(l+1)}}[\mathbf{k}\times\nabla_k]Y_{lm}\left(\frac{\mathbf{k}}{k}\right),$$

$$\mathbf{Y}_{l,l-1,m}\left(\frac{\mathbf{k}}{k}\right)=\sqrt{\frac{l}{2l+1}}\,\mathbf{Y}^{(-1)}_{lm}+\sqrt{\frac{l+1}{2l+1}}\,\mathbf{Y}^{(1)}_{lm}$$
$$=\frac{1}{k\sqrt{2l+1}}\left\{\sqrt{l}\,\mathbf{k}-\frac{1}{\sqrt{l}}[\mathbf{k}\times[\mathbf{k}\times\nabla_k]]\right\}Y_{lm}\left(\frac{\mathbf{k}}{k}\right).$$

After some transformations we then find

$$S^{(2)}_{l0}=\frac{2l+1}{4\pi l(l+1)}\left\{P'_l\frac{[\mathbf{p}\times\mathbf{k}][\mathbf{q}\times\mathbf{v}]}{kv}-P''_l\frac{([\mathbf{q}\times\mathbf{v}]\mathbf{k})([\mathbf{p}\times\mathbf{k}]\mathbf{v})}{k^2v^2}\right\}, \tag{A.3}$$

$$S^{(2)}_{l,-1}=\frac{1}{4\pi kv}[lP_l(\mathbf{kp})(\mathbf{vq})+P'_l(kv)^{-1}\{(\mathbf{kp})([\mathbf{v}\times\mathbf{q}][\mathbf{v}\times\mathbf{k}])$$
$$+(\mathbf{vq})([\mathbf{k}\times\mathbf{p}][\mathbf{k}\times\mathbf{v}])+l^{-1}([[\mathbf{k}\times\mathbf{p}]\times\mathbf{k}][[\mathbf{v}\times\mathbf{q}]\times\mathbf{v}])\}$$
$$+l^{-1}P''_l(kv)^{-2}([\mathbf{k}\times\mathbf{p}][\mathbf{k}\times\mathbf{v}])([\mathbf{v}\times\mathbf{q}][\mathbf{v}\times\mathbf{k}])], \tag{A.4}$$

$$S^{(3)}_{l}=\frac{\sqrt{2l+1}}{4\pi v}\left\{\sqrt{l}\,P_l(\mathbf{qv})+\frac{1}{\sqrt{l}}P'_l\,\frac{[\mathbf{v}\times\mathbf{q}][\mathbf{v}\times\mathbf{k}]}{kv}\right\},$$
$$P^{(n)}_l\equiv P^{(n)}_l(\mathbf{kv}/kv). \tag{A.5}$$

B. CALCULATION OF THE INTEGRALS $J_l(k)$

To compute the integrals entering in the formula for the probability, we must know the singular points of the "radial" part of the Fourier transform of the nuclear potential $J_l(k)$.

By definition

$$J_l(k)=\frac{(2\pi)^3}{i\sqrt{\omega k}}\int_0^\infty H^{(1)}_{l+\frac{1}{2}}(\omega r)\,I_{l+\frac{1}{2}}(kr)\,rdr$$
$$=\frac{(2\pi)^3}{i\sqrt{\omega k}}\lim_{\lambda\to\infty}\int_0^\lambda H_{l+\frac{1}{2}}(\omega r)\,I_{l+\frac{1}{2}}(kr)\,rdr.$$

Using the antiderivative and the identity

$$\lim_{\lambda\to\infty}\frac{1-e^{i\lambda x}}{x}=\frac{\mathrm{P}}{x}-i\pi\delta(x),$$

where P designates the principal value, we obtain

$$J_l(k)=-\frac{(4\pi)^2}{\omega^{l+1}}k^l\left\{\frac{\mathrm{P}}{k^2-\omega^2}+i\pi\delta(k^2-\omega^2)\right\}$$
$$=-\frac{(4\pi)^2}{\omega^{l+1}}\frac{k^l}{k^2-\omega^2-i\varepsilon}.$$

C. CALCULATION OF THE BASIC INTEGRALS

The formula for the probability contains integrals of the form

$$(K^{(n)},\mathbf{K}^{(n)})=\mathrm{Re}\int\frac{J_l(k)\,P^{(n)}_l\left(\frac{\mathbf{kv}}{kv}\right)(1,\mathbf{k})\,d^3k}{[(\mathbf{k}-\mathbf{v})^2+\lambda^2][(\mathbf{k}-\mathbf{v}+\mathbf{p}_s)^2-\mathbf{p}^2-i\varepsilon]}\qquad(s=1,2). \tag{C.1}$$

Using the Feynman identity

$$\frac{1}{ab}=\int_0^1 dz/[az+b(1-z)]^2,$$

we find

$$(K,\mathbf{K})=\mathrm{Re}\int_0^1 dz\left\{(1,\mathbf{V})\frac{\partial}{\partial\Lambda^2}+(0,1)\,\mathbf{V}\frac{\partial}{\partial V^2}\right\}R, \tag{C.2}$$

$$R=\int\frac{J_l(k)\,P_l\left(\frac{\mathbf{kv}}{kv}\right)d^3k}{(\mathbf{k}-\mathbf{V})^2-\Lambda^2},\qquad \mathbf{V}=\mathbf{v}-\mathbf{p}z,$$
$$\Lambda^2=\mathbf{p}^2z^2-\lambda^2(1-z)+i\varepsilon z. \tag{C.3}$$

Choosing the z axis in the direction of $\mathbf{V}$ and integrating over the angles in (C.3), we obtain

$$R=P_l\left(\frac{\mathbf{Vv}}{Vv}\right)\frac{2\pi}{V}\int_0^\infty J_l(k)\,Q_l\left(\frac{k^2+V^2-\Lambda^2}{2kV}\right)kdk,$$

$$Q_l(x)=\frac{1}{2}\int_0^1\frac{P_l(t)}{x-t}dt,$$

where Q_l are the Legendre functions of the second kind.

Since the integrand is an even function, we can extend the integration from $-\infty$ to $+\infty$ and consider the contour in the upper half plane. Calculating the residues with the help of Appendix B and using the identity

$$\ln\frac{(k+V)^2-\Lambda^2}{(k-V)^2-\Lambda^2}=\int_{-V+\Lambda}^{V+\Lambda}\frac{dt}{k-t}+\ln\frac{k+V+\Lambda}{k-V+\Lambda}$$

[the second term is regular in the upper half-plane,

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*The first and second sums were computed by Berestetskiĭ, Dolginov, and Ter-Martirosyan[5] for the particular case $\mathbf{k} = \mathbf{v}$ and $\mathbf{p} = \mathbf{q}$.

see formula (C.3)], we obtain

$$R = P_l\left(\frac{\mathbf{Vv}}{Vv}\right)\frac{\pi^2 i}{V}\left\{\int_{-V+\Lambda}^{V+\Lambda} J_l(k)\,P_l\left(\frac{k^2+V^2-\Lambda^2}{2kV}\right)kdk\right.$$

$$\left. + \frac{(4\pi)^2}{\omega}Q_l\left(\frac{\omega^2+V^2-\Lambda^2}{2\omega V}\right)\right\} \qquad (C.4)$$

Substituting (C.4) in (C.2) and separating out the term which contains Λ in the denominator and diverges for $\lambda \to 0$, we integrate this term by parts over z with $dU = dz/\Lambda$ and obtain

$$(K, \mathbf{K}) = \mathrm{Re}\,\pi^2 i\left\{-(1, \mathbf{V})\,F\Big|_{z=0}\frac{1}{p}\ln\left[i\frac{\lambda}{p}+\frac{\lambda^2}{2p^2}\right]\right.$$

$$\left. + (K,\mathbf{K})_0\right\},$$

$$(K, \mathbf{K})_0 = (1, \mathbf{V})\,F\big|_{z=1}\frac{1}{p}\ln\left[2+\frac{\lambda^2}{2p^2}\right]$$

$$-\int_0^1 dz\,[(1, \mathbf{V})\,F]'\ln\left(\sqrt{z^2-\frac{\lambda^2}{p^2}(1-z)}\right.$$

$$\left. + z + \frac{\lambda^2}{2p^2}\right) + \int_0^1 dz\left\{(1, \mathbf{V})\frac{P_l}{V}\left[\frac{1}{2}J_l(k)\,P_l(x)\Big|_{-V+\Lambda}^{V+\Lambda}\right.\right.$$

$$\left. - \frac{1}{2V}\int_{-V+\Lambda}^{V+\Lambda} J_l(k)\,P_l'(x)\,dk + \frac{\partial}{\partial\Lambda^2}\frac{(4\pi)^2}{\omega^2}Q_l(y)\right]$$

$$\left. + (0,1)\,\mathbf{V}\frac{\partial}{\partial V^2}\left[\frac{P_l}{V}\left(\int_{-V+\Lambda}^{V+\Lambda} J_l(k)\,P_l(x)\,kdk + \frac{(4\pi)^2}{\omega}Q_l(y)\right)\right]\right\}. \qquad (C.5)$$

Here

$$P_l \equiv P_l\left(\frac{\mathbf{Vv}}{Vv}\right), \qquad x \equiv \frac{k^2+V^2-\Lambda^2}{2kV}, \qquad y \equiv \frac{\omega^2+V^2-\Lambda^2}{2\omega V},$$

$$F = \tfrac{1}{2}P_l\,(J_l(k)\,P_l(k)\,|_{V+\Lambda} - J_l(k)\,P_l(x)\,|_{-V+\Lambda}).$$

$(K, \mathbf{K})_0$ does not contain divergent parts and converges uniformly with respect to λ. The singularities of the type

$$\frac{1}{(V+\Lambda)^2-\omega^2}, \qquad \frac{1}{(V-\Lambda)^2-\omega^2},$$

$$\ln\frac{(V+\Lambda)^2-\omega^2}{(V-\Lambda)^2-\omega^2}, \qquad \ln\frac{(V+\omega)^2-\Lambda^2}{(\omega-V)^2-\Lambda^2}$$

under the integral do not give any imaginary contributions on account of the relations

$$(V-\Lambda)^2 < (V+\Lambda)^2 < (p_1+p_2)^2 < \omega^2,$$

$$(\omega+V)^2 > (\omega-V)^2 > p^2z^2 > \Lambda^2 \qquad (\lambda \to 0).$$

$(K, \mathbf{K})_0$ is therefore real for $\lambda \to 0$.

Finally we have

$$(K,\mathbf{K}) = \frac{\pi^3}{2p}J_l(v)(1, \mathbf{v}). \qquad (C.6)$$

It is easily seen that replacing $\mathbf{v}$ in the argument of the Legendre polynomial in (C.1) by an arbitrary vector $\mathbf{u}$ leads to

$$\mathrm{Re}\int\frac{J_l(k)\,P_l\left(\frac{\mathbf{ku}}{ku}\right)(1, \mathbf{k})\,d^3k}{[(\mathbf{k}-\mathbf{v})^2+\lambda^2][(\mathbf{k}-\mathbf{v}+\mathbf{p})^2-\mathbf{p}^2-i\varepsilon]} = \frac{\pi^3}{2p}J_l(v)P_l\left(\frac{\mathbf{vu}}{vu}\right)(1, \mathbf{v}). \qquad (C.7)$$

Differentiating both sides of this equation once and twice with respect to u_α/u and setting $\mathbf{u} = \mathbf{v}$, we obtain

$$\mathrm{Re}\int\frac{J_l(k)P_l'\left(\frac{\mathbf{kv}}{kv}\right)(1, \mathbf{k})}{[1]\,[2]}\frac{k_\alpha}{k}d^3k = \frac{\pi^3}{2p}J_l(v)(1, \mathbf{v})\frac{v_\alpha}{v}, \qquad (C.8)$$

$$\mathrm{Re}\int\frac{J_l(k)P_l''\left(\frac{\mathbf{kv}}{kv}\right)(1, \mathbf{k})}{[1]\,[2]}\frac{k_\alpha}{k}\frac{k_\beta}{k}d^3k = \frac{\pi^3}{2p}J_l(v)(1, \mathbf{v})\frac{v_\alpha}{v}\frac{v_\beta}{v}. \qquad (C.9)$$

It follows from formulas (C.1), (C.6), (C.8), and (C.9) that the value of the integral just calculated can be formally obtained from the function under the integral by replacing the integral sign by $\pi^3/2$, changing the denominator of the expression under the integral to p, and replacing $\mathbf{k}$ by $\mathbf{v}$ in the numerator of all functions.

By this method one can also compute an arbitrary tensor $K^{\alpha\beta\ldots\sigma}$ with a rank which is conformity with the convergence of the integral

$$K^{\alpha,\beta\ldots\sigma} = \mathrm{Re}\int\frac{J_l(k)\,P_l\,(\mathbf{kv}/kv)\,k_\alpha k_\beta\ldots k_\sigma}{[1]\,[2]}d^3k. \qquad (C.10)$$

In the calculation of $K^{\alpha\beta\ldots\sigma}$ we must use the identity

$$\frac{\overbrace{k_\alpha k_\beta\ldots k_\sigma}^{m}}{[(\mathbf{k}-\mathbf{V})^2-\Lambda^2]^2} = \overbrace{V_\alpha V_\beta\ldots V_\sigma}^{m}\left[\left(\frac{\partial}{\partial V^2}+\frac{\partial}{\partial\Lambda^2}\right)\int d\Lambda^2\right]^{m-1}$$

$$\times\left(\frac{\partial}{\partial V^2}+\frac{\partial}{\partial\Lambda^2}\right)\frac{1}{(\mathbf{k}-\mathbf{V})^2-\Lambda^2}.$$

As a result we obtain

$$K^{\alpha,\beta\ldots\sigma} = \frac{\pi^3}{2p}J_l(v)\,v_\alpha v_\beta\ldots v_\sigma. \qquad (C.11)$$

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Translated by R. Lipperheide
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