AN ASYMPTOTICALLY EXACT SOLUTION FOR THE MODEL HAMILTONIAN OF THE THEORY OF SUPERCONDUCTIVITY

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Submitted to JETP editor February 13, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) 39, 120-129 (July, 1960)

It is shown that it is possible to satisfy with asymptotic exactness the entire chain of equations for Green's functions constructed on the basis of the model Hamiltonian of Bardeen, Cooper, and Schrieffer. Thus the asymptotic exactness of the known solution for the superconducting state is proved without the use of perturbation theory. It is shown that the trivial solution that corresponds to the normal state must be rejected at temperatures below the critical temperature.

INTRODUCTION

In a previous paper[1] it has been shown on the basis of the model Hamiltonian of Bardeen, Cooper, and Schrieffer (BCS)[2] that the thermodynamic functions of a superconducting system, which were obtained by a variation method in BCS, are asymptotically exact for $V \to \infty$, $N/V = \text{const}$ (V is the volume of the system, and N the number of particles). This conclusion was based on the fact that each term of the perturbation-theory series, by means of which the correction to that solution is calculated, is asymptotically small for $V \to \infty$. Certain objections can, however, be raised against such a proof. In fact, if in the model Hamiltonian of BCS one takes as the zeroth approximation the Hamiltonian of noninteracting particles, then each term of the thermodynamic perturbation theory will also be asymptotically small, whereas it is well known that the existing theory of superconductivity is based on the inclusion of precisely these terms. On the basis of a partial summation of the diagrams (ladder approximation), Prange[3] has even expressed a doubt that a solution of the superconducting type with a gap in the spectrum of elementary excitations exists at all for the model Hamiltonian of BCS.

In the present paper we show how to obtain the same asymptotically exact solution as in references 2 and 1 without resorting to perturbation theory. Besides this it will be shown that for temperatures Θ below the phase-transition temperature Θ_0 the solution that appears for $V \to \infty$ and corresponds to the nonsuperconducting state (the trivial solution) must be rejected as failing to satisfy the necessary conditions for the exact Green's functions. Arguments about the absence of the trivial solution for $\Theta < \Theta_0$ have already been put forward by Wentzel[4] and by Thouless.[5] Thouless based his argument on an examination of a special model that admits of exact solution.

2. THE MODEL AND APPROXIMATING HAMILTONIANS IN THE THEORY OF SUPERCONDUCTIVITY

The model Hamiltonian of BCS is of the form

$$H = \sum_f T_f a_f^+ a_f - \frac{1}{2V}\sum_{ff'} J(f, f')\, a_f^+ a_{-f}^+ a_{-f'} a_{f'} + \nu\mathfrak{A},$$
$$\mathfrak{A} = \frac{1}{2}\sum_f w_f (a_{-f} a_f + a_f^+ a_{-f}^+), \quad \nu \geqslant 0, \tag{2.1}$$

where $f = (\mathbf{k}, \sigma)$, $-f = (-\mathbf{k}, -\sigma)$; σ is the spin index, which takes the values $+\frac{1}{2}$ and $-\frac{1}{2}$, and $\mathbf{k}$ is the momentum; $T_f = k^2/2m - \mu$, and μ is the chemical potential; and a_f and a_f^+ are operators that satisfy the commutation relations of Fermi statistics. $J(f, f')$ and w_f are real functions which have the properties

$$J(f, f') = J(f', f) = -J(-f, f'), \quad w_{-f} = -w_f. \tag{2.2}$$

For example,

$$J(f, f') = \frac{1}{2} J(\mathrm{k}, \mathrm{k}')\,[\delta(\sigma - \sigma') - \delta(\sigma + \sigma')], \quad J(\mathrm{k}, \mathrm{k}')$$
$$= J(\mathrm{k}', \mathrm{k}) = J(-\mathrm{k}, \mathrm{k}') \tag{2.3}$$

[$\delta(\sigma - \sigma')$ is the Kronecker symbol].

In the Hamiltonian (2.1) we have introduced a supplementary operator $\nu\mathfrak{A}$, which will play an auxiliary part in the selection of the required solutions (see later argument, Sec. 4). In the final results we shall set $\nu = 0$.

Following reference 1, we introduce certain real functions A_f and write the Hamiltonian (1) in the form

$$H = H_0 + H_1, \tag{2.4}$$

$$H_0 = \sum_f H_f = \sum_f \{\tfrac{1}{2} A_f (\nu w_f - C_f) + T_f a_f^+ a_f + \tfrac{1}{2} C_f (a_f^+ a_{-f}^+ + a_{-f} a_f)\}, \tag{2.5}$$

$$H_1 = -\frac{1}{2V} \sum_{ff'} J(f, f') B_f^+ B_{f'}, \tag{2.6}$$

$$B_f = a_{-f} a_f - A_f, \tag{2.7}$$

$$C_f = \nu w_f - \frac{1}{V} \sum_f J(f, f') A_{f'}. \tag{2.8}$$

We now set

$$A_f = \langle a_{-f} a_f \rangle_0 = \langle a_f^+ a_{-f}^+ \rangle_0, \tag{2.9}$$

where the symbol $\langle \ldots \rangle_0$ means that the averaging is taken over a grand ensemble with the Hamiltonian H_0. (We note that the operator H_0 does not satisfy the condition that the total number of particles $N = \Sigma_f a_f^+ a_f$ is conserved, and therefore in general $A_f \neq 0$). Then

$$\langle B_f \rangle_0 = 0 \tag{2.10}$$

and it can be shown (cf. reference 1) that the operator H_1, treated as a perturbation, gives an asymptotically small contribution for $V \to \infty$ (N/V = const) in each term of the thermodynamic perturbation theory. Neglecting the term H_1 in Eq. (2.4) for this reason, we get the following expression for the thermodynamic potential:

$$\Omega = -\Theta \ln \mathrm{Sp}\, \{e^{-H_0/\Theta}\}. \tag{2.11}$$

The Hamiltonian H_0, which we shall hereafter call the approximating Hamiltonian, is a quadratic form in the operators a_f, a_f^+, and can be diagonalized by means of a linear transformation (cf. reference 6):

$$a_f = u_f \alpha_f + v_f \alpha_{-f}^+, \quad u_f^2 + v_f^2 = 1,$$

$$u_f = u_{-f}, \quad v_f = -v_{-f}. \tag{2.12}$$

We get as the result

$$H_0 = U + \sum_f E_f \alpha_f^+ \alpha_f, \tag{2.13}$$

$$\Omega = U - \Theta \sum_f \ln (1 + e^{-E_f/\Theta}), \tag{2.14}$$

$$U = \tfrac{1}{2} \sum_f \{A_f (\nu w_f - C_f) + T_f - E_f\}. \tag{2.15}$$

The energy of an elementary excitation is given by

$$E_f = \sqrt{T_f^2 + C_f^2}. \tag{2.16}$$

The parameters of the canonical transformation (2.12) are connected with C_f and E_f by the relations

$$u_f^2 - v_f^2 = T_f / E_f, \qquad u_f v_f = -C_f / 2E_f. \tag{2.17}$$

Carrying out the averaging over the grand ensemble with the Hamiltonian (2.13) explicitly, we get the following expressions for the quantities A_f and the mean occupation numbers $\bar{n}_f$:

$$A_f = \langle a_{-f} a_f \rangle_0 = -(C_f / 2E_f) \tanh (E_f / 2\Theta), \tag{2.18}$$

$$\bar{n}_f = \langle a_f^+ a_f \rangle_0 = \tfrac{1}{2} \{1 - (T_f / E_f) \tanh (E_f / 2\Theta)\}. \tag{2.19}$$

Substituting Eq. (2.18) in Eq. (2.8), we get the equation for C_f:

$$C_f = \nu w_f + \frac{1}{2V} \sum_{f'} J(f, f') \tanh (E_{f'} / 2\Theta) C_{f'} / E_{f'}. \tag{2.20}$$

These same results can be obtained starting from the equations for the two-time (retarded or advanced) Green's functions constructed for the approximating Hamiltonian H_0:

$$i dG_f (t - t') / dt = \delta (t - t') + T_f G_f (t - t') + C_f \Gamma_f (t - t'),$$

$$-i d\Gamma_f (t - t') / dt = T_f \Gamma_f (t - t') - C_f G_f (t - t'), \tag{2.21}$$

where, for example, the retarded functions are given by (cf. references 7, 8)

$$G_f (t - t') = \langle\langle a_f (t); a_f^+ (t') \rangle\rangle_0^{adv} = -i\theta (t - t') \langle [a_f (t); a_f^+ (t')]_+ \rangle_0,$$

$$\Gamma_f (t - t') = \langle\langle a_{-f}^+ (t); a_f^+ (t') \rangle\rangle_0^{adv} = -i\theta (t - t') \langle [a_{-f}^+ (t); a_f^+ (t')]_+ \rangle_0 \tag{2.22}$$

and $[\ldots]_+$ is the anticommutator. The system of equations (2.21) can be solved by means of spectral representations, which supplement it with the necessary boundary conditions. (Analogous equations for the causal Green's functions have been considered by Gor'kov.[9]) The system (2.21) is closed and reduces to (2.20). For $\Theta < \Theta_0$, where Θ_0 is the temperature of the transition from the superconducting to the normal state, this equation has two solutions, one of which, $C_f = 0$, is trivial. The nontrivial solution goes to zero for $\Theta = \Theta_0$. If we adopt for $\Theta < \Theta_0$ the nontrivial solution, which corresponds to the lowest value of the thermodynamic potential (2.14), the thermodynamic functions will give a good description of the properties of superconducting systems.

3. THE ASYMPTOTICALLY EXACT SOLUTION OF THE CHAIN OF EQUATIONS FOR THE GREEN'S FUNCTIONS

We shall now prove, without resorting to perturbation theory, that Eq. (2.14) is indeed asymptotically exact for $V \rightarrow \infty$ (N/V = const) as the expression for the thermodynamic potential Ω of the system with the model Hamiltonian (2.1), and (in Sec. 4) that for $\Theta < \Theta_0$ and $\nu = 0$ the trivial solution $C_f = 0$ of Eq. (2.20) must be rejected.

Starting from the Hamiltonian (2.1), we get for the operator $a_f(t)$ in the Heisenberg representation the equation

$$i\frac{da_f}{dt} = T_f a_f - \frac{1}{V}\sum_{f'} J(f, f') a^+_{-f} a_{-f'} a_{f'} + \nu w_f a^+_{-f}. \qquad (3.1)$$

Let us consider Green's functions of the form

$$G_{f,\mathfrak{M}}(t-t') = \langle\langle a_f(t)\mathfrak{M}(t);\, a^+_f(t')\rangle\rangle, \qquad (3.2)$$

$$\Gamma_{f,\mathfrak{N}}(t-t') = \langle\langle a^+_{-f}(t)\mathfrak{N}(t);\, a^+_f(t')\rangle\rangle, \qquad (3.3)$$

where the operators $\mathfrak{M}$ and $\mathfrak{N}$ are products of pairs of operators $a^+_q a_q$, $a^+_g a^+_{-g}$, and $a_{-h}a_h$, for example,

$$\mathfrak{M} = a^+_{q_1} a_{q_1} \ldots a^+_{q_k} a_{q_k} \ldots a^+_{g_i} a^+_{-g_i} \ldots a_{-h_j} a_{h_j} \ldots, \qquad (3.4)$$

and where all the indices f, $q_1, \ldots, q_k, \ldots$, $g_i, \ldots, h_j, \ldots$ are different. We note that for the Hamiltonian (3.1) the operator for the total number of particles, $N = \Sigma a^+_f a_f$, is not an integral of the motion, and therefore the Green's functions (3.3) are different from zero.

Using Eq. (3.1), we get for the functions $G_{f,\mathfrak{M}}$ and $\Gamma_{f,\mathfrak{N}}$ the equations

$$idG_{f,\mathfrak{M}}/dt = \delta(t-t')\langle[a_f\mathfrak{M},\, a^+_f]_+\rangle + T_f G_{f,\mathfrak{M}} - \frac{1}{V}\sum_{f'} J(f,f')\langle\langle a^+_{-f} a_{-f'} a_{f'}\mathfrak{M}; a^+_f(t')\rangle\rangle + \nu w_f \Gamma_{f,\mathfrak{M}} + \langle\langle a_f\, i\frac{d\mathfrak{M}}{dt};\, a^+_f(t')\rangle\rangle, \qquad (3.5)$$

$$id\Gamma_{f,\mathfrak{N}}/dt = \delta(t-t')\langle[a^+_{-f}\mathfrak{N},\, a^+_f]_+\rangle - T_f\Gamma_{f\mathfrak{N}} - \frac{1}{V}\sum_{f'} J(f, f')\langle\langle a^+_{f'} a^+_{-f'} a_f\mathfrak{N};\, a^+_f(t')\rangle\rangle + \nu w_f G_{f\mathfrak{N}} + \langle\langle a^+_{-f}\, i\frac{d\mathfrak{N}}{dt};\, a^+_f(t')\rangle\rangle \qquad (3.6)$$

(for brevity we omit the argument t in the operators).

These equations are an infinite system of coupled equations. We shall assume that the summation in (3.5) and (3.6) is taken over those values of f' that are equal neither to $\pm f$ nor to any of the indices of the operators involved in $\mathfrak{M}$ and $\mathfrak{N}$. We are then making in each equation an error of the order $1/V$, which is admissible. But then the Green's functions under the summation signs in (3.5) and (3.6) will belong to the same class as the functions $G_{f,\mathfrak{M}}$ and $\Gamma_{f,\mathfrak{N}}$, and consequently the chain of equations for these functions will be a closed one.

We shall now show that we can satisfy the entire chain (3.5) and (3.6) to within terms of order $1/V$ if we carry out the averaging in (3.2) and (3.3) not with the model Hamiltonian (2.1), but with the approximating Hamiltonian (2.5), i.e., if we set

$$G_{f,\mathfrak{M}} = \langle\mathfrak{M}\rangle_0 G_f, \qquad G_f = \langle\langle a_f(t)\, a^+_f(t')\rangle\rangle_0; \qquad (3.7)$$

$$\Gamma_{f,\mathfrak{N}} = \langle\mathfrak{N}\rangle_0 \Gamma_f, \qquad \Gamma_f = \langle\langle a^+_{-f}(t)\, a^+_f(t')\rangle\rangle_0, \qquad (3.8)$$

where, for example,

$$\langle\mathfrak{M}\rangle_0 = \bar{n}_{q_1}\ldots\bar{n}_{q_k}\ldots A^*_{g_i}\ldots A_{h_j}\ldots \qquad (3.9)$$

and $\bar{n}_q$ and A_g are defined by the relations (2.18) and (2.19). Then the entire chain of equations reduces to the pair of equations (2.21) for G_f and Γ_f.

In fact, substituting (3.7) and (3.8) in (3.5) and (3.6), we get

$$\langle\mathfrak{M}\rangle_0\, idG_f/dt = \langle\mathfrak{M}\rangle_0\,\delta(t-t') + T_f\langle\mathfrak{M}\rangle_0 G_f - \frac{1}{V}\sum_{f'} J(f, f')\langle a_{-f'} a_{f'}\mathfrak{M}\rangle_0\Gamma_f + \nu w_f\langle\mathfrak{M}\rangle_0\Gamma_f + i\langle d\mathfrak{M}/dt\rangle_0 G_f, \qquad (3.10)$$

$$\langle\mathfrak{N}\rangle_0\, id\Gamma_f/dt = -T_f\langle\mathfrak{N}\rangle_0\Gamma_f - \frac{1}{V}\sum_{f'} J(f, f')\langle a^+_{f'} a^+_{-f'}\mathfrak{N}\rangle_0 G_f + \nu w_f\langle\mathfrak{N}\rangle_0 G_f + \langle id\mathfrak{N}/dt\rangle_0\Gamma_f. \qquad (3.11)$$

We now use the fact that

$$\langle a_{-f'} a_{f'}\mathfrak{M}\rangle_0 = A_{f'}\langle\mathfrak{M}\rangle_0, \qquad \langle a^+_{f'} a^+_{-f'}\mathfrak{N}\rangle_0 = A_{f'}\langle\mathfrak{N}\rangle_0.$$

Then, according to (3.1) and (2.18)–(2.20), we have apart from terms of order $1/V$

$$\langle ida^+_q a_q/dt\rangle_0 = \frac{1}{V}\sum_{q'} J(q, q')\langle a^+_q a^+_{-q} a_{-q'} a_{q'} - a^+_{q'} a^+_{-q'} a_{-q} a_q\rangle + \nu w_f\langle a^+_q a^+_{-q} - a_{-q} a_q\rangle_0 = 0,$$

$$-\langle ida^+_g a^+_{-g}/dt\rangle_0 = 2T_g A_g - \frac{1}{V}\sum_{g'} J(g, g')(1-2\bar{n}_g)A_{g'} + \nu w_g(1-2\bar{n}_g) = 0$$

and consequently

$$\langle id\mathfrak{M}/dt\rangle_0 = 0, \qquad \langle id\mathfrak{N}/dt\rangle_0 = 0. \qquad (3.12)$$

Thereupon (3.10) and (3.11) reduce to the system (2.21).

Thus we see that (3.7) and (3.8) are an asymptotically exact solution (for $V \rightarrow \infty$, N/V = const) of the entire chain of equations for the Green's functions. We have convinced ourselves [see Eqs. (3.7)–(3.9)] that all the Green's functions for the model Hamiltonian (2.1) and the approximating Hamiltonian (2.5) coincide, and consequently these Hamiltonians are equivalent in an asymptotic sense.

We have here considered Green's functions (3.2) and (3.3) which, in general, do not satisfy the condition of conservation of the number of particles (the number of creation operators is not equal to the number of annihilation operators), and for $\nu = 0$ [cf. Eq. 2.1)] averages of the type $\langle a_{-f} a_f \rangle$, where the averaging is over a grand ensemble with the Hamiltonian H, are equal to zero. We can, however, obtain all of the results of the theory of superconductivity by starting with a chain of equations for Green's functions in which the numbers of creation and annihilation operators are equal to each other.

Let us consider Green's functions of the forms

$$G_{f,\mathfrak{M}}(t-t') = \langle\langle a_f(t)\,\mathfrak{M}(t);\ a_f^+(t')\rangle\rangle, \quad (3.13)$$

$$\Gamma_{f,f',\mathfrak{N}}(t-t') = \langle\langle a_{-f}^+(t)\,a_{-f'}(t)\,a_{f'}(t)\,\mathfrak{N}(t);\ a_f^+(t')\rangle\rangle, \quad (3.14)$$

where the operators $\mathfrak{M}$ and $\mathfrak{N}$ are made up of products of operators of the form $a_q^+ a_q a_g^+ a_{-g}^+ a_{-h} a_h$:

$$\mathfrak{M} = \ldots a_q^+ a_q \ldots a_g^+ a_{-g}^+ a_{-h} a_h \ldots \quad (3.15)$$

Assuming that all the indices of the operators occurring in the Green's functions (3.13) and (3.14) are different, we have the following closed chain of equations:

$$idG_{f,\mathfrak{M}}/dt = \delta(t-t')\langle\mathfrak{M}\rangle + T_f G_{f,\mathfrak{M}} - \frac{1}{V}\sum_{f''} J(f, f'')\langle\langle a_{-f}^+ a_{-f''} a_{f''}\mathfrak{M};\ a_f^+(t')\rangle\rangle + \langle\langle a_f\, i\frac{d\mathfrak{M}}{dt};\ a_f^+(t')\rangle\rangle,$$

$$id\Gamma_{f,f',\mathfrak{N}}/dt = -T_f\Gamma_{f,f',\mathfrak{N}} - \frac{1}{V}\sum_{f''} J(f, f'')\langle\langle a_f a_{f''}^+ a_{-f''}^+ a_{-f'} a_{f'}\mathfrak{N};\ a_f^+(t')\rangle\rangle + \langle\langle a_{-f}^+\, i\frac{da_{-f'}a_{f'}}{dt}\mathfrak{N};\ a_f^+(t')\rangle\rangle + \langle\langle a_{-f}^+ a_{-f'} a_{f'}\, i\frac{d\mathfrak{N}}{dt};\ a_f^+(t')\rangle\rangle \quad (3.16)$$

(the summation index f′ is not equal to any of the other indices).

It is easy to verify that the system (3.16) can be satisfied, apart from terms of order 1/V, if we perform the averaging in (3.13) and (3.14) with respect to the approximating Hamiltonian, i.e., if we set

$$G_{f,\mathfrak{M}} = \langle\mathfrak{M}\rangle_0 G_f, \quad \Gamma_{f,f',\mathfrak{N}} = \langle\mathfrak{N}\rangle_0 A_{f'}\Gamma_f. \quad (3.17)$$

The system (3.16) then again reduces to the system of equations (2.21).

By repeating these same arguments one can show that the chain for the many-time causal Green's functions of the form

$$G(t_1, \ldots, t_n) = \langle T a_{f'_1}(t_1)\ldots a_{f_l}(t_l)\, a_{f_{l+1}}^+(t_{l+1})\ldots a_{f_n}^+(t_n)\rangle, \quad (3.18)$$

where T is the chronological-ordering operator and the averaging is taken with respect to the model Hamiltonian (2.1) ($\nu \neq 0$ or $\nu = 0$), can also be solved with asymptotic exactness.

Let us now examine the solutions of (2.20). We set

$$C_f = C\varphi_f = (-1)^{\sigma+1/2} C\varphi_k, \quad \varphi_{k_0} = 1,$$
$$w_f = J_f = \tfrac{1}{2}(-1)^{\sigma+1/2} J_k, \quad (3.19)$$

where k_0 is the magnitude of the momentum at the Fermi surface, C is the size of the gap at the Fermi surface, $J_k = J_{k,k_0}$, and $J_{k,k'}$ is the kernel $J(\mathbf{k}, \mathbf{k}')$ averaged over the angular variables. Setting $k = k_0$ in (2.20) and using (3.19), we have

$$C = \nu J + \frac{C}{2V}\sum_f J_f \tanh\frac{E_f}{2\Theta}\frac{\varphi_f}{E_f}, \quad (3.20)$$

where $J = \frac{1}{2} J_{k_0}$. Multiplying (3.20) by J_f/J and subtracting the result from (2.20), we get an equation for φ_f:

$$\varphi_f = \frac{J_f}{J} + \frac{1}{2V}\sum_{f'}\left\{J(f, f') - \frac{J_f J_{f'}}{J}\right\}\tanh\frac{E_{f'}}{2\Theta}\frac{\varphi_{f'}}{E_{f'}}. \quad (3.21)$$

Unlike the corresponding expression in (2.20) the integrand in (3.21) already has no singularities as $C \to 0$ and $\Theta \to 0$.

For sufficiently small interaction parameter (cf. references 1 and 6), $C \sim \Theta \sim e^{-1/\rho}$ are small quantities ($\rho \ll 1$), and we can set $C = 0$ and $\Theta = 0$ in (3.21). As the result we then get a linear inhomogeneous equation for φ_f:[10]

$$\varphi_f = \frac{J_f}{J} + \frac{1}{2V}\sum_{f'}\left\{J(f, f') - \frac{J_f J_{f'}}{J}\right\}\frac{\varphi_{f'}}{|T_{f'}|}, \quad (3.22)$$

from which, confining ourselves to the first step of iteration, we have

$$\varphi_f \approx J_f / J. \quad (3.23)$$

Substituting (3.23) in (3.20), we bring the equation for the gap C into the following form:

$$C = \nu J + \frac{CJ}{2V}\sum_f \frac{J_f^2}{J^2}\frac{\tanh(E_f/2\Theta)}{E_f}. \quad (3.24)$$

We note that in the case of a factorizable kernel

$$J(f, f') = J_f J_{f'}/J, \quad J_{-f} = -J_f, \quad (3.25)$$

the function (3.23) is an exact solution of (3.21) as well as (3.22). Consequently (3.24) will in this case be valid also for strong interaction. For definiteness we shall hereafter assume that the interaction is factorizable.

Equation (3.24) has two solutions. One of them goes over for $\nu \to 0$ into the well known solution of (2.20) (cf. references 1 and 2), which gives a gap in the spectrum of the elementary excitations (for small interaction ρ we have $C \sim e^{-1/\rho}$), and the other goes to zero for $\nu \to 0$ [the "trivial" solution of Eq. (2.20)].

Let us find the solution that goes to zero as $\nu \to 0$. In doing so we confine ourselves to the case of zero temperature, $\Theta = 0$. The extension to the case of nonzero temperature presents no difficulties (cf. reference 10), and does not change the character of the solution.

Let us set $\Theta = 0$ in (3.24) and go over from the summation over $f = (\mathbf{k}, \sigma)$ to integration over the variable $\xi = k^2/2m - \mu$ ($\xi \equiv T_f$), assuming that the Fermi energy is much larger than the mean Debye frequency ω ($\omega/\mu \ll 1$) at which the kernel in (3.20) changes rapidly. We get

$$C = \nu J + \rho C \int_0^\infty d\xi \varphi^2(\xi) / \sqrt{\xi^2 + C^2\varphi^2(\xi)}, \qquad (3.26)$$

where [cf. Eq. (2.3)].

$$\rho = mk_0 J/\pi^2, \qquad \varphi^2(\xi) = (J_f/J)^2.$$

For small C (we are looking for the solution $C \to 0$ as $\nu \to 0$) we can set $\varphi(\xi) \approx \varphi(0) = 1$ in the radicand in (3.26). Integrating by parts and neglecting the small quantity C/ω where possible, we get

$$C = \nu J - C\rho \ln (C/\tilde{\omega}), \qquad (3.27)$$

$$\tilde{\omega} = \omega \exp\left\{-\int_0^\infty \ln \frac{2\xi}{\omega} \frac{d\varphi^2(\xi)}{d\xi} d\xi\right\}. \qquad (3.28)$$

Solving the equation (3.27), we find

$$C \approx \nu J / \rho \ln \frac{\nu J}{\rho\tilde{\omega}} \to 0 \text{ for } \nu \to 0 \ (\nu > 0). \qquad (3.29)$$

For weak interaction ($\rho \ll 1$), C/ω is small, and on setting $\nu \to 0$ in the same equation (3.27), we get the well known expression for the gap in the spectrum of elementary excitations of superconductors:

$$C = \tilde{\omega} e^{-1/\rho}. \qquad (3.30)$$

4. THE UNACCEPTABILITY OF THE TRIVIAL SOLUTION AT TEMPERATURES BELOW THE CRITICAL TEMPERATURE

We shall now show that the "trivial" solution (3.29) must be rejected. For this purpose we use the following exact property of the average of the operator $\mathfrak{A}$ that appears in the Hamiltonian (2.1):

$$\frac{d}{d\nu} \langle \mathfrak{A} \rangle \leqslant 0. \qquad (4.1)$$

This property can be established in the following way. Regarding the operator $\Delta\nu\mathfrak{A}$ as a small perturbation, we have for the increment of the average value (cf. reference 8)

$$\Delta \langle \mathfrak{A} \rangle = (\Delta\nu / 2\pi) \langle\langle \mathfrak{A} | \mathfrak{A} \rangle\rangle_{E=0}, \qquad (4.2)$$

where $\langle\langle \mathfrak{A} | \mathfrak{A} \rangle\rangle_E$ is the Fourier component of the corresponding retarded Green's function $\langle\langle \mathfrak{A}(t); \mathfrak{A}(t') \rangle\rangle$. For $\Delta\nu \to 0$ we get from (4.2)

$$\frac{d}{d\nu} \left\langle \frac{\mathfrak{A}}{V} \right\rangle = \frac{1}{2\pi} \left\langle\left\langle \frac{\mathfrak{A}}{V} \middle| \mathfrak{A} \right\rangle\right\rangle_{E=0}. \qquad (4.3)$$

On the other hand,

$$\langle\langle \mathfrak{A} | \mathfrak{A} \rangle\rangle_{E=0} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} J(\omega) (e^{\omega/\Theta} - 1) \omega^{-1} d\omega, \qquad (4.4)$$

where $J(\omega)$ is the spectral intensity of the two-time correlation function $\langle \mathfrak{A}(t) \mathfrak{A}(t') \rangle$.[7] Since $\mathfrak{A}^+ = \mathfrak{A}$, we have $J(\omega) > 0$. The second factor in the integrand in (4.4) is also always positive, and from this there follows the property (4.1).

One comment must be made about the formula (4.3), namely that the averaging in its right member must be done not with the approximating Hamiltonian H_0, but with the complete Hamiltonian H. In fact, the quantity whose derivative is taken in the left member of (4.3) is given by

$$\langle V^{-1}\mathfrak{A} \rangle = \frac{1}{V} \sum_f J_f \langle a_{-f} a_f \rangle \qquad (w_f = J_f), \qquad (4.5)$$

which remains finite as $V \to \infty$. In the right member of (4.3), on the other hand, we have

$$\langle\langle V^{-1}\mathfrak{A} | \mathfrak{A} \rangle\rangle = \frac{1}{4V} \sum_{f, f'} J_f J_{f'} \langle\langle a_f^+ a_{-f}^+ + a_{-f} a_f \,|\, a_{f'}^+ a_{-f'}^+ + a_{-f'} a_{f'} \rangle\rangle. \qquad (4.6)$$

The summation in (4.6) is taken over two indices. Therefore in Eq. (4.6) the terms of order $1/V$ that are dropped in averaging over the ensemble with the Hamiltonian H_0 give a finite contribution and must be taken into account. If we were to do the averaging in the right member of (4.3) with the Hamiltonian H_0, we would get zero, whereas the left member, as we shall see, is different from zero. In the left member of (4.3) we obviously can do the averaging with the approximating Hamiltonian H_0.

Using the formula (2.18) (averaging with the Hamiltonian H_0) and the solution (3.23), we rewrite the expression (4.5) in the form

$$\langle V^{-1}\mathfrak{A} \rangle = -\gamma J = -\frac{CJ}{2V} \sum_f \varphi_f^2 \frac{\tanh(E_f/2\Theta)}{E_f}. \qquad (4.7)$$

According to Eq. (3.24),

$$C = (\nu + \gamma) J, \qquad (4.8)$$

and for the quantity γ we get the transcendental equation

$$\gamma = \frac{(\gamma + \nu) J}{2V} \sum_f \varphi_f^2 \frac{\tanh(E_f/2\Theta)}{E_f},$$

$$E_f = \sqrt{T_f^2 + (\nu + \gamma)^2 J_f^2}. \qquad (4.9)$$

The condition (4.1) can written in the form

$$d\gamma / d\nu \geqslant 0. \tag{4.10}$$

Calculating this derivative by using (4.9), we have

$$\frac{d\gamma}{d\nu} = \Big[\frac{J}{2V}\sum_f \varphi_f^2 \frac{T_f^2}{E_f^3} \tanh\frac{E_f}{2\Theta} + \frac{(\nu+\gamma)^2 J^3}{4\Theta V}\sum_f \varphi_f^4 E_f^{-2} \cosh^{-1}\frac{E_f}{2\Theta}\Big] \times \Big[1 - \frac{J}{2V}\sum_f \varphi_f^2 \frac{T_f^2}{E_f^3}\tanh\frac{E_f}{2\Theta} - \frac{(\nu+\gamma)^2 J^3}{4\Theta V}\sum_f \varphi_f^4 E_f^{-2}\cosh^{-1}\frac{E_f}{2\Theta}\Big]^{-1} \geqslant 0. \tag{4.11}$$

Let us now set $\nu \to 0$. The first factor in the expression (4.11) is always positive. For $\gamma \neq 0$ ($C \neq 0$, $\Theta < \Theta_0$) the second factor is also positive, since by using (4.9) we can put it in the form

$$\frac{JC^2}{2V}\sum_f \varphi_f^4 E_f^{-3} \frac{\sinh(E_f/\Theta) - E_f/\Theta}{2\cosh^2(E_f/2\Theta)} > 0.$$

For the trivial solution $\gamma = 0$ ($C = 0$) the second factor in the expression (4.11) is given by

$$1 - \frac{J}{2V}\sum_f \varphi_f^2 \frac{\tanh(|T_f|/2\Theta)}{|T_f|} = \frac{J}{2V}\sum_f \varphi_f^2 \frac{\tanh(|T_f|/2\Theta) - \tanh(|T_f|/2\Theta_0)}{|T_f|},$$

where the phase-transition temperature Θ_0 is determined from Eq. (4.9) under the condition $\gamma = 0$ or $C = 0$ (the condition that the gap in the spectrum (2.16) of the elementary excitations vanishes):

$$1 = \frac{J}{2V}\sum_f \varphi_f^2 \frac{\tanh(|T_f|/2\Theta_0)}{|T_f|}. \tag{4.12}$$

Consequently, in this case the second factor is positive for $\Theta > \Theta_0$ and negative for $\Theta < \Theta_0$; that is, for $\Theta = \Theta_0$ the trivial solution $C = 0$ ($\gamma = 0$) does not satisfy the relation (4.1), which is a direct consequence of the spectral representation, and therefore this solution must be rejected at temperatures below the critical temperature.

Thus the exact criterion (4.1), which is obtained without neglect of terms of order $1/V$, enables us to reject the superfluous solution $C = 0$ that was obtained as a result of the passage to the limit $V \to \infty$ ($N/V = \text{const}$).

In conclusion we formulate briefly the results we have obtained.

1) It has been shown that with asymptotic accuracy, dropping terms of order $1/V$ from the equations, one can satisfy the entire chain of equations for the Green's functions constructed on the basis of the BCS model Hamiltonian. In this connection we verify that one can use instead of the model Hamiltonian a simpler approximating Hamiltonian, which is a quadratic form in the operators.

2) It has been shown that at temperatures below the critical temperature the "trivial solution" does not satisfy the necessary conditions for the exact Green's functions and must be rejected.

3) Thus without resorting to perturbation theory we again confirm the asymptotic accuracy of the results obtained in references 1 and 2.

From a rigorous mathematical point of view, however, certain objections can be raised against the proofs presented here. It still has to be shown that when terms of order $1/V$ are dropped from the equations the solution also changes by an infinitesimal amount. One of the writers[11] has carried out all the necessary estimations by an entirely different method and has shown that the fractional discrepancies between the eigenvalues of the model Hamiltonian and the approximating Hamiltonian, and also between the corresponding Green's functions, will vanish in the limit $V \to \infty$ ($N/V = \text{const}$).

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Translated by W. H. Furry
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