CYCLOTRON RESONANCE IN GERMANIUM AND SILICON AND THE EFFECT OF NEGATIVE EFFECTIVE MASSES

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We consider cyclotron resonance in semiconductors with degenerate valence bands. We have determined the frequency spectrum (effective mass spectrum) for the case where the magnetic field is along the [001] axis. We have shown that there is a branch of negative cyclotron-resonance frequencies in germanium and silicon. By solving the transport equation we have obtained an expression for the absorbed power and we have considered the problem of negative absorption.

1. INTRODUCTION

DOUSMANIS and co-workers[1] have given in a recently published paper experimental data on negative effective masses as obtained by cyclotron resonance in germanium. Using a circularly polarized high-frequency field, these authors observed, with the sign of the magnetic field negative (corresponding to resonance on electrons), a new resonance localized in a frequency interval which was appreciably different from the known values of electron-resonance frequencies. (The magnetic field was in these experiments along the [001] axis; since the electronic energy spectrum in germanium is described by a set of ellipsoids of revolution situated along the [111] axes, only one frequency value can occur in these experiments.)

The new resonance occurs not as an absorption but as an emission resonance, in strong contradistinction from those observed earlier. Strictly speaking it is observed as a gap in the absorption background which occurs when $H = 0$.

Krõmer[2] has somewhat earlier drawn attention to the fact that the energy spectrum of the "heavy" holes in germanium and silicon allows a change in sign of the second derivatives of the energy with respect to the quasi-momentum, so that the components of the inverse-effective-mass tensor m_{ik}^{-1} can become negative in some region of phase space. He expressed the hypothesis that negative values of the components of m_{ik}^{-1} should manifest themselves in the occurrence of a negative resistivity under well defined circumstances. Guided by Krõmer's arguments, Dousmanis and co-workers, concluded that they observed resonance on negative effective masses, in the sense of the inverse-effective-mass tensor. From our further exposition it will become clear that this point of view is, strictly speaking, incorrect.

One must note, among the other effects observed in reference 1, a new clearly resolved resonance, which corresponds to a positive effective mass smaller than the ones usually ascribed to the "light" holes.

The notes by Kaus[3] and Mattis and Stevenson[4] appeared in connection with reference 1. The former considered the problem of a negative resistance and its connection with negative effective masses. The whole consideration neglected the magnetic field, and the problem of the cyclotron resonance was not considered at all. In the second note the authors superficially analyzed the problem of negative absorption during cyclotron resonance, introducing purely formally two kinds of particles, with positive and negative effective mass.

The new results obtained by Dousmanis and co-workers force us to give a careful analysis of the whole problem connected with hole cyclotron resonance in semiconductors with degenerate bands, such as germanium and silicon. This is the more necessary as all results mentioned here are essentially not included in the analysis given up to now (see, for instance, references 5 to 8). For the sake of simplicity, all considerations in this paper will be for the case where the magnetic field H is parallel to the [001] axis. It is easy in principle to generalize this to the case of an arbitrary direction of the magnetic field.

2. QUASI-PARTICLE HODOGRAPHS: FREQUENCY (EFFECTIVE-MASS) SPECTRUM

Under the conditions which are usually realized for cyclotron resonance experiments, the inequality $kT/\hbar\omega_0 \gg 1$ is satisfied, where ω_0 is the frequency of the electric field. Since most carriers are due to the external energy source, the average energy of the particles is as a rule larger than kT. The conditions for the quasi-classical approximation are thus known to be well satisfied. We shall take this into account and analyze the hodographs—the trajectories of the quasi-particles in a magnetic field in quasi-momentum space—for degenerate valence bands in germanium and silicon. The dispersion law can in this case, if referred to the cubic axes, be written in the form[5-8]

$$\varepsilon = \frac{1}{2m_0}\{Ap^2 \pm [B^2p^4 + C^2(p_x^2p_y^2 + p_y^2p_z^2 + p_x^2p_z^2)]^{1/2}\}, \quad (2.1)$$

where $\mathbf{p}$ is the quasi-momentum of the particles, m_0 the free electron mass, and A, B, and C constants (the plus sign refers to the "light" hole band; the minus sign to the "heavy" hole band).

Let the magnetic field be along the z axis. We change to a cylindrical system of coordinates and introduce the notation

$$x = \varepsilon[p_z^2/2m_0]^{-1}, \quad y^2 = p_\perp^2/p_z^2. \quad (2.2)$$

We have then instead of (2.1)

$$x = A(y^2+1) \pm [B^2(y^2+1)^2 + C^2(^1/_8 y^2(1-\cos 4\varphi) + y^2)]^{1/2}. \quad (2.1')$$

The quantities ϵ and p_z, and therefore also x, are clearly conserved in a magnetic field. It follows from (2.1′) that similarly shaped trajectories will correspond to a fixed value of x in the planes perpendicular to p_z. To find the explicit form of the hodographs, we solve (2.1′) for y^2. After some simple transformations we find

$$y^2 = \tfrac{1}{2}\gamma_1^{-1}(-\gamma_2 \pm \sqrt{\gamma_2^2 - 4\gamma_1\gamma_3}); \quad (2.3)$$

$$\gamma_1 = A^2 - B^2 - \tfrac{1}{8}C^2 + \tfrac{1}{8}C^2\cos 4\varphi, \quad (2.4)$$

$$\gamma_2 = 2(A^2 - B^2) - C^2 - 2Ax,$$

$$\gamma_3 = A^2 - B^2 + x^2 - 2Ax.$$

We shall analyze these solutions by using the experimental values of A, B, and C for Ge and Si. There are clearly no trajectories for small x. They begin at x_1, which is the solution of the equation $\gamma_2^2 = 4\gamma_3\gamma_1$ ($\varphi = \pi/4$), in the form of four symmetric points on the bisectors. When x increases further, the area of each of the four hodographs increases up to the point x_2, when the hodographs touch the p_x and p_y axes. The value of x_2 can be found from the equation $\gamma_2^2 = 4\gamma 3\gamma_1$ ($\varphi = 0$). In the range between x_1 and x_2 there are thus four hodographs which possess mirror symmetry, with respect to the planes through the bisector (type I hodographs); the cubic symmetry is realized only by the four together (see Fig. 1).

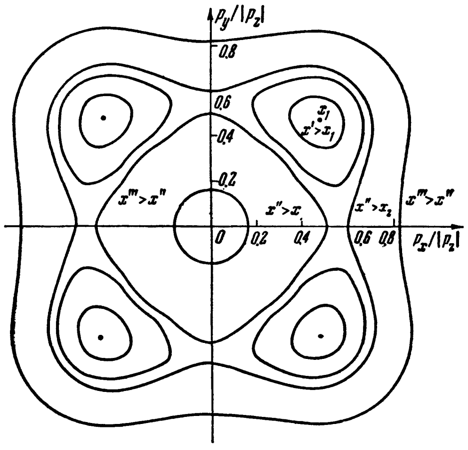

FIG. 1

When $x > x_2$ Eq. (2.3) has two solutions. We are led to two kinds of hodographs with cubic symmetry (see Fig. 1), where the area of one kind increases steadily with increasing x (type II hodographs) while the area of the second kind contracts to a point when $x = x_3$ (type III) for $x_3 = A - B$. When $x > x_3$, the double-valuedness ceases but appears again at $x = x_4 = A + B$, corresponding to hodographs of a new kind (type IV). These begin in the point y = 0 and increase steadily with increasing x. One can show that the first three hodographs refer to the "heavy" hole band and the last one to the "light" hole band. We note here a result, which is important for what follows, that when the dispersion law (2.1) holds along with hodographs which are the boundaries of an area that increases with increasing x (or ϵ for fixed p_z), there exist also hodographs with the opposite x dependence (type III).

We study now the frequency spectrum corresponding to all types of hodographs. If we consider the motion of quasi-particles in p-space for closed energy surfaces, we find easily the following expression for the frequency (see, for instance, reference 9 and also reference 10)

$$\frac{1}{\omega(\varepsilon, p_z)} = -\frac{1}{2\pi}\frac{c}{eH}\frac{\partial S(\varepsilon, p_z)}{\partial \varepsilon}, \quad (2.5)$$

where S is the area enclosed in p-space by a hodograph with a given value of ϵ and p_z. Using the notation (2.2), we get for the case under consideration

$$\frac{1}{\omega(x)} = -\frac{1}{2\pi}\frac{cm_0}{eH}\frac{\partial S'}{\partial x} \quad \text{or} \quad \frac{\omega_c}{\omega(x)} = \frac{1}{2\pi}\frac{\partial S'}{\partial x}, \quad (2.6)$$

where ω_c is the cyclotron frequency corresponding to a free positron, and

$$S' = \oint y^2 d\varphi. \quad (2.7)$$

We can write for the effective mass defined by the cyclotron resonance

$$m^*(x)/m_0 = (1/2\pi)\,\partial S'/\partial x. \tag{2.8}$$

We must draw attention to the fact that both the frequency and the effective mass are functions of x only. In other words, we shall have in p-space conical surfaces with their vertices at the point $p = 0$, corresponding to fixed values of m^* or ω/ω_c.

We consider type I hodographs corresponding to the range $x_1 < x < x_2$. We change variables: $\varphi = \pi/4 + \varphi'$. The expression for S' is then of the form

$$S' = \int_{-\varphi_1'}^{\varphi_1'} \frac{1}{\gamma_1}\sqrt{\gamma_2^2 - 4\gamma_1\gamma_3}\, d\varphi',$$

where φ_1' is determined from the condition that the expression under the radical sign vanishes

$$\varphi_1' = \tfrac{1}{2}\arcsin\sqrt{\gamma_4/\gamma_3 C^2},$$

$$\gamma_4 = \gamma_2^2 - 4\gamma_3(A^2 - B^2) + \gamma_3 C^2. \tag{2.9}$$

We introduce instead of φ' a variable ψ through the relation

$$\sqrt{\gamma_3 C^2/\gamma_4}\sin 2\varphi' = \sin 2\psi.$$

The limits of integration are then equal to $\pm\pi/4$. Differentiating with respect to x under the integral sign we get after a transformation

$$\frac{\partial S'}{\partial x} = \frac{2A(-\gamma_2)}{\sqrt{\gamma_3 C^2}\,(A^2 - B^2 - C^2/4)}\,\Pi_1(\vartheta, k) + \frac{4(A - x)}{\sqrt{\gamma_3 C^2}}\,K(k). \tag{2.10}$$

Here K and Π_1 are respectively the complete elliptic integrals of first and third kind. The arguments ϑ and k are determined by the expressions

$$\vartheta = C^2k^2/(4A^2 - 4B^2 - C^2), \qquad k = \sqrt{\gamma_4/\gamma_3 C^2}. \tag{2.11}$$

We can use one of the Legendre transformations[11] to change the complete elliptic integral of the third kind to incomplete elliptic integrals of the first and second kind. As a result, only tabulated functions will enter in the right hand side of (2.10) (see, for instance, reference 12).

An analysis of the expression obtained shows that the frequency ω decreases monotonically in the interval x_1 to x_2 tending to zero as $x \to x_2$. Let us find the limiting value of the frequency corresponding to x_1. As $x \to x_1$, we have $k \to 0$ and $\Pi_1(\vartheta, k) \to \pi/2$, $K(k) \to \pi/2$. The result is

$$\frac{\omega_c}{\omega_1} = \frac{1}{2\sqrt{\gamma_3(x_1)C^2}}\left[2(A - x_1) - \frac{A\gamma_2(x_1)}{A^2 - B^2 - C^2/4}\right].$$

We use the values of A, B, and C for germanium and silicon, which are given in the most recent papers[6,8]

	A	B	C	
Ge:	13.1	8.3	12.5	
Si:	4.0	1.1	4.1	(2.12)

We find then for ω_1

$$\text{Ge:}\quad \omega_1/\omega_c \approx 3.0, \quad m_1^*/m_0 \approx 0.33;$$
$$\text{Si:}\quad \omega_1/\omega_c \approx 1.56, \quad m_1^*/m_0 \approx 0.64. \tag{2.13}$$

One can show that (2.10) tends logarithmically to infinity as $x \to x_2$. This occurs because the condition $\partial\epsilon/\partial p_\perp = 0$ for $\varphi = 0, \pi/2, \pi, 3\pi/2$ is realized for $x = x_2$ in a saddle point. In that sense we have a situation similar to the one that occurs in metals when one changes over from closed to open surfaces (see, for instance, reference 9).

We consider now type II hodographs. We write the expression for the frequency as

$$\omega_c/\omega = (2\pi)^{-1}\int_0^{2\pi}(\partial y^2/\partial x)_\varphi\, d\varphi. \tag{2.14}$$

Using (2.3) we obtain after a number of transformations the expression ($x_2 < x < x_3$ and $x_4 < x < \infty$)

$$\frac{\omega_c}{\omega} = \frac{A}{\sqrt{(A^2 - B^2 - C^2/8)^2 - (C^2/8)^2}} + \frac{4(A - x)}{\pi\sqrt{\gamma_4}}K(k) - \frac{2A\gamma_2}{\pi\sqrt{\gamma_4}}\frac{1}{A^2 - B^2 - C^2/4}\Pi_1(\vartheta, k), \tag{2.15}$$

$$\vartheta = C^2/(4A^2 - 4B^2 - C^2), \qquad k = \sqrt{\gamma_3 C^2/\gamma_4}. \tag{2.16}$$

For $x_3 < x < x_4$ we have

$$\frac{\omega_c}{\omega} = \frac{A}{\sqrt{(A^2 - B^2 - C^2/8)^2 - (C^2/8)^2}} + \frac{4}{\pi}\frac{(A - x)}{\sqrt{\gamma_4 + |\gamma_3|C^2}}K(k) - \frac{2A\gamma_2}{\pi(A^2 - B^2)\sqrt{\gamma_4 + |\gamma_3|C^2}}\Pi_1(\vartheta, k),$$

$$k = \sqrt{|\gamma_3|C^2/(\gamma_4 + |\gamma_3|C^2)}, \qquad \vartheta = -C^2/4(A^2 - B^2). \tag{2.17}$$

A study of Eqs. (2.15) and (2.17) shows that the frequency increases monotonically from zero at $x = x_2$ to a value ω_∞ as $x \to \infty$. As $x \to x_2 + 0$, the period tends logarithmically to infinity.

Let us determine the value ω_∞. As $x \to \infty$ [see (2.16)]

$$k \to \sqrt{C^2/(C^2 + 4B^2)}, \tag{2.18}$$

and the frequencies and effective masses of Ge and Si are respectively equal to

$$\text{Ge:}\quad \omega_\infty/\omega_c \approx 3.6, \quad m_\infty^*/m_0 \approx 0.28;$$

$$\text{Si:}\quad \omega_\infty/\omega_c \approx 2.16, \quad m_\infty^*/m_0 \approx 0.46. \tag{2.19}$$

For type III hodographs the integration of (2.14) leads to the following final expression

$$\frac{\omega_c}{\omega} = -\left\{\frac{4(A - x)}{\pi\sqrt{\gamma_4}}K(k) - \frac{A}{\sqrt{(A^2 - B^2 - C^2/8)^2 - (C^2/8)^2}} + \frac{2A(-\gamma_2)}{\pi\sqrt{\gamma_4}}\frac{1}{A^2 - B^2 - C^2/4}\Pi_1(\vartheta, k)\right\}. \tag{2.20}$$

Here ϑ and k are defined according to (2.16).

The expression within the curly brackets is always positive in the range where type III hodographs occur. We are thus led to a branch of negative frequencies and, hence, of negative effective masses [defined in the sense of (2.8)]. We must note that the sign of the effective mass in (2.8) depends on the first derivative $\partial\epsilon/\partial p_\perp$ and is not directly connected with the sign of the second derivatives.

The absolute magnitude of ω increases steadily in the region considered, from zero (at $x = x_2$) to a limiting value ω_3 corresponding to the disappearance of this kind of hodograph ($x = x_3$). At $x = x_3 \equiv A - B$ the quantity k vanishes and one checks easily that

$$\Pi_1(\vartheta, 0) = \pi / 2\sqrt{1+\vartheta}. \qquad (2.21)$$

Substituting this value into (2.20) we get after some transformations the following simple formula for ω_3

$$\omega_3 / \omega_c = -(C^2 + 2B^2 - 2AB) / 2B. \qquad (2.22)$$

This expression depends only on the constants in the energy spectrum. Using the values (2.12) we find

$$\text{Ge:} \quad \omega_3/\omega_c \approx -4.6, \quad m_3^*/m_0 \approx -0.22;$$

$$\text{Si:} \quad \omega_3/\omega_c \approx -4.7, \quad m_3^*/m_0 \approx -0.21. \qquad (2.23)$$

For type IV hodographs the frequency is determined by the same expression (2.20). Now, however, $x_4 < x < \infty$, and the expression within the curly brackets is always negative. We are led to a branch of positive frequencies corresponding to the "light" hole band. Hodographs of this type appear first at $x = x_4 \equiv A + B$ with a finite frequency ω_4. When x increases further, the frequency decreases steadily to a value ω'_∞.

At $x = x_4$ the quantity k tends again to zero and Π_1 attains the value (2.21). After an appropriate transformation we are led to the expression

$$\omega_4 / \omega_c = (C^2 + 2B^2 + 2AB) / 2B. \qquad (2.24)$$

For germanium and silicon we have respectively

$$\text{Ge:} \quad \omega_4/\omega_c \approx 30.8, \quad m_4^*/m_0 \approx 0.032;$$

$$\text{Si:} \quad \omega_4/\omega_c \approx 12.7, \quad m_4^*/m_0 \approx 0.078. \qquad (2.25)$$

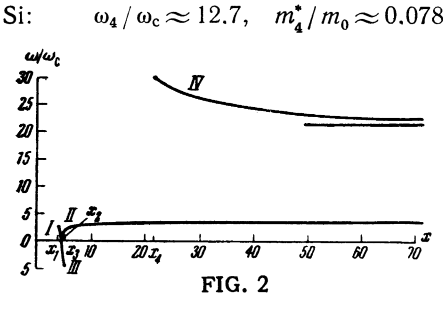

FIG. 2

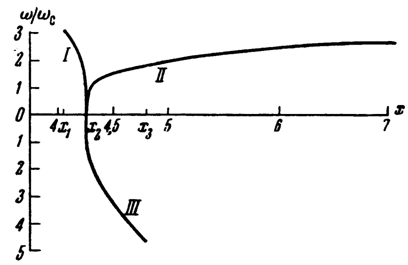

FIG. 3

As $x \to \infty$ the quantity k tends to the value (2.17) and for ω'_∞ we find from (2.20)

$$\text{Ge:} \quad \omega'_\infty/\omega_c \approx 22.3, \quad m^{*\prime}_\infty/m_0 \approx 0.045;$$

$$\text{Si:} \quad \omega'_\infty/\omega_c \approx 5.7, \quad m^{*\prime}_\infty/m_0 \approx 0.18. \qquad (2.26)$$

In Figs. 2 and 3 we have given the whole frequency spectrum corresponding to the valence band of germanium with **H** along the [001] axis. Clearly the behavior of the cyclotron resonance depends essentially upon the form of this spectrum. In particular, the appearance in reference 1 of negative effective masses is intimately connected with the negative branch of frequencies corresponding to type III hodographs. The presence of three free ends of frequency branches, where the frequency attains a maximum absolute value, is noteworthy.

3. SOLUTION OF THE TRANSPORT EQUATION

As was noted in the preceding section, the condition for quasi-classical behavior is satisfied under the normal experimental conditions for cyclotron resonance. We restrict ourselves therefore to a consideration of the classical transport equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f + \frac{\partial f}{\partial \mathbf{p}}\left(e\mathbf{E} + \frac{e}{c}[\mathbf{v}\times\mathbf{H}]\right) = L(f - f^{(0)}) + N - M,$$
$$\mathbf{v} = \partial\varepsilon/\partial\mathbf{p}. \qquad (3.1)$$

Here L is a linear operator corresponding to the collision integral, and N and M are terms describing respectively the creation and absorption of quasi-particles and referred to a unit volume (M is generally speaking a functional that depends on the distribution function of all the kinds of quasi-particles).

The need for introducing N and M into Eq. (3.1) is connected with the fact that, because of the low temperature, the cyclotron resonance experiments are performed on non-equilibrium carriers which are produced either by illumination of the semiconductor or by secondary ionization when the carriers collide with impurity atoms. In order to be specific we shall assume that the carriers are excited by illumination. We can write the solution of Eq. (3.1) in the form

$$f = f_0(\mathbf{r}, \mathbf{p}) + f_1(\mathbf{r}, \mathbf{p}, t),$$

where f_0 is the stationary solution corresponding to $\mathbf{E} = 0$. Assuming that the diffusion length is large compared to the mean free path and that the electrical field is sufficiently small, we can use the fact that the operator L is linear to write down the following set of equations

$$\mathbf{v}\nabla f_0 + \frac{e}{c}[\mathbf{v}\times\mathbf{H}]\frac{\partial f_0}{\partial \mathbf{p}} = L(f_0 - f^{(0)}) + N - M, \quad (3.2)$$

$$\frac{\partial f_1}{\partial t} + \frac{e}{c}[\mathbf{v}\times\mathbf{H}]\frac{\partial f_1}{\partial \mathbf{p}} + e\mathbf{E}\frac{\partial f_0}{\partial \mathbf{p}} = L(f_1). \quad (3.3)$$

We can easily transform Eq. (3.3) into

$$\frac{\partial f_1}{\partial t} + \omega\frac{\partial f_1}{\partial \psi} + e\mathbf{E}\frac{\partial f_0}{\partial \mathbf{p}} = L(f_1), \quad (3.4)$$

where ω is the frequency of the rotation of the quasi-particles in the azimuthal plane, which is determined from (2.6), and ψ is defined by the expression

$$\psi = \frac{m_0}{m^*}\int_0^{\varphi}\left(\frac{\partial y^2}{\partial x}\right)_{\varphi, p_z} d\varphi.$$

The solution of Eq. (3.2) when the creation and absorption terms (and the appropriate boundary conditions) are present leads to a value of f_0 which is in general a function of $\mathbf{p}$. It is therefore necessary to find the solution of (3.4) for an arbitrary dependence $f_0(\mathbf{p})$. ($\mathbf{r}$ enters into the equation purely as a parameter and will not be specially designated in the following.)

The problem of solving the transport equation for an arbitrary dispersion law $\epsilon(\mathbf{p})$ and of applying it to galvanomagnetic phenomena in metals was first considered consistently in a paper by I. M. Lifshitz, Azbel', and Kaganov[13] and also by McClure.[14] We use in the following a method for solving (3.4) which is along the same principles as the method given in these papers.

At the temperatures under consideration the carriers are scattered by impurities. We use this fact and introduce instead of $L(f_1)$ approximately $-f_1/\tau(\mathbf{p})$. Taking into consideration that $\mathbf{E} \sim e^{i\omega_0 t}$ and thus $f_1 \sim e^{i\omega_0 t}$, we change (3.4) into

$$\omega\frac{\partial f_1}{\partial \psi} + f_1\frac{1 + i\omega_0\tau}{\tau} + e\mathbf{E}\frac{\partial f_0}{\partial \mathbf{p}} = 0.$$

A solution of this equation which satisfies the obvious periodicity condition can be written in the form

$$f_1 = -\frac{e\mathbf{E}}{\omega}\int_{-\infty}^{\psi}\frac{\partial f_0}{\partial \mathbf{p}}\exp\left\{-\frac{1}{\omega}\int_{\psi'}^{\psi}\frac{d\psi}{\tau}(1 + i\omega_0\tau)\right\}d\psi'. \quad (3.5)$$

We introduce instead of τ some value of this quantity averaged over the hodograph, according to

$$\oint d\psi/\tau(\mathbf{p}) = 1/\tau' \qquad (\tau' = \tau(p_z, \varepsilon)).$$

We expand $\partial f_0/\partial \mathbf{p}$ in a Fourier series in ψ

$$\partial f_0/\partial \mathbf{p} = \sum_{m=-\infty}^{\infty}\mathbf{F}\, e^{im\psi}. \quad (3.6)$$

After some simple manipulations we get then

$$f_1 = -e\mathbf{E}\sum_{m=-\infty}^{\infty}\mathbf{F}_m\tau' e^{im\psi}/(1 + i\omega_0\tau' + im\omega\tau').$$

We can write the corresponding expression for the current in the form

$$\mathbf{j} = -2\frac{e^2E_\alpha}{(2\pi\hbar)^3}\sum_{m=-\infty}^{\infty}\int d^3p\tau'\mathbf{v}F_{m\alpha}e^{im\psi}/(1 + i\omega_0\tau' + im\omega\tau'). \quad (3.7)$$

(Summation over repeated Greek indices is implied throughout.)

We expand $\mathbf{v}$ in a Fourier series in ψ and perform in (3.7) the integration over ψ. Returning again to the integration over the whole phase volume, we get

$$\mathbf{j} = -2\frac{e^2E_\alpha}{(2\pi\hbar)^3}\sum_{m=-\infty}^{\infty}\int d^3p\tau'\mathbf{v}_m^* F_{m\alpha}/(1 + i\omega_0\tau' + im\omega\tau'). \quad (3.7')$$

Let us determine the power absorbed per unit volume of the semiconductor. We consider the case of circular polarization of the electrical field in a plane perpendicular to the magnetic field, i.e., $E_y = iE_x$. Omitting the intermediate steps, we can then write the final expression in the form

$$P = \tfrac{1}{2}\mathrm{Re}(\mathbf{j}\mathbf{E}^*) =$$
$$= -2\frac{e^2E_0^2}{(2\pi\hbar)^3}\sum_{m=1}^{\infty}\int d^3p\tau'\{[1 + (m\omega\tau')^2 - (\omega_0\tau')^2]^2$$
$$+ 4(\omega_0\tau')^2\}^{-1}\times\{(v_{mx}F_{mx} + v_{my}^*F_{my})[1 + (m\omega\tau')^2 + (\omega_0\tau')^2]$$
$$+ 2m\omega_0\omega\tau'^2\mathrm{Im}(v_{mx}F_{my} - v_{my}^*F_{mx})\} \quad (3.8)$$

Equation (3.8) enables us to analyze the problem for any type of $\mathbf{p}$-dependence of f_0.

We restrict ourselves to the case where the light is incident along the direction of the magnetic field (as was the case in the experiment by Dousmanis and co-workers[1]). Taking into account that $\omega_0\tau' \gg 1$ and that in (3.8) the main contribution comes from carriers with ω near to ω_0, we can assume then that

$$f_0 = f_0(\varepsilon, p_z). \quad (3.9)$$

Equation (3.8) simplifies then and becomes equal to

$$P = -2\frac{e^2E_0^2}{(2\pi\hbar)^3}\sum_{m=1}^{\infty}\int d^3p\left(\frac{\partial f_0}{\partial \varepsilon}\right)_{p_z}\frac{\tau'}{[1 + (m\omega\tau')^2 - (\omega_0\tau')^2]^2 + 4(\omega_0\tau')^2}$$
$$\times\{(|v_{mx}|^2 + |v_{my}|^2)[1 + (m\omega'\tau')^2 + (\omega_0\tau')^2]$$
$$+ 4m\omega_0\omega\tau'^2\,\mathrm{Im}(v_{mx}v_{my})\}. \quad (3.10)$$

When the hodographs possess a symmetry axis of the fourth order, one can transform the expression obtained into the form

$$P = -4\frac{e^2E_0^2}{(2\pi\hbar)^3}\sum_{m=0}^{\infty}\int d^3p v_{2m+1,x}^2 L_{2m+1}\left(\frac{\partial f_0}{\partial \varepsilon}\right)_{p_z}, \quad (3.11)$$

$$L_{2m+1}=\tau'\frac{1+[\omega_0+(-1)^m(2m+1)\omega]^2\tau'^2}{[1+(2m+1)^2(\omega\tau')^2-(\omega_0\tau')^2]^2+4(\omega_0\tau')^2}. \qquad (3.12)$$

We use the expressions for the velocity components, which can easily be found from the equations of motion

$$v_x=(1/m^*)(\partial p_y/\partial\psi)_{p_z,\varepsilon},\quad v_y=-(1/m^*)(\partial p_x/\partial\psi)_{p_z,\varepsilon}.$$

Taking into account that all types of hodographs possess mirror symmetry, we find

$$v_{nx}/\left[\frac{|p_z|}{m_0}\right]=\frac{n}{2\pi}\frac{\omega}{\omega_c}\int_0^{2\pi}\frac{p_y}{|p_z|}\sin n\psi d\psi\equiv f(x),$$

$$v_{ny}/\left[\frac{|p_z|}{m_0}\right]=-\frac{in}{2\pi}\frac{\omega}{\omega_c}\int_0^{2\pi}\frac{p_x}{|p_z|}\cos n\psi d\psi\equiv f'(x). \qquad (3.13)$$

Let us expand $y = p_\perp/|p_z|$ in a cosine Fourier series (when considering type I hodographs we assume that the origin is chosen on the bisectors in the points where the hodographs start)

$$y=\sum_{m=0}^{\infty}y_m(x)\cos m\varphi. \qquad (3.14)$$

One shows easily that the Fourier components of the velocity are determined by the quantities $y_n(x)$.

We change in (3.10) and (3.11) to the integration variables ϵ, p_z, and ψ. Taking into account the fact that $\omega(x)$ is not single-valued for the "heavy" hole band we have

$$\int d^3p\ldots=2\pi\int_{-\infty}^{\infty}dp_z\left\{4\int_{x_1p_z^2/2m_0}^{x_2p_z^2/2m_0}m^*d\varepsilon\ldots+\int_{x_2p_z^2/2m_0}^{\infty}m^*d\varepsilon\ldots\right.$$
$$\left.+\int_{x_2p_z^2/2m_0}^{x_3p_z^2/2m_0}(-m^*)d\varepsilon\ldots\right\}; \qquad (3.15)$$

and for the "light" hole band

$$\int d^3p\ldots=2\pi\int_{-\infty}^{\infty}dp_z\int_{x_4p_z^2/2m_0}^{\infty}m^*d\varepsilon\ldots \qquad (3.15')$$

[In (3.15) the m^*'s are in the sequence of integrals over $d\epsilon$ determined with respect to the first, second, and third branch of the frequency spectrum.] The infinite upper limit was chosen in the conventional way: we assume that the distribution function vanishes with increasing $p_z(\epsilon)$ before any deviations from (2.1) in the dispersion law become apparent.

We substitute (3.13) into (3.10) and (3.11) and use (3.15). One sees easily that the integrand turns then out to be proportional to the frequency. As a result, singularities occur sometimes in the resonance spectrum curve, not only at extrema of the frequency, but also near maximum values at the limits of the frequency-spectrum branches. To be sure, the realization of such a possibility, together with the shape of the singularity, depends essentially on the steepness of the change in frequency and on the character of the behavior of the complete integrand near these points. In particular, the position is alleviated when the second end point of the branch corresponds to zero frequency.

We have shown in the preceding section that for the dispersion law (2.1) there are three such points in the frequency spectrum. If we turn to the cyclotron resonance spectrum in germanium, which was obtained in reference 1, it turns out that ω_3 is the same as the observed resonance frequency for negative effective masses, and that ω_4 lies near the frequency of the second new resonance, which corresponded to holes with m^* lower than the effective mass of the "light" holes. [We are dealing here with the normally determined values — they are the same as the m'_∞ in (2.26)].

The frequency ω_1 is near ω_∞, the frequency of the usual "heavy" holes [compare (2.13) and (2.19)]. Since the latter corresponds on the spectral curve to a wide resonance with a large intensity, the resonance at frequencies near ω_1 cannot appear explicitly. It is, however, possible that there are some singularities in the resonance curve near ω_∞ which are connected just with the first frequency branch.

We note that since the hodographs connected with the first frequency branch do not possess a symmetry axis, even harmonic frequencies may occur under well defined conditions near the frequency of the "heavy" holes.

4. CHARACTER OF THE CYCLOTRON SPECTRUM

Let us briefly analyze the conditions under which the usual increase in absorption near a resonance can be replaced by a decrease (with respect to the level of absorption for $H = 0$) or even change over to an emission spectrum. It can be seen explicitly from (3.10) and (3.11) that P is always positive if $\partial f_0/\partial\epsilon < 0$. In order that the sign of P change it is thus at any rate necessary that there be in p space a region with the opposite character of level population. This condition is, however, not sufficient. The fact is that the inevitable decrease of f_0 at sufficiently large energies leads in the most favorable case to the simultaneous occurrence of regions with $\partial f_0/\partial\epsilon > 0$ and $\partial f_0/\partial\epsilon < 0$. Larger values of the energy correspond, however, to a larger phase volume. In the case under consideration this leads, in particular, to $P > 0$ for any form of the function $f_0(\epsilon)$* (see below), if f_0 is a function of the energy only.

For the sake of simplicity we shall consider only the first harmonic. Using (3.10), (3.11), (3.12), and (3.15) we can write the expression for the "heavy" hole band in the form

$$P_1 = -\frac{4\pi e^2 E_0^2}{m_0 (2\pi\hbar)^3}\int_0^\infty dp_z p_z^2 \Big\{4\int_{x_1 p_z^2/2m_0}^{x_2 p_z^2/2m_0}\left(\frac{\partial f_0}{\partial \varepsilon}\right)_{p_z}\frac{\omega}{\omega_c}\, y_0^2 L_1' d\varepsilon$$

$$+\int_{x_2 p_z^2/2m_0}^{\infty}\left(\frac{\partial f_0}{\partial \varepsilon}\right)_{p_z}\frac{\omega}{\omega_c}\, y_0^2 L_1' d\varepsilon$$

$$+\int_{x_2 p_z^2/2m_0}^{x_3 p_z^2/2m_0}\left(\frac{\partial f_0}{\partial \varepsilon}\right)_{p_z}\left(-\frac{\omega}{\omega_c}\right) y_0^2 L_1' d\varepsilon\Big\}. \tag{4.1}$$

[The expression for L_1' can be obtained by comparing (4.1) with (3.10) and taking (3.13) into account.] If f_0 is not an even function of p_z (in the given point in the volume of the semiconductor) we must understand by f_0 in (4.1) (and in the following) the quantity $\frac{1}{2}[f_0(-p_z) + f_0(p_z)]$.

We integrate by parts (with respect to ϵ) in the expression obtained. One sees easily that the integrands tend to zero for all finite limits of integration. Changing from ϵ to the variable x [see (2.2)] we get then

$$P_1 = \frac{4\pi e^2 E_0^2}{m_0 (2\pi\hbar)^3}\int_0^\infty dp_z p_z^2 \Big\{4\int_{x_1}^{x_2} f_0 \frac{\partial}{\partial x}\left[\frac{\omega}{\omega_c}\, y_0^2 L_1'\right] dx$$

$$+\int_{x_2}^{\infty} f_0 \frac{\partial}{\partial x}\left[\frac{\omega}{\omega_c}\, y_0^2 L_1'\right] dx + \int_{x_2}^{x_3} f_0 \frac{\partial}{\partial x}\left[\left(-\frac{\omega}{\omega_c}\right) y_0^2 L_1'\right] dx\Big\}. \tag{4.2}$$

Since f_0 is essentially positive, the negative contribution to p_1 can only be connected with regions where the derivative

$$\frac{\partial}{\partial x}\left[\left|\frac{\omega}{\omega_c}\right| y_0^2 L_1'\right] = \left|\frac{\omega}{\omega_c}\right|\left[L_1'\frac{dy_0^2}{dx} + y_0^2\frac{\partial}{\partial x}\left[\left|\frac{\omega}{\omega_c}\right| L_1'\right]\right] \tag{4.3}$$

becomes negative.

Let us consider the region of negative frequencies ($x_2 < x < x_3$). In that region (and only in that region) $\partial^2 y_0^2/\partial x < 0$. If we take into account that $y_0(x) \to 0$ as $x \to x_3$, it follows from (4.3) that the integrand in the last integral in (4.2) becomes negative near x_3. When ω_3 approaches ω_0 this can, for a well defined form of the function f_0, correspond to a decrease in p_1 or even to a negative value of that quantity.

We give here an approximate discussion that enables us to make some observations about the corresponding form of the function f_0. To do this we consider a value of the magnetic field which selects an interval of negative frequencies far from the end point of the branch. Let $\omega_0\tau' \gg 1$ and let the integrand in (4.1) change sufficiently smoothly in the range of x defined by the resonance denominator in L_1'. Neglecting the first two integrals and changing first the order of integration in (4.1), we find after some manipulations

$$P_1 \approx \text{const}\cdot\frac{y_0^2(x_0)}{x_0^{5/2}}\left|\frac{\omega(x_0)}{\omega_c}\right|\cdot\left|\frac{d\omega}{dx}\right|^{-1}\int_0^\infty d\varepsilon\varepsilon^{3/2}\left(-\frac{\partial f_0}{\partial\varepsilon}\right)_{p_z}\Bigg|_{p_z^2=2m_0\varepsilon/x_0}. \tag{4.4}$$

Here x_0 is the value of x for which the frequency ω is equal to ω_0.

One can consider f_0 in (4.1) to be a function of p_z^2. Taking this into account, we have

$$\left(-\frac{\partial f_0}{\partial\varepsilon}\right)_{p_z}\Bigg|_{p_z^2=2m_0\varepsilon/x_0} = -\frac{df_0}{d\varepsilon} + \frac{2m_0}{x_0}\left(\frac{\partial f_0}{\partial p_z^2}\right)_\varepsilon\Bigg|_{p_z^2=2m_0\varepsilon/x_0}.$$

We substitute this expression into (4.4) and integrate the first term by parts

$$P_1 \sim \int_0^\infty d\varepsilon\left\{\frac{3}{2}\varepsilon^{1/2} f_0 + \frac{2m_0\varepsilon^{5/2}}{x_0}\left(\frac{\partial f_0}{\partial p_z^2}\right)_\varepsilon\right\}\Bigg|_{p_z^2=2m_0\varepsilon/x_0}.$$

It can be clearly seen from this expression that if f_0 depends on the energy only, p_1 will always be positive. For an emission spectrum to occur in the frequency interval considered, it is thus at any rate necessary, that, together with the regions where $(\partial f_0/\partial\epsilon)_{p_z} > 0$, there exist a region where $(\partial f_0/\partial p_z^2)_\epsilon < 0$, or more precisely where

$$\frac{4}{3}\frac{m_0\varepsilon}{x_0}\left(\frac{\partial \ln f_0}{\partial p_z^2}\right)_\varepsilon\Bigg|_{p_z^2=2m_0\varepsilon/x_0} < -1. \tag{4.5}$$

We note that, other things being equal, inequality (4.5) is best satisfied for the minimum values of x_0.

The condition obtained here is physically connected with the necessity of a steep decrease in the number of particles which give a contribution to p_1 in that energy range where $(\partial f_0/\partial\epsilon)_{p_z} < 0$. (It is clear that an equivalent result might be obtained if the dispersion law (2.1) were changed in that energy range in such a way that the given interval in the frequency spectrum would vanish. Compare a similar remark in reference 4.)

We must note that the ability to emit rather than absorb energy in cyclotron resonance experiments for an arbitrary initial non-equilibrium distribution is, apparently, not an exclusive privilege of the negative frequency range. In particular, an approximate analysis similar to the one given in the foregoing might also be used for the first frequency branch. (Although the first term in (4.3) is positive, the second term, which can change sign, will lead to a negative value of the integrand

*We assume all the time that f_0 vanishes with increasing energy before the dispersion law changes appreciably.

when the distribution has an inverse character.) We require here, to be sure, additional limitations on the function f_0, which are connected, moreover, with the fact that the second branch possesses also the same frequency interval.

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