THE MICROSCOPIC THEORY OF THE FERMI LIQUID

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A connection is established between the Fermi excitation energies as defined by the Landau theory of the Fermi liquid and by the many-particle quantum theory at zero temperature.

AT the present time a number of papers[1-4] discuss the microscopic derivation of the fundamental premises of Landau's general theory of the Fermi liquid.[5] Landau[3] obtained a most essential result by establishing the exact microscopic meaning of the function $f(\mathbf{p}, \mathbf{p}')$, which plays a fundamental part in the theory of the Fermi liquid and determines the variation of the quasi-particle energies ϵ_p under infinitesimal changes in their distribution functions $n(p)$

$$\delta\varepsilon_p = \mathrm{Sp}_\sigma \int f(\mathbf{p}, \mathbf{p}')\, \delta n(\mathbf{p}')\, d\tau'. \tag{1}$$

As regards the quasi-particle energy, the following must be kept in mind. In the theory of the Fermi liquid, ϵ_p is defined as the variational derivative of the total energy of the system with respect to the distribution function $n(p)$

$$\varepsilon_p = \delta E / \delta n(p). \tag{2}$$

On the other hand, in the general many-particle quantum theory, it has been shown[6,7] that the energies and attenuations of the Fermi excitations of the system (quasi-particles or holes), for momenta close to the boundary, are determined by the poles of the one-particle Green's function in the momentum representation

$$G(p, \varepsilon) = [\varepsilon_p^0 - \varepsilon - \Sigma(p, \varepsilon)]^{-1},$$

$$\varepsilon_p^0 = p^2/2m, \tag{3}$$

along the lower half-plane (for quasi-particles) or the upper half-plane (for holes). From this we obtain the following equations for the energy ϵ_p and the attenuation Γ_p of a quasi-particle:

$$\varepsilon_p^0 - \varepsilon_p - \Sigma_0(p, \varepsilon) = 0; \tag{4}$$

$$\Gamma_p = \Sigma_1(p, \varepsilon_p) \,/\, [1 + (\partial\Sigma_0/\partial\varepsilon)_{\varepsilon=\varepsilon_p}], \tag{4a}$$

where $\Sigma_0 = \mathrm{Re}\Sigma$, $\Sigma_1 = \mathrm{Im}\Sigma$, and Σ is the compact part of the self energy of the particle. Obviously, it is to be expected that the quantity ϵ_p, defined by the relation (2), should agree with the root of Eq. (4). The aim of the present communication is to prove this statement.

We shall make some preliminary observations. It has been shown[6,8] that the imaginary part of the Green's function, and therefore also the attenuation, change sign at $\epsilon = \mu$, where $\mu = \partial E/\partial N$ is the chemical potential of the system. To supplement this theorem, which is completely general in character, we shall make the assumption that the imaginary part of the self energy is continuous at $\epsilon = \mu$, i.e.,

$$\lim_{\varepsilon\to\mu} \Sigma_1(p, \varepsilon) = 0 \tag{5}$$

[we note that this assumption serves as the basis for the derivation of (4) and (4a)].

Let us now consider the equation $\epsilon_p = \mu$ (relative to p), which, according to (4), can also be written in the form

$$\varepsilon_p^0 - \mu - \Sigma(p, \mu) = 0. \tag{6}$$

The root (p_0) of this equation is the limiting momentum for the quasi-particles $\epsilon(p_0) = \mu$.

Using an expansion in powers of the interaction constant, Hugenholtz and van Hove[9] (see also reference 7) showed that p_0 coincides with the limiting Fermi momentum for an ideal gas:

$$p_0 = (3\pi^2 N)^{1/3} \quad (\hbar = 1), \tag{7}$$

where N is the number of particles in the system, whose volume is taken to be unity. This theorem may prove to be incorrect under conditions in which perturbation theory is inapplicable. For instance, for systems displaying superconductivity, Eq. (6) has in general no real solution*.[10] In our work it has been assumed that perturbation theory is applicable. We also assume that the forces acting between particles are paired.

*In this connection we note that it follows from Landau's results, obtained without the aid of perturbation theory (see footnote 3 of reference 4), that if (6) has a real root, it coincides with (7).

Let us now undertake the quantum-mechanical determination of $\delta E/\delta n(p)$ (we assume that $T = 0$), starting from the assumptions given above. In doing so we shall use, in a definite manner, the idea used by Klein and Prange[7] for the proof of the cited theorem of Hugenholtz and van Hove.

Let us consider any diagram which makes a contribution to the total energy of the system, for example one of those in the figure. The dotted lines correspond to the expression

$$iu(k) = i\int U(x) \exp(ikx)\, d^4x, \quad U(x) = v(\mathbf{x})\,\delta(x_0),$$

where $v(\mathbf{x})$ is the potential of the interaction between the particles, the vertex corresponds to a delta function expressing the law of conservation of the 4-momentum, and the full line corresponds to the Green's function of the non-interacting particles

$$G_0(\mathbf{p}, \varepsilon) = (\varepsilon_p^0 - \varepsilon - i\delta)^{-1},$$

$$\delta \to +0 \text{ for } p > p_0; \quad \delta \to -0 \text{ for } p < p_0. \tag{8}$$

To obtain the energy of the system as a functional of the occupation numbers of the quasi-particles, we express (8) in the form

$$G(p, \varepsilon) = P\frac{1}{\varepsilon_p^0 - \varepsilon} + i\pi(1 - 2n(p))\,\delta(\varepsilon_p^0 - \varepsilon), \tag{9}$$

where $n(p)$ are the occupation numbers of the non-interacting particles at $T = 0$

$$n(p) = \begin{cases} 1 & (p < p_0) \\ 0 & (p > p_0) \end{cases}. \tag{10}$$

In what follows it is important that the occupation numbers of the non-interacting particles coincide with the occupation numbers of the quasi-particles in the ground state and in states differing infinitesimally from the ground state.

The expression for the total energy will have the form

$$E = E_0 - S(\Omega), \tag{11}$$

where E_0 is the energy of the non-interacting fermions, which can be expressed in the form

$$E_0 = \int \varepsilon_p^0 n(p)\, d^3p; \quad d^3p = dp_x\, dp_y\, dp_z / (2\pi)^3, \tag{12}$$

and $S(\Omega)$ is the sum of the contributions of all the corrected diagrams. The quantity Ω corresponding to an n-th order diagram has the following form:

$$\Omega = \alpha(-1)^l(-i)^n \int d^4k_1 \ldots \int d^4k_n \int d^4q_1 \ldots \int d^4q_{2n} \prod_{j=1}^{2n-1} \delta_j$$
$$\times\, u(k_1) \ldots u(k_n)\, G_0(q_1) \ldots G_0(q_{2n}), \tag{13}$$

where $d^4q = dq_x\, dq_y\, dq_z\, dq_0 / (2\pi)^4$, l is the number of closed loops, δ_j is the delta function located at one of the vertices, and the factor α can take on one of the two values 1 or $\frac{1}{2}$ (see below).

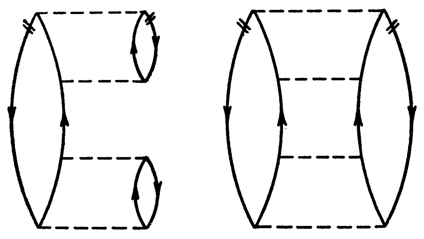
Each of the Green's functions $G_0(q) = G_0(\mathbf{q}, q_0)$ in (13) depends on the occupation numbers $n(\mathbf{q})$. Hence Ω is a functional of $n(\mathbf{q})$. Upon evaluating $\delta\Omega/\delta n(\mathbf{q})$ we obtain 2n terms, some one of the functions $G_0(q_i)$ being varied in each term. In the remaining quantities, $n(\mathbf{q})$ is equal to its value at $T = 0$ [see Eq. 10)]. Thus, each of the 2n components has the form

$$-i\frac{\delta}{\delta n(\mathbf{p})}\int G_0(\mathbf{q}, q_0)\,\Sigma_R^{(i)}(\mathbf{q}, q_0)\, d^4q, \quad i = 1, \ldots, 2n, \tag{14}$$

where in performing the variation it is only the function G_0 which need be assumed to depend on $n(\mathbf{q})$. Substituting into this the expression for G_0 from (9) and performing the variation, we obtain instead of (14),

$$-\int \Sigma_R^{(i)}(\mathbf{p}, \varepsilon)\,\delta(\varepsilon - \varepsilon_p^0)\, d\varepsilon = -\Sigma_R^{(i)}(\mathbf{p}, \varepsilon_p^0). \tag{15}$$

The function $\Sigma_R^{(i)}(\mathbf{p}, \epsilon)$, from its definition contained in (14), is represented by a diagram obtained by eliminating one of the full lines in the total energy diagram (see the figure). Consequently $\Sigma_R^{(i)}$ corresponds to one of the diagrams of particle self energy (generally speaking, a non-compact one).



a b

We now observe that in some of the total-energy diagrams it is possible to obtain one and the same self-energy diagram by eliminating either of two different lines (see b in the figure). This effect is compensated by the factor α in (13), which, as Klein and Prange have shown[7] is equal to $\frac{1}{2}$ for such diagrams, and to unity in the remaining cases. Finally, it can be stated that variation of $S(\Omega)$ yields the totality of all possible self-energy diagrams for the particles, each diagram being counted only once, i.e.,

$$\delta S(\Omega) / \delta n_p = -\Sigma_R(p, \varepsilon_p^0), \tag{16}$$

where the index R indicates that non-compact as well as compact diagrams enter into (16). In view of (11), (12), and (16) we obtain

$$\delta E/\delta n(p) = \varepsilon_p^0 - \Sigma_R(p, \varepsilon_p^0). \tag{17}$$

For Σ_R we may write the expansion

$$\Sigma_R(p, \varepsilon) = \Sigma + \Sigma G_0 \Sigma + \Sigma G_0 \Sigma G_0 \Sigma + \ldots, \tag{18}$$

where $\Sigma(p, \epsilon)$ is the compact portion of the self energy of the particle. The right-hand side of (18) has a singularity at $\epsilon = \epsilon_p^0$. Nevertheless, the quantity $\Sigma_R(p, \epsilon_p^0)$ can be assigned a definite value by following the method described in the appendix to Klein and Prange's paper.[7]

Let us first put $p = p_0$. In this case the imaginary part of the self energy (and consequently the attenuation) is equal to zero. It follows from (15) that when $\epsilon = \epsilon_{p_0}^0$, $\Sigma_R(p_0, \epsilon)$ must be taken to mean*

$$\lim_{\varepsilon \to \varepsilon_{p_0}^0} \Sigma_R(p_0, \varepsilon) = \int \Sigma_R(p_0, \varepsilon)\, \delta(\varepsilon - \varepsilon_{p_0}^0)\, d\varepsilon. \tag{19}$$

As shown in the appendix to reference 7, it follows from (18) and (19) that

$$\Sigma_R(p_0, \varepsilon_{p_0}^0) = \Sigma(p_0, \varepsilon_{p_0}), \tag{20}$$

where ϵ_{p_0} is the energy of a quasi-particle with a momentum equal to the limiting momentum p_0. Consequently,

$$\delta E / \delta n\, (p_0) = \varepsilon_{p_0}, \tag{21}$$

where ϵ_{p_0} satisfies Eq. (4), q.e.d.

Turning to the case $p \neq p_0$, we note that here the attenuation is not equal to zero. The concept of quasi-particle energy is meaningful only when the attenuation, and therefore also the imaginary part of the self energy, are sufficiently small and can be neglected. According to our assumption of the continuity of $\mathrm{Im}\, \Sigma$ at $\epsilon \sim \mu$, this will be the case if p is sufficiently close to p_0. Neglecting the imaginary part of Σ for these values of p, and repeating the considerations which led to (20) and (21), we obtain

$$\delta E / \delta n\, (p) \approx \varepsilon_p,$$

where ϵ_p satisfies Eq. (4), and p is sufficiently close to p_0. When $p = p_0$ this equation becomes exact.

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*In view of the singularity of $\Sigma_R(p_0, \varepsilon)$ at $\varepsilon = \varepsilon_{p_0}^0$, formula (19) is by no means a trivial consequence of the definition of the δ-function.

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