APPLICATION OF DISPERSION RELATION TECHNIQUES TO A STUDY OF THE SIMPLEST GREEN'S FUNCTIONS IN MESODYNAMICS

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An approximate system of dispersion equations for the meson and nucleon Green's functions and the vertex function is studied in pseudoscalar charge-symmetric mesodynamics. The asymptotic behavior of these functions, corresponding to infinite values of Z_2^{-1} and Z_3^{-1}, is determined in the weak-coupling case.

1. INTRODUCTION

In this paper we present certain results of a study within the framework of the dispersion relation approach, of the single-particle Green's functions and the vertex function in pseudoscalar charge-symmetric mesodynamics. The analogous problems as they appear in electrodynamics have been discussed previously.[1,2]* The main purpose of this type of investigation is a clarification of the asymptotic behavior of the Green's functions. This would make it possible to decide on the renormalizability properties of the theory and on the connection between the dispersion relation and Lagrangian formulations of quantum field theory. However, the analysis of the simplest approximate set of dispersion equations shows that in all probability there exists no effective expansion parameter (even in the weak coupling case) in the asymptotic region which would make it possible to estimate the contributions from many-particle Green's functions. For the strong coupling case the problem of boundary conditions which would determine a unique solution (see Sec. 3) remains unsolved.

It should also be mentioned that, in contrast to the approximate dispersion equations for scattering problems (obtained on the basis of the Mandelstam representation[6]) where one studies matrix elements on the energy shell with free external lines ($p_i^2 = m_i^2$), for a study of contributions to the asymptotic region of single particle Green's functions and the vertex function due to higher order approximations it is necessary to know the analytic structure of matrix elements with virtual ($p_i^2 \neq m_i^2$) external lines.

*In a similar vein are the papers by Redmond and Uretsky[3] and by Bogolyubov, Logunov, and Shirkov[4] (see also reference 5).

In Sec. 2 the approximate equations are derived for the meson Green's function $\Delta(q^2)$, nucleon Green's function $G(p)$, and the vertex Green's function $F_i(p, p')$. The derivation is based on the analytic structure of these functions and the unitarity condition. Dispersion relations are proven for $F_i(p, p')$ as a function of p^2 in the physical region ($p'^2 = m^2$, $q^2 = (p-p')^2 = \mu^2$). In Sec. 3 is discussed the approximate set of equations which reduces to the solution of the Hilbert problem.[7] A unique solution is obtained by making use of the boundary condition (13c) and the requirement that the solution, if it can be expanded in a power series in the coupling constant (weak coupling), should coincide with the corresponding perturbation theory series. The asymptotic behavior of $F_i(p, p')$ in p^2 and q^2 is found in the weak coupling case. The results obtained are discussed briefly in Sec. 4. Section 5 consists of a mathematical appendix.

2. DERIVATION OF THE EQUATIONS

1. The dispersion relations for the meson Green's function $\Delta(x-x')\delta_{ij} = i\langle 0 | T\varphi_i(x) \times \varphi_j(x') | 0\rangle$ and the nucleon Green's function $G_{\alpha\beta}(x-x') = i\langle 0 | T\psi_\alpha(x)\overline{\psi}_\beta(x') | 0\rangle$ can be derived without particular use of the causality principle (see, e.g., reference 8) and have the form*

$$\Delta(q^2) = \frac{1}{(2\pi)^4}\int e^{iqx}\Delta(x)\,dx = \frac{1}{\mu^2 - q^2 - i\varepsilon} + \int_{(3\mu)^2}^{\infty} \frac{\rho(q'^2)\,dq'^2}{q'^2 - q^2 - i\varepsilon}, \qquad (1)$$

$$G(p) = \frac{1}{(2\pi)^4}\int e^{ipx}G(x)\,dx = \frac{1}{m - \hat{p} - i\varepsilon} + \int_{(m+\mu)^2}^{\infty} \frac{\rho_1(p'^2) + \hat{p}\rho_2(p'^2)}{p'^2 - p^2 - i\varepsilon}\,dp'^2. \qquad (2)$$

*We assume here that $\Delta(q^2)$ and $G(p)$ fall off as their arguments increase; $x^2 = x_0^2 - \mathbf{x}^2$; $i, j = 1, 2, 3$ are the isotopic indices of the meson; α, β (= 1, ... 8) are the spinor and isotopic indices of the nucleon.

Here $\varphi(x)$ and $\psi(x)$ are meson and nucleon field operators in the Heisenberg representation; m and μ are the nucleon and meson masses; ρ and ρ_λ are real spectral functions with the properties: $\rho \geq 0$, $\rho_2\sqrt{p^2} \pm \rho_1 > 0$, $\rho_2 > 0$; all quantities are renormalized quantities.

Making use of the equations of motion

$$(\Box_x - \mu^2)\varphi_i(x) = j_i(x), \qquad (i\hat{\nabla}_x - m)\psi(x) = \eta(x)$$

(where

$$j(x) = i\frac{\delta S}{\delta\varphi_{in}(x)}S^*, \qquad \eta(x) = i\frac{\delta S}{\delta\bar{\psi}_{in}(x)}S^*,$$

S being the S-matrix operator: $|n^{out}\rangle = s|n^{in}\rangle$, where the states $|n^{in}\rangle$ correspond to incoming and the states $|n^{out}\rangle$ to outgoing waves) as well as of the expansion in terms of a complete set of states $|n^{in}\rangle$ (or $|n^{out}\rangle$) we find

$$\rho(q^2) = \tfrac{1}{3}(2\pi)^3 \,\mathrm{Sp}\sum_n (\mu^2 - q^2)^{-2}\langle 0|j_i(0)|n^{in}\rangle\langle n^{in}|j_i(0)|n\rangle \times \delta(p_n - q), \tag{3}$$

$$\rho_1(p^2) + \hat{p}\rho_2(p^2) = (2\pi)^3\sum_n (m - \hat{p})^{-2}\langle 0|\eta(0)|n^{in}\rangle \times \langle n^{in}|\bar{\eta}(0)|n\rangle\,\delta(p_n - p). \tag{4}$$

Thus, in order to find ρ and ρ_λ for all values of the arguments, one would have to know all the matrix elements $\langle 0|j|n\rangle$ and $\langle 0|\eta|n\rangle$; in practice this is an insoluble problem. The following expressions are found from covariance considerations for the simplest matrix elements $\langle 0|j_i|p_1,p_2\rangle$ and $\langle 0|\eta|p_1,q_1\rangle$, where $|p_1,p_2\rangle$ is a state of a nucleon (p_1) and antinucleon (p_2), and $|p_1,q_1\rangle$ is a state of a nucleon (p_1) and a meson (q_1):

$$\langle 0|j_i|p_1,p_2\rangle = \bar{U}_+(\mathbf{p}_1)\gamma_5\tau_i F(q^2)U_-(\mathbf{p}_2), \qquad q = p_1 + p_2; \tag{5}$$

$$(2q_{10})^{1/2}\langle 0|\eta|p_1q_1\rangle = \gamma_5\tau_i(F_1(p^2) + \hat{p}F_2(p^2))U_-(\mathbf{p}_1), \qquad p = p_1 + q_1; \tag{6}$$

where $U_\mp$ are solutions of the Dirac equation for a particle and antiparticle, and F and F_λ ($\lambda = 1, 2$) are invariant functions.*

Introducing Eqs. (5) and (6) into (3) and (4) we obtain

$$\rho(q^2) = (\mu^2 - q^2)^{-2}q^2(16\pi^2)^{-1} (1 - 4m^2/q^2)^{1/2}|F(q^2)|^2\,\theta(q^2 - 4m^2) + \sum_n{}'\langle\ldots\rangle, \tag{7}$$

*The function $F(q^2)$ may be called the mesonic form factor of the nucleon.

$$\rho_1(p^2) + \hat{p}\rho_2(p^2) = \frac{3}{16\pi^2p^2}(p^2 - (m+\mu)^2)^{1/2}(p^2 - (m-\mu)^2)^{1/2} \times\Big\{\frac{\mu^2 m}{(m^2-p^2)^2}(|F_1|^2 + p^2|F_2|^2) + \frac{1}{2}\Big(1 - \frac{\mu^2(p^2+m^2)}{(p^2-m^2)^2}\Big) \times(F_1^*F_2 + F_2^*F_1) + \hat{p}\Big[\frac{1}{2p^2}\Big(1 - \frac{\mu^2(p^2+m^2)}{(p^2-m^2)^2}\Big)(|F_1|^2 + p^2|F_2|^2) + \frac{m\mu^2}{(p^2-m^2)^2}(F_1^*F_2 + F_2^*F_1)\Big]\Big\} + \sum_n{}''\langle\ldots\rangle. \tag{8}$$

The $\Sigma'\langle\ldots\rangle$ in Eq. (7) and the $\Sigma''\langle\ldots\rangle$ in Eq. (8) denote the contributions of all remaining terms to ρ and ρ_λ respectively. Since each term in Σ' and Σ'' gives a positive contribution to ρ and ρ_2, the study of the asymptotic behavior of the first terms on the right side of Eqs. (7) and (8) gives important information on the renormalizability character of the theory[9] (see also subsection 4).* Consequently, as a first approximation, one may start the study of the properties of ρ and ρ_λ from an analysis of $F(q^2)$ and $F_\lambda(p^2)$.

2. The analytic properties of F and F_λ follow from the analytic properties of the vertex Green's function†

$$F_i^c(p,p') = i\int e^{ipx-iqy}(\Box_y - \mu^2)(i\hat{\nabla}_x - m) \times\langle 0|T\psi(x)\varphi_i(y)|p'\rangle\,dxdy = (2\pi)^4\delta(p-q-p')\,i\int e^{ipx} \times(i\hat{\nabla}_x - m)\langle 0|T\psi(x)j_i(0)|p'\rangle\,dx = (2\pi)^4\delta(p-q-p') \times i\int e^{-iqx}(\Box_y - \mu^2)\langle 0|T\eta(0)\varphi_i(y)|p'\rangle\,dy. \tag{9}$$

It is not difficult to see that [in the following we omit the factor $(2\pi)^4\delta(p-p'-q)$ from $F^c(p,p')$]

$$\bar{U}(\mathbf{p})F_i^c(p,p') \to \langle p|j_i(0)|p'\rangle = \bar{U}(\mathbf{p})\gamma_5\tau_i F(q^2)U(\mathbf{p}'), \quad p^2 \to m^2, \; p_0 > 0; \tag{10}$$

$$F_i^c(p,p') \to (2q_0)^{1/2}\langle 0|\eta(0)|p', q_i\rangle = \gamma_5\tau_i(F_1(p^2) + \hat{p}F_2(p^2))U(\mathbf{p}'), \quad q^2 \to \mu^2, \; q_0 > 0. \tag{11}$$

In the general case

$$F_i^c(p,p') = \gamma_5\tau_i(F_1(p^2,q^2) + \hat{p}F_2(p^2,q^2))U(\mathbf{p}'), \tag{12}$$

*We observe that the nucleon + antinucleon state is not the lowest energy state in $\sum_n$ in Eq. (3); therefore a study of this state does not lead to any quantitative results whatever about the behavior of $\Delta(q^2)$ in the region $q^2 \ll m^2$. Apparently states $|n\rangle$ in Eq. (3) with 3, 5 . . . (up to $n = 2m/\mu$) mesons play a dominant role in this region. However an analysis of these states is beset by great mathematical difficulties. The meson + nucleon state in Eq. (4) is the lowest energy state [since $p_n^2 > (m+\mu)^2$] and so the contribution of this state may be decisive in determining the behavior of G(p) in the region $|p^2| < (m + 2\mu)^2$.

†The vertex Green's function should be distinguished from the vertex part [see below, Eq. (14)].

and it follows from a comparison of Eqs. (10), (11), and (12) that

$$F(q^2) = F_1(m^2, q^2) + mF_2(m^2, q^2), \tag{13a}$$

$$F_\lambda(p^2) = F_\lambda(p^2, \mu^2), \tag{13b}$$

$$F(\mu^2) = F_1(m^2, \mu^2) + mF_2(m^2, \mu^2) = g, \tag{13c}$$

with g the renormalized coupling constant.

The vertex Green's function $F_i^c(p, p')$ and the vertex part $\Gamma_i^5(p, p')$ are related by

$$F_i^c(p, p') = (\mu^2 - q^2)\Delta(q^2)(m - \hat{p})G(p)\Gamma_5^i(p, p')U(\mathbf{p}'). \tag{14}$$

Let us begin by studying the analytic properties of $F_i^c(p, p')$ as a function of q^2. For $p^2 < 0$

$$F_i^c(p, p') = F_i^+(p, p') = i\int e^{-iqx}(\Box_x - \mu^2)\langle 0|\theta(x)[\varphi_i(x), \eta(0)]_-|p'\rangle dx, \tag{15a}$$

i.e., F_i^c coincides with the retarded Green's function. Let us also introduce the advanced Green's function

$$F_i^-(p, p') = -i\int e^{-iqx}(\Box - \mu^2)\langle 0|\theta(-x)[\varphi_i(x), \eta(0)]_-|p'\rangle dx. \tag{15b}$$

In the system $\mathbf{p}' = 0$, as a consequence of the identities

$$q_0 = \frac{p^2 - q^2 - m^2}{2m}, \qquad p_0 = \frac{p^2 + m^2 - q^2}{2m},$$
$$|\mathbf{p}| = |\mathbf{q}| = \left(\frac{(q^2 - m^2 - p^2)^2 - 4p^2m^2}{4m^2}\right)^{1/2}, \tag{16}$$

the functions F_i^+ and F_i^- can be directly analytically continued into the upper and lower half planes respectively of the complex variable $z = q^2$. Since on the real axis we have $F_i^+(p, q^2) = F_i^-(p, q^2)$ for $q^2 < (3\mu)^2$, a single analytic function $F_i(p, z)$ exists, regular in the entire complex plane cut along $(3\mu)^2 \leq q^2 < \infty$* with

$$\lim_{z \to q^2 \pm i\varepsilon} F_i(p^2, z) = F_i^\pm(p^2, z). \tag{17}$$

In an entirely analogous manner one proves that for $q^2 < 0$ there exists a single analytic function $F_i(z', q^2)$ regular in the entire complex $z' = p^2$ plane except for the cut $(m + \mu)^2 \leq p^2 < \infty$ on the real axis. Here

$$\lim_{z' \to p^2 \pm i\varepsilon} F_i(z', q^2) = F_i^\pm(z', q^2) = \pm i\int e^{ipx}(i\hat{\nabla}_x - m) \times \langle 0|\theta(\pm x)[\psi(x), j_i(0)]_-|p'\rangle dx,$$

$$F_i^+(p^2, q^2) = F_i^c(p, p'), \qquad q^2 < 0 \quad (\operatorname{Im} z = \operatorname{Im} z' = 0). \tag{18}$$

*Strictly speaking it is the invariant functions $F_1(p^2, z)$ and $F_2(p^2, z)$ of Eq. (12) that have these analyticity properties.

Since for $\operatorname{Re} z < (3\mu)^2$ and $\operatorname{Re} z' < (m + \mu)^2$ the functions $F_i(p^2, z)$ and $F_i(z', q^2)$ coincide it follows that they represent a single analytic function $F_i(z', z)$ of two complex variables z′ and z.

For real values of z and z′ the expression

$$-\tfrac{1}{2} i\tau_i\gamma_5[F_1^+(p^2, q^2) - F_1^-(p^2, q^2) + \hat{p}(F_2^+(p^2, q^2) - F_2^-(p^2, q^2))]$$

is equal, for $p^2 < 0$, to

$$\tfrac{1}{2}(2\pi)^4 \sum_n \langle 0|j(0)|n\rangle\langle n|\eta(0)|p'\rangle\delta(p_n - q). \tag{19}$$

and, for $q^2 < 0$, to

$$\tfrac{1}{2}(2\pi)^4 \sum_n \langle 0|\eta(0)|n\rangle\langle n|j(0)|p'\rangle\delta(p_n - p) \tag{20}$$

The expression (20), which is the absorptive part of $F(p^2, q^2)$ for $q^2 < 0$, can be analytically continued to the physical point $q^2 = \mu^2$ provided that $p^2 \geq (m + \mu)^2$. Indeed, the matrix element

$$(2\pi)^4\langle n|j(0)|p'\rangle\delta(p_n - p) = \int e^{-iqx}\langle n|j(x)|p'\rangle dx$$

goes over as $q^2 \to \mu^2$, $q_0 > 0$ [since, as a consequence of conservation of nucleon number, $p_n^2 = p^2 = (p' + q)^2 \geq (m + \mu)^2$]* into the matrix element for the scattering of a meson by a nucleon into a final state n in the physical region of total energy $p_0 = (p_0' + q_0)$. Thus it is rigorously proved that the functions $F_\lambda^+(p^2)$ [see Eqs. (6) and (13b)] are boundary values of analytic functions $F_\lambda(z) = F_\lambda(z, q^2)(q^2 \to \mu^2)$ regular in the entire complex plane except for the cut $(m + \mu)^2 \leq \operatorname{Re} z < \infty$ $(\operatorname{Im} z = 0)$.

As regards the absorptive part (19), it is not possible in the general case to prove the validity of analytic continuation to the point $p^2 = m^2$ for an arbitrary ratio of m to μ starting from only the covariance, causality and spectrum properties of the theory,[11] although such proof is possible in any order of perturbation theory.[12] In what follows we shall neglect the terms in Eq. (20) with $p_n^2 < 4m^2$ so that in our approximation $F(q^2)$ [see Eqs. (5) and (13a)] is the boundary value of an analytic function regular in the entire complex plane except for the cut $4m^2 \leq \operatorname{Re} z < \infty$. We do not explicitly write out the dispersion relations for $F(q^2)$ and $F_\lambda(p^2)$ since this requires a knowledge of the asymptotic behavior of $F(z)$ and $F_\lambda(z)$ as $|z| \to \infty$. This question is clarified in Sec. 3.

3. In essence, Eqs. (19) and (20) express the unitarity conditions for the vertex Green's function $F_i(p, p')$, which make it possible to relate $F_i(p, p')$ to other matrix elements. It is much more difficult to decide now what is a reasonable

*Electromagnetic and weak interactions are ignored.

approximation than in the case of ρ and ρ_λ, since the contributions of the various terms to the summation over n in Eqs. (19) and (20) have different signs. At this time we can suggest nothing better than a restriction to the simplest terms in Eqs. (19) and (20), i.e., keeping the term corresponding to the nucleon-antinucleon pair (in the intermediate state) in Eq. (19), and the term corresponding to a nucleon and meson in Eq. (20).* We then obtain, in place of Eqs. (19) and (20), the approximate equations (see Sec. 5, item 1)

$$F^+(q^2) - F^-(q^2) = iA(q^2)(F^+(q^2) + F^-(q^2)), \quad (21)$$

$$F_1^+(p^2) - F_1^-(p^2) + \hat{p}(F_2^+(p^2) - F_2^-(p^2)) = i(A_1(p^2) + \hat{p}A_2(p^2))[F_1^+(p^2) + F_1^-(p^2) + \hat{p}(F_2^+(p^2) + F_2^-(p^2))],$$

$$A(q^2) = \mathrm{Re}\,A_0(q^2)/(1 - \mathrm{Im}\,A_0(q^2)), \quad (22)$$

where $A_0(q^2)$ is the partial amplitude for elastic nucleon-antinucleon scattering in the singlet spin and triplet isotopic spin S-state in the barycentric frame $(\mathbf{q} = 0)$;

$$A_1(p^2) = {}^1/_2(a_s + a_p), \quad \sqrt{p^2}A_2(p^2) = {}^1/_2(a_s - a_p),$$

$$a_{s,p}(p^2) = \mathrm{Re}\,A_{s,p}(p^2)/(1 - \mathrm{Im}\,A_{s,p}(p^2)),$$

with $A_{s,p}(p^2)$ the partial amplitude for elastic meson-nucleon scattering in their barycentric frame $(\mathbf{p} = 0)$ in the S- and P-state respectively with total spin and isospin $1/2$.

It is easy to see on the basis of conservation laws precisely why these states entered into Eqs. (21) and (22). We leave out the details of the calculations. Let us only note that in the derivation of Eqs. (21) and (22) we took into account the fact that $F^+(q^2) = (F^-(q^2))^*$ and $F_\lambda^+(p^2) = (F_\lambda^-(p^2))^*$. This reality condition on the absorptive parts, Eqs. (19) and (20), can be satisfied in the approximation under discussion by taking half of the sum over $|n^{in}\rangle$ and $|n^{out}\rangle$ states on the right side of Eqs. (19) and (20).

If the contributions from inelastic processes are ignored, then

$$A(q^2) = \tan\delta_0(q^2), \qquad a_{s,p}(p^2) = \tan\delta_{s,p}(p^2),$$

where $\delta_0(q^2)$ and $\delta_{s,p}(p^2)$ are the phase shifts in the above-indicated states.

In Born approximation†

$$A(q^2) = A^b(q^2) = -\frac{3g^2}{16\pi}(1 - \zeta^{-1})^{1/2}, \quad (23)$$

$$A_1^b(p^2) = -\frac{g^2}{16\pi}\left(\frac{3\xi - 1}{2\xi} + \frac{\ln\xi}{(1-\xi)}\right), \quad (24)$$

$$A_2^b(p^2) = \frac{g^2}{16\pi m}\left(\frac{\xi + 1}{2\xi^2} + \frac{\ln\xi}{(1-\xi)}\right). \quad (25)$$

Here we have for simplicity put $\mu = 0$ and introduced the dimensionless variables $\zeta = q^2/4m^2$ and $\xi = p^2/m^2$; for ξ and $\zeta \gg 1$

$$A^b \to -3g^2/16\pi,$$

$$A_1^b \to -3g^2/32\pi, \qquad A_2^b \to -g^2\ln\xi/16\pi m\xi.$$

3. STUDY OF THE APPROXIMATE SYSTEM

1. Let us assume $A(q^2)$ and $A_\lambda(p^2)$ to be known functions, independent of the unknown functions Δ, G and F_1^c. Then the approximate system of equations (1), (2), (7), (8), (21), and (22)* separates and in fact reduces to a study of the last two relations, i.e., to the solution of the homogeneous Hilbert problem:[7] to find an analytic function F(z) (analogously for F_λ) regular in the entire plane except for the cut and of finite order at infinity, given the boundary condition on the cut $F^+(q^2) = K(q^2)F^-(q^2)$ where $K(q^2)(1 + iA(q^2))/(1 - iA(q^2))$ is a known function which vanishes nowhere.

As is well known, the solution of such a problem is not unique. To obtain a unique solution additional boundary conditions must be imposed on F(z). In our case one such additional boundary condition consists of the relation (13c). However in the approximation under consideration this is not sufficient. If the validity of perturbation theory for $g \to 0$ (weak coupling) is assumed then one can demand that the solution that was found should coincide with the corresponding perturbation theory expression. If on the other hand (as is quite permissible) perturbation theory has no region of applicability (i.e., the solution cannot be expanded in a series near g = 0) then the question of additional boundary conditions becomes much more complex and has not been solved so far.† Nevertheless the absence of additional constants and unphysical singularities may be taken as additional criteria for determining a unique solution.

2. The solution of Eq. (21), satisfying the stated conditions has the form

$$F(z) = g\exp\left[\frac{(z-\mu^2)}{\pi}\int_{4m^2}^{\infty}\frac{\tan^{-1}A(z')\,dz'}{(z'-z)(z'-\mu^2)}\right]. \quad (26)$$

*The asymptotic vanishing of $F(q^2)$ and $F_\lambda(p^2)$ with increasing q^2 and p^2 would provide some justification for the legitimacy of this approximation. However, (see Sec. 3), in the weak coupling approximation, when the asymptotic behavior can be determined explicitly, the functions do not vanish. On the other hand, for small $p^2 < (m + 2\mu)^2$, the indicated approximation may turn out to be not bad for $F_\lambda(p^2)$.

†The expression $A^b(q^2)$ is given in reference 13.

*We assume that the terms Σ' and Σ'' are omitted from Eqs. (7) and (8).

†This question is of general importance for the theory of dispersion equations.[14]

It is easy to see that the asymptotic behavior of $F(z)$ as $|z| \to \infty$ is determined by the properties of the integral

$$J = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\tan^{-1} A(z')}{z' - \mu^2} dz'.$$

If the integral J converges then $F(z) \to \text{const}$ as $|z| \to \infty$. If the integral diverges* then there are two possibilities:

a) $\tan^{-1} A(z) \to \text{const} = \alpha > 0$. In this case $F(z)$ falls off asymptotically. In particular for $\text{Re}\, z \to -\infty$ ($\text{Im}\, z = 0$), $F(z) \to g|z|^{-\alpha}$,

b) $\tan^{-1} A(z) \to \text{const} = \beta < 0$. Then $F(z)$ increases asymptotically: $F(z) \to g|z|^{-\beta}$ for $\text{Re}\, z \to -\infty$ ($\text{Im}\, z = 0$).

It is interesting that for $A(z) \to \pm\infty$ as $z \to \infty$ the asymptotic behavior of $F(z)$ is independent of $A(z)$: $F(z) \to g|z|^{\mp 1/2}$.

If it is assumed that perturbation theory is valid for $F(z)$ in the weak coupling case ($g \to 0$) then, taking into account that $A^b(z') \ll 1$, we obtain by replacing $\tan^{-1} A^b(z')$ by $A^b(z')$ and expanding the exponent in Eq. (26) in a series, the following

$$F(z) \approx g\left[1 + \frac{(z-\mu^2)}{\pi} \int_{4m^2}^{\infty} \frac{A^b(z')\,dz'}{(z'-z)(z'-\mu^2)}\right]. \tag{27}$$

Expression (27) [with $A^b(q^2)$ given by (23)] coincides with the renormalized vertex Green's function calculated to terms of order $\sim g^3$. The asymptotic behavior of $F(z)$ in the weak coupling case not using perturbation theory is found by substituting expression (23a) into Eq. (26):

$$F(z) \to g\left(\frac{|z|}{4m^2}\right)^{3g^2/16\pi^2}, \quad \text{Re}\, z \to -\infty, \quad \text{Im}\, z = 0. \tag{28}$$

Thus, in the weak coupling case, $F(q^2)$ increases asymptotically.

3. Equation (20), equivalent to the Hilbert problem for two analytic functions $F_1(z)$ and $F_2(z)$, cannot be solved in closed form. It can be reduced to the solution of two Fredholm equations † with singular kernels.[7] An exact solution of this system may turn out to be not a bad approximation to reality in the region $|p^2| < (m + 2\mu)^2$ and would express the dependence of the vertex function on the values of the meson-nucleon scattering phase shifts in the $S_{1/2}$ and $P_{1/2}$ states. We do not discuss this question, but consider instead the asymptotic solution of Eq. (22) for $p^2 > m^2$. In this case, if we let $\mu^2 = 0$ (and discard terms of order μ^2/p^2 and higher) we can find an exact solution for the system (22) satisfying the boundary conditions formulated above. As is shown in Sec. 5 (item 2) the solution has the form

$$F_1^{\pm}(p^2) + \hat{p}F_2^{\pm}(p^2) = \sum_{r=1}^{2} C_r \exp\left\{\frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} \frac{\varphi_1^r(z') + \hat{p}\varphi_2^r(z')}{z' - p^2 \mp i\varepsilon} dz'\right\}, \tag{29}$$

$$\varphi_1^{1,2}(p^2) = \tan^{-1}\left(-\frac{1 + A_2'^2 - A_1^2}{2A_1} \pm \frac{[(1 + A_2'^2 + A_1^2)^2 - 4A_1^2 A_2'^2]^{1/2}}{2|A_1|}\right), \tag{29a}$$

$$\sqrt{p^2}\,\varphi_2^{1,2}(p^2) = \tan^{-1}\left(-\frac{1 + A_1^2 - A_2'^2}{2A_2'} \pm \frac{[(1 + A_1^2 + A_2'^2)^2 - 4A_1^2 A_2'^2]^{1/2}}{2|A_2'|}\right), \tag{29b}$$

$$A_2'(p^2) = \sqrt{p^2}\, A_2(p^2).$$

The constants C_1 and C_2 are related by

$$\sum_{r=1}^{2} C_r \exp\left\{\frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} \frac{\varphi_1^r(z) + m\varphi_2^r(z)}{z - m^2} dz\right\} = g, \tag{30}$$

which leaves the ratio C_1/C_2 arbitrary. This arbitrariness can be removed by requiring agreement with renormalized perturbation theory for $g \to 0$. Otherwise the arbitrariness in the choice of a solution remains. It is important to emphasize that in the weak coupling case (A_1 and $A_2' \ll 1$) only the term $\sim C_1$ enters into the corresponding expressions for $F_1(p^2)$ and $F_2(p^2)$ calculated by perturbation theory. Setting $C_2 \equiv 0$ in Eqs. (29) and (30) and expanding in powers of A_1^b and A_2^b we find (for $g \to 0$) the expression

$$F_1(p^2) + \hat{p}F_2(p^2) \approx g\left(1 + \frac{p^2 - m^2}{\pi} \int_{(m+\mu)^2}^{\infty} \frac{A_1^b(z)\,dz}{(z - p^2)(z - m^2)} + \frac{\hat{p}}{\pi} \int_{(m+\mu)^2}^{\infty} \frac{A_2^b(z)\,dz}{z - p^2} - \frac{m}{\pi} \int_{(m+\mu)^2}^{\infty} \frac{A_2^b(z)\,dz}{z - m^2}\right), \tag{31}$$

which coincides with the result of renormalized perturbation theory accurate to terms of order g^3. The asymptotic behavior for weak coupling not by perturbation theory is given by

$$p^2 \to -\infty \begin{cases} F_1(p^2) \sim \left(\dfrac{|p^2|}{m^2}\right)^{3g^2/32\pi^2}, & (32a) \\ F_2(p^2) \sim \left(\ln\left|\dfrac{p^2}{m^2}\right|\right)^2 \left(\left|\dfrac{p^2}{m^2}\right|\right)^{3g^2/32\pi^2 - 1}. & (32b) \end{cases}$$

Thus, in the weak coupling approximation, $F_1(p^2)$ increases asymptotically and $F_2(p^2)$ decreases. If Eqs. (29) and (30) with $C_1 \equiv 0$ are taken as the solution, then F_1 and F_2 fall off asymptotically ($\text{Re}\, z \to -\infty$, $\text{Im}\, z = 0$) in the weak coupling approximation:

*We assume that $\int_{4m^2}^{\infty} \frac{\tan^{-1} A(z')dz'}{z'^2}$ converges.

†These equations follow from the dispersion relations for $F_\lambda(z)$ obtained under the assumption of a definite asymptotic behavior for these functions.

$$F_1(z) \sim |z|^{-\pi^{-1}\tan^{-1}(32\pi/3g^2)}, \quad (33a)$$

$$F_2(z) \sim |z|^{-\pi^{-1}\tan^{-1}(32\pi/3g^2)-1/2}, \quad (33b)$$

in disagreement with perturbation theory.

4. To conclude this section we discuss briefly the role of inelastic processes in the determination of the asymptotic behavior of $F(q^2)$. If the omitted terms in Eq. (21) are denoted by $\Psi(q^2)$ then we are led to the inhomogeneous Hilbert problem:[7]

$$F^+(q^2) = \frac{1 + iA(q^2)}{1 - iA(q^2)} F^-(q^2) + \frac{\Psi(q^2)}{1 - iA(q^2)}. \quad (34)$$

A solution of Eq. (34), satisfying condition (13c) and not involving additional constants, is given by

$$F^{\pm}(q^2) = F_0^{\pm}(q^2)\left(\frac{q^2-\mu^2}{\pi}\int\limits_{(3\mu)^2}^{\infty}\frac{\widetilde{\Psi}(z)\,dz}{(z-\mu^2)(z-q^2\mp i\varepsilon)} + 1\right),$$

$$\widetilde{\Psi}(z) = \Psi(z)\,(1 + A^2(z))^{-1/2}\,|F_0^{\pm}(z)|^{-1}, \quad (35)$$

with $F_0^{\pm}(z)$ the solution of the corresponding homogeneous problem [Eq. (26)].

From a physical point of view it seems most natural to assume that $A(q^2) \to 0$ as $q^2 \to \infty$. In that case $F_0(q^2)$ approaches a constant and only one possibility exists for making $F(q^2)$ in Eq. (35) fall off asymptotically:

$$\widetilde{J} \equiv \frac{1}{\pi}\int\limits_{(3\mu)^2}^{\infty}\frac{\widetilde{\Psi}(z)\,dz}{z-\mu^2} = 1, \quad (36)$$

i.e., $\Psi(q^2)$ must vanish for $q^2 \to \infty$.* In an analogous manner the necessary conditions are found for $F(q^2)$ to decrease asymptotically for a different possible behavior of $A(q^2)$. The contribution of inelastic processes to $F_\lambda(p^2)$ may be similarly considered and a condition of the type (36) is easily obtained by introducing an inhomogeneous term $\Psi_1(p^2) + \hat{p}\Psi_2(p^2)$ into Eq. (22); however, all this is not necessary since at the present time nothing is known about the asymptotic behavior of partial waves, nor about the other matrix elements that contribute to $\Psi(q^2)$ and $\Psi_\lambda(p^2)$.

*At the same time, condition (36) may impose limitations on possible values of the constant g.

4. DISCUSSION

1. The renormalization constants Z_2 and Z_3 are expressed in terms of the spectral functions $\rho(q^2)$ and $\rho_2(p^2)$ by [see Eqs. (7) and (8)]:

$$Z_3^{-1} = 1 + \int\limits_{(3\mu)^2}^{\infty}\rho(q^2)\,dq^2, \quad (37)$$

$$Z_2^{-1} = 1 + \int\limits_{(m+\mu)^2}^{\infty}\rho_2(p^2)\,dp^2. \quad (38)$$

In the approximation considered here the convergence of the above integrals is determined, according to Eqs. (7) and (8), by the asymptotic behavior of $F(q^2)$ and $F_\lambda(p^2)$: if these functions vanish at infinity sufficiently fast so that the integrals in Eqs. (37) and (38) converge then Z_2 and Z_3 will be finite; otherwise Z_2^{-1} and Z_3^{-1} are divergent constants. It is important to note here that the latter result cannot be improved upon by taking into account higher approximations for ρ and ρ_2 so that the falling off of $F(q^2)$ and $F_\lambda(p^2)$ is a necessary (but not sufficient) condition for finiteness of the renormalization constants Z_2 and Z_3. As has been seen in the previous section the asymptotic behavior of $F(q^2)$ and $F_\lambda(p^2)$ can be determined only very crudely, since in the strong coupling case ($g \sim 1$) it is not clear what serves as an expansion parameter, and in the weak coupling case in the lowest approximation an increasing asymptotic behavior is obtained for $F(q^2)$ [see Eq. (28)] and $F_\lambda(p^2)$ [see Eqs. (32a, b)]. We are thus led to the conclusion that in the first approximation for weak coupling the constants Z_2^{-1} and Z_3^{-1} diverge. It follows from the dispersion relations for $\Delta(q^2)$ and $G(p)$ that in this approximation these functions decrease at infinity slower than the corresponding free particle Green's functions Δ_0 and G_0 [the explicit asymptotic behavior is easy to derive starting from Eqs. (1), (2), (7) and (8), (28), (32a) and (32b)].

2. It is important to note the following: independent of whether $F(q^2)$ and $F_\lambda(p^2)$ increase or decrease asymptotically it is impossible in the approximation under consideration to run into an internal inconsistency of the "zero-charge" type,[15] characteristic of certain approximate solutions of the Schwinger-Dyson equations. This is related to the fact that the vertex part $\Gamma_5^i(p, p')$, determined by the relation (14) for given Δ, G and $F_\lambda^C(p, p')$, decreases for increasing values of its arguments as is easy to deduce from Eqs. (1), (2), (7), (8) and (14).*

3. It seems to us that as a consequence of the great mathematical difficulties that arise when one attempts to estimate the importance of higher order approximations in the dispersion relation approach, as well as due to the absence of an effective expan-

*It follows from the work of Lehmann, Symanzik, and Zimmerman[9] that in the approximation in which inelastic processes are ignored the falling off of $\Gamma_5(p, p')$ as a function of q^2 (for $p^2 = p'^2 = m^2$) and $p^2(p'^2 = m^2, q^2 = \mu^2)$ is a necessary and sufficient condition for the absence of unphysical singularities ("ghost states") in $\Delta(q^2)$ and $G(p)$.

sion parameter in the asymptotic region, progress on the question of internal consistency of such an approach and its relation to the Lagrangian formulation of quantum field theory will have to await the application of new, in particular statistical, methods of calculation of the many-particle matrix elements which play an important role at high energies.

5. MATHEMATICAL APPENDIX

1. In order to obtain Eqs. (21) and (22) we made use of the partial wave expansion for the nucleon-antinucleon and meson-nucleon elastic scattering amplitudes, $\langle p_1, l; p_1', l' | \bar{U}_+^r(\mathbf{p})\eta(0) | p'r' \rangle$ and $\langle q_1^j, p_1 | j(0) | p' \rangle$. If terms that do not contribute to $F^+(q^2) - F^-(q^2)$ and $F_\lambda^+(p^2) - F_\lambda^-(p^2)$ are omitted this expansion looks as follows (in the barycentric frame for the corresponding process):

$$(q^2/2)^{1/2}(4\pi)^{-1}\langle p_1, l; p_1', l' | \bar{U}_+^r(\bar{\mathbf{p}})\eta(0) | p', r' \rangle$$
$$= |\mathbf{p}|^{-1}\delta_{rr'}\delta_{ll'}\sum_l (2l+1) A_l(q^2) P_l(\cos\theta)$$
$$+ \text{ isotopic triplet terms} \qquad (39)$$

$$\tau_j (2\omega)^{1/2} E(E+\omega)^{-1}\langle p_1, r_1; q', j | j_i(0) | p', r \rangle$$
$$= |\mathbf{q}|^{-1}\tau_i \sum_l ((l+1) A_{l+}(p^2) P_{l+}^{r_1, r}(\cos\theta')$$
$$+ l A_{l-}(p^2) P_{l-}^{r_1, r}(\cos\theta')). \qquad (40)$$

The notation in Eq. (39) is as follows: $q = p + p' = p_1 + p_1'$, $\mathbf{q} = 0$, θ is the angle between $\mathbf{p}$ and $\mathbf{p}_1$; $A_l(q^2)$ is the partial wave amplitude for nucleon-antinucleon scattering in the singlet state with orbital angular momentum l and total isospin 1; $(r, l)(r', l') = 1, 2$ are the indices denoting the antinucleon and nucleon states; $P_l(\cos\theta)$ are Legendre polynomials. If the inelastic processes are small then $A_l(q^2) \approx \exp(-i\delta_l(q^2)) \sin\delta_l(q^2)$ with $\delta_l(q^2)$ the phase shift in the corresponding state.

The notation in Eq. (40) is as follows: $p = p' + q = p_1 + q_1$, $\mathbf{p} = 0$; $p_0' = p_{10} = E$ is the nucleon energy; $q_0 = q_{10} = \omega$ is the meson energy; θ' is the angle between $\mathbf{p}'$ and $\mathbf{p}_1$; $A_{l\pm}(p^2)$ are the partial wave meson-nucleon scattering amplitudes in states of total angular momentum $I = l \pm 1/2$ and isospin $1/2$ with $A_{0+}(p^2) \equiv A_s(p^2)$, $A_{1-}(p^2) \equiv A_p(p^2)$; $P_{l\pm}^{r, r'}(\cos\theta')$ are the angular polynomials introduced in reference 16.

$$A_{l\pm}(p^2) = \exp\{-i\delta_{l\pm}(p^2)\}\sin\delta_{l\pm}(p^2)$$

in the energy region $(m+\mu)^2 \le p^2 = (E+\omega)^2 \le (m+2\mu)^2$, where $\delta_{l\pm}(p^2)$ is the phase shift in the corresponding state.

2. We now prove that Eq. (29), together with Eqs. (29a) and (29b), constitutes a solution of Eq. (22) for $\mu^2 = 0$. For $\mu^2 \neq 0$ we obtain from Eq. (29), making use of Eq. (16) with $q^2 = \mu^2$

$$F^+(\hat{p}) - F^-(\hat{p}) = i\Big\{\tan[(\varphi_1(p^2) + \hat{p}\varphi_2(p^2))\,\theta(p^2 - (m+\mu)^2)]$$
$$+ \tan\Big[\mathbf{e}\gamma((m+\mu)^2 - p^2)^{1/2}(p^2 - (m-\mu)^2)^{1/2}$$
$$\times \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{\varphi_2(z)\,dz}{z^2 - p^2}\,\theta((m+\mu)^2 - p^2)\,\theta(p^2 - (m-\mu)^2)\Big]\Big\}$$
$$\times (F^+(\hat{p}) + F^-(\hat{p})),$$

$$F^\pm(\hat{p}) = F_1^\pm(p^2) + \hat{p}F_2^\pm(p^2), \qquad \theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}; \qquad (41)$$

where $\mathbf{e}$ is a unit vector in the direction of $\mathbf{p}$. It is seen that for $\mu \to 0$ the second term on the right side of Eq. (41) vanishes and we obtain Eq. (22) under the condition

$$\tan[\varphi_1(p^2) + \hat{p}\varphi_2(p^2)] = A_1(p^2) + \hat{p}A_2(p^2),$$

from which one obtains Eqs. (29a) and (29b). It is elementary to prove that at infinity the solution (29) is of finite order if it is taken into account that $\varphi_2(z)$ falls off not slower than $1/\sqrt{z}$ as $z \to \infty$ [see Eq. (29b)].

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