SECOND-ORDER EQUATIONS FOR SPINOR FIELDS

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Second order equations for fermions are investigated on the basis of the Lagrangian formalism. It is shown that an additional degree of freedom of the particles is due to two orientations of an imaginary electric dipole moment, which, however, may have a real radiation correction.

SECOND order equations for spinor wave functions have been proposed by Feynman and Gell-Mann[1] and have been discussed in different connections in a series of papers.[2-4] The present paper is devoted to the quantization of a spinor field satisfying a second order equation and to the interpretation of the additional degree of freedom.

Let the field be described by the spinor $\psi(x)$ with the usual transformation properties. Then we can write the Lagrangian which leads to the Klein-Gordon equation in the following form:

$$L = \frac{1}{m}\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_\mu\gamma_\nu\frac{\partial\psi}{\partial x_\nu} - m\bar{\psi}\psi, \qquad (1)$$

$$m^{-1}(\Box - m^2)\psi(x) = 0. \qquad (2)$$

On introducing the interaction with the electromagnetic field by means of the substitution $\partial_\mu \to \partial_\mu - ieA_\mu$, we can obtain the Lagrangian which leads to the equation of Feynman and Gell-Mann (provided that $\gamma_5\psi = \psi$).

For the time being we restrict ourselves to the discussion of the free field and calculate the dynamical variables in accordance with the usual rules:

the energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{m}\frac{\partial\bar{\psi}}{\partial x_\rho}\gamma_\rho\gamma_\mu\frac{\partial\psi}{\partial x_\nu} + \frac{1}{m}\frac{\partial\bar{\psi}}{\partial x_\nu}\gamma_\mu\gamma_\rho\frac{\partial\psi}{\partial x_\rho} - \delta_{\mu\nu}L \qquad (3a)$$

the current vector

$$J_\mu = \frac{i}{m}\bar{\psi}\gamma_\mu\gamma_\nu\frac{\partial\psi}{\partial x_\nu} - \frac{i}{m}\frac{\partial\bar{\psi}}{\partial x_\nu}\gamma_\nu\gamma_\mu\psi \qquad (3b)$$

and the spin angular momentum tensor

$$S_{\lambda,\mu\nu} = \frac{i}{2m}\bar{\psi}\sigma_{\mu\nu}\gamma_\lambda\gamma_\rho\frac{\partial\psi}{\partial x_\rho} - \frac{i}{2m}\frac{\partial\bar{\psi}}{\partial x_\rho}\gamma_\rho\gamma_\lambda\sigma_{\mu\nu}\psi \qquad (3c)$$

We note that $S_{0,0i} \neq 0$ ($i = 1, 2, 3$), while in the theory based on the Dirac equation $S_{0,0i} \equiv 0$.

We now write $\psi(x)$ in the form of a superposition of plane waves:

$$\psi(x) = (2\pi)^{-3/2}\sum_{\lambda=1}^{4}\int\frac{dp}{\sqrt{2E_p}}[a^\lambda(\mathbf{p})u\ (\mathbf{p})e^{-ipx} + b^{*\lambda}(\mathbf{p})v\ (\mathbf{p})e^{ipx}]. \qquad (4)$$

Following the idea of Feynman and Gell-Mann we choose the basis spinors $u^\lambda(\mathbf{p})$ and $v^\lambda(\mathbf{p})$ to be the eigenfunctions of the projection operators $\frac{1}{2}(1 \pm \gamma_5)$ and $\frac{1}{2}(1 \pm \sigma_p)$ (here $\sigma_p = \boldsymbol{\sigma}\mathbf{p}/p$):

$$\begin{aligned}
&{}^1/_4(1+\gamma_5)(1+\sigma_p)u^1(\mathbf{p}) = u^1(\mathbf{p}),\\
&\quad {}^1/_4(1+\gamma_5)(1+\sigma_p)v^1(\mathbf{p}) = v^1(\mathbf{p}),\\
&{}^1/_4(1+\gamma_5)(1-\sigma_p)u^2(\mathbf{p}) = u^2(\mathbf{p}),\\
&\quad {}^1/_4(1+\gamma_5)(1-\sigma_p)v^2(\mathbf{p}) = v^2(\mathbf{p}),\\
&{}^1/_4(1-\gamma_5)(1+\sigma_p)u^3(\mathbf{p}) = u^3(\mathbf{p}),\\
&\quad {}^1/_4(1-\gamma_5)(1+\sigma_p)v^3(\mathbf{p}) = v^3(\mathbf{p}),\\
&{}^1/_4(1-\gamma_5)(1-\sigma_p)u^4(\mathbf{p}) = u^4(\mathbf{p}),\\
&\quad {}^1/_4(1-\gamma_5)(1-\sigma_p)v^4(\mathbf{p}) = v^4(\mathbf{p}).
\end{aligned} \qquad (5)$$

If we assume that the operation of strong reflection of a spinor is carried out by means of the matrix $i\gamma_5$, then the following relation must hold

$$i\gamma_5 u^\lambda(\mathbf{p}) = \sigma_p v^\lambda(\mathbf{p}). \qquad (6)$$

The spinors $u^\lambda(\mathbf{p})$ and $v^\lambda(\mathbf{p})$ can be expressed in terms of $u^\lambda(0)$ and $v^\lambda(0)$ by means of the Lorentz transformation matrix:

$$\begin{aligned}
u^\lambda(\mathbf{p}) &= \frac{E_p + p + m + (E_p + p - m)\gamma_5\sigma_p}{2\sqrt{m(E_p + p)}}u^\lambda(0),\\
v^\lambda(\mathbf{p}) &= \frac{E_p + p + m + (E_p + p - m)\gamma_5\sigma_p}{2\sqrt{m(E_p + p)}}v^\lambda(0).
\end{aligned} \qquad (7)$$

Formulas (5) — (7) enable us to obtain the relations for normalization, orthogonality and summation over the index λ:

$$\bar{u}^\mu(\mathbf{p})u^{\lambda+2}(\mathbf{p}) = m\delta_{\mu\lambda}, \quad \bar{v}^\mu(\mathbf{p})v^{\lambda+2}(\mathbf{p}) = -m\delta_{\mu\lambda}; \qquad (8a)$$

$$\sum_{\lambda=1}^{4}u^\lambda(\mathbf{p})\bar{u}^{\lambda+2}(\mathbf{p}) = mI, \quad \sum_{\lambda=1}^{4}v^\lambda(\mathbf{p})\bar{v}^{\lambda+2}(\mathbf{p}) = -mI; \qquad (8b)$$

$$\sum_{\lambda=1}^{4} u^{\lambda}(\mathbf{p})\bar{u}^{\lambda}(\mathbf{p}) = \hat{p}, \qquad \sum_{\lambda=1}^{4}{}' v^{\lambda}(\mathbf{p})\bar{v}^{\lambda}(\mathbf{p}) = \hat{p}. \tag{8c}$$

Here and later the index $\lambda + 2$ is equal to 1 and 2 for λ equal to 3 and 4.

We now consider the quantization of the free spinor field. From the canonical commutation relations it follows that

$$[\psi_\alpha(x), \bar{\psi}_\beta(y)]_+ = -mi\delta_{\alpha\beta}D(x-y),$$
$$[\psi_\alpha(x), \psi_\beta(y)]_+ = [\bar{\psi}_\alpha(x), \bar{\psi}_\beta(y)] = 0. \tag{9}$$

These relations hold if

$$a^{\lambda+2}(\mathbf{p})\,a^{*\mu}(\mathbf{q}) + a^{*\mu}(\mathbf{q})\,a^{\lambda+2}(\mathbf{p}) = b^{\lambda+2}(\mathbf{p})\,b^{*\mu}(\mathbf{q}) + b^{*\mu}(\mathbf{q})\,b^{\lambda+2}(\mathbf{p}) = \delta_{\lambda\mu}\delta(\mathbf{p}-\mathbf{q}) \tag{10}$$

and all the other anticommutators are equal to zero.

As may be seen from (10) the asterisks on the operators cannot denote Hermitian conjugation and, therefore, the norm of the state amplitude Ψ defined simply by $\Psi^*\Psi$ will not be positive definite. However, the operator η which has the properties

$$\eta = \eta^*, \ \eta^2 = 1, \ \eta a^{\lambda}(\mathbf{p})\,\eta = a^{\lambda+2}(\mathbf{p}),$$
$$\eta b^{\lambda}(\mathbf{p})\,\eta = b^{\lambda+2}(\mathbf{p}), \ \eta\Psi_0 = \Psi_0, \tag{11}$$

generates a positive definite metric in Hilbert space of the state amplitudes: $\Psi^*_\eta\Psi > 0$; the observable quantities correspond to self-adjoint operators which now satisfy the condition $A = \eta^*A^*\eta$.

The operators for the energy-momentum, the charge, and the spin pseudovector are self-adjoint, as can be shown by utilizing relations (3), (4) and (8a). But the single particle states are also characterized by the imaginary eigenvalues ($\pm i/2$) of the component along the direction of motion of the polar vector $S_{0i} = \int S_{0,0i}dx$. This vector may be called the polar spin; with it is associated an imaginary dipole moment of the particle, which, however, may have a real radiation correction (if there exist interactions which are not invariant with respect to the combined inversion).

The Lagrangian for the interaction with the electromagnetic field has the form

$$L' = \frac{ie}{m}\left(\bar{\psi}\frac{\partial\psi}{\partial x_\mu} - \frac{\partial\bar{\psi}}{\partial x_\mu}\psi\right)A_\mu - \frac{e}{2m}\bar{\psi}\sigma_{\mu\nu}\psi F_{\mu\nu} + \frac{e^2}{m}A^2_\rho\bar{\psi}\psi. \tag{12}$$

By following the rules for constructing the scattering matrix in the case of derivative coupling,[5] one can prove the unitarity of the S matrix and carry out calculations for specific effects. In spite of its peculiar form which is close to the electrodynamics of scalar particles, the theory leads to the usual results. In Brown's paper[2] dealing with the Compton effect the approach to the problem is equivalent to ours if we assume that in nature there exist states corresponding only to "right-handed" (or "left-handed") particles. This assumption is not necessary; it may be shown (for example by considering Møller scattering) that the Pauli exclusion principle applies, as before, to states for which only the real quantum numbers coincide.

[1] R. Feynman and M. Gell-Mann, Phys. Rev 109, 193 (1958).
[2] L. Brown, Phys. Rev. 111, 957 (1958).
[3] G. Marx, Nuclear Phys. 9, 337 (1959).
[4] A. Barut, Ann. Phys. 5, 95 (1958).
[5] P. T. Matthews, Phys. Rev. 76, 1657 (1949).

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