SINGLE-PHOTON ANNIHILATION AND ELECTRON-PAIR CREATION IN A MEDIUM

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Single-photon annihilation and electron-pair creation $e^+ + e^- \rightleftarrows \gamma$ can occur in a dispersive medium with a refractive index less than unity. For such processes to occur, the photon energy must lie within the interval $2m \leq \omega \leq N^{1/3}$, where m is the electron rest-mass energy, and N the electron density in the medium. In this paper we have evaluated the probability for such processes. This effect is similar in its physics to the Vavilov-Cerenkov effect. The processes discussed are the dominant ones in the energy range $2m < \omega < 10m$.

1. If no outside particles are involved, the creation of an electron-positron pair by a γ quantum and, conversely, the transformation of a pair into one quantum,

$$e^+ + e^- \leftrightarrows \gamma \tag{1}$$

is impossible in vacuo because of the momentum conservation law. It is, however, easy to check that the momentum conservation law is observed for the processes (1) in a dispersive medium with refractive index $n(\omega) < 1$:

$$\mathbf{k} = \mathbf{p}_1 + \mathbf{p}_2; \quad \omega = E_1 + E_2; \; k = \omega n(\omega), \tag{2}$$

and these phenomena can thus occur.

The energies of the quanta involved in the processes (1) must have not only a lower bound, but also an upper one. The upper limit of the energy of the quanta is determined by the obvious requirement $\lambda \gtrsim l$, where $\lambda = 1/\omega$ is the wavelength divided by 2π, and l the mean distance between particles. It is evident that when $\lambda < l$ the concepts of a continuous medium and of a refractive index independent of the position cease to be correct. Thus, if the average interparticle distance is $l \sim N^{-1/3}$, where N is the number of electrons per unit volume, the applicability of the "refractive index" concept will impose on the photon energy a limit: $\omega \lesssim l^{-1} \approx N^{1/3}$, whence it follows that the threshold for pair formation must correspond to a particle density of the order of $N \gtrsim 8\lambda_C^{-3} \approx 1.4 \times 10^{32}\ \mathrm{cm}^{-3}$, where $\lambda_C = 1/m$ is the electron Compton wavelength divided by 2π.

Such high densities of matter are not even assumed to exist in the interior regions of the sun or other normal stars. However, the possibility of the existence, in the central regions of white dwarfs, of matter with a density of the order of nuclear densities or even higher has often been discussed in the literature.[1-3] The idea of the existence of a hyper-dense state of matter ("prestellar matter") has particularly been emphasized by Ambartsumyan.[4]

For the processes (1) to occur in a dispersive medium, the photon energy must therefore satisfy the condition

$$2m \lesssim \omega \lesssim N^{1/3}. \tag{3}$$

The temperature of the medium plays here an important role.[5] At sufficiently low temperatures we are dealing with a strongly degenerate electron gas in which the process $\gamma \to e^+ + e^-$ is forbidden by the Pauli principle. We shall evaluate in the following the probability for a single-photon annihilation and for the creation of electron pairs, assuming that all the necessary conditions for the occurrence of these phenomena in a medium exist in nature.

2. We consider first pair production by a γ quantum. We have for the matrix element for such a process[6-8]

$$S_{if}^{(1)} = -e\left(\frac{\mu}{2\alpha\omega n^2 V}\right)^{1/2} \int \bar{\psi}_1 \gamma_\nu e^{i\tilde{k}\tilde{x}} \psi_2 \, d^4x, \tag{4}$$

where $n^2 = \epsilon\mu$, V is the normalization volume, γ_ν is the component of the matrix $\tilde{\gamma}$ in the direction of the polarization of the photon (a tilde indicates a four-dimensional vector), $\tilde{k} = (\mathbf{k}, i\omega)$ is the four-momentum of the photon, ψ_1 and ψ_2 are the electron and positron wave functions,

$$\psi_1 = \frac{u_1(\mathbf{p}_1)}{\sqrt{V}} \exp\{i\tilde{p}_1 \tilde{x}\}, \; \psi_2 = \frac{u_2(\mathbf{p}_2)}{\sqrt{V}} \exp\{-i\tilde{p}_2 \tilde{x}\}, \tag{5}$$

and, finally

$$\alpha = 1 + (\omega/n)\, dn/d\omega. \tag{6}$$

Using (5) we get from (4) for the transition probability per unit time

$$dW = \frac{\mu e^2}{(2\pi)^2\, 2\alpha\omega n^2} \left| \bar{u}_1(\mathbf{p}_1)\,\gamma_\nu u_2(-\mathbf{p}_2) \right|^2 \delta(\tilde{k} - \tilde{p}_1 - \tilde{p}_2)\, d^3p_1\, d^3p_2. \tag{7}$$

We average next over the polarization of the quantum, sum over the spin components of the electron and the positron, also integrate over the positron momentum, and get from (7)

$$dW = \frac{\mu e^2}{(2\pi)^2\, 2\alpha\omega n^2}\, \frac{E_1E_2 + m^2 - p_1p_2\cos\theta_1\cos\theta_2}{E_1E_2} \times \delta(\omega - E_1 - E_2)\, d^3p_1, \tag{8}$$

where θ_1 is the angle between $\mathbf{p}_1$ and $\mathbf{k}$; θ_2 is the angle between $\mathbf{p}_2$ and $\mathbf{k}$, and $d^3p_1 = 2\pi p_1 E_1 dE_1 \sin\theta_1\, d\theta_1$.

We must still integrate over the electron momentum. Integration over θ_1 in (8) is equivalent to the substitution

$$\delta(\omega - E_f)\sin\theta_1\, d\theta_1 \to E_2/kp_1,$$

where $E_f = E_1 + E_2$ is the energy of the final state of the system

$$E_f = E_1 + (E_1 + k^2 - 2kp_1\cos\theta_1)^{1/2}.$$

We get as a result

$$dW = \frac{\mu e^2}{4\pi\alpha\omega^2 n^3}(E_1E_2 + m^2 - p_1p_2\cos\theta_1\cos\theta_2)\, dE_1. \tag{9}$$

E_2, θ_1, and θ_2 are now functions of E_1

$$\cos\theta_1 = E_1/np_1 - k(1-n^2)/2n^2p_1,$$
$$\cos\theta_2 = E_2/np_2 - k(1-n^2)/2n^2p_2. \tag{10}$$

Eliminating θ_1, θ_2, and E_2 from (9) we get

$$dW = \frac{\mu e^2}{4\pi\alpha\omega^2 n^5}\left[(n^2-1)(\omega E_1 - E_1^2) + n^2m^2 + \frac{1-n^4}{4}\,\omega^2\right] dE_1. \tag{9'}$$

Integrating (9′) over E_1 between the limits m and $\omega - m$ we get the total probability of annihilation of a photon accompanied by the formation of an electron pair

$$W = \frac{\mu e^2}{4\pi\alpha\omega^2 n^5}(\omega - 2m) \times (1-n^2)\left[\frac{1}{12}(\omega-2m)^2 + \frac{1}{4}n^2\omega^2 + \frac{n^2}{1-n^2}m^2\right]. \tag{11}$$

To obtain the total probability of creation of a pair by a quantum per unit path length we must divide (11) by the phase velocity of the light, $1/n$.

In order to get the complete picture of the probability evaluated here, one must know the dispersion law for very high densities of matter. This problem was considered by the present author in reference 5. It was shown there that at very high temperatures $\kappa T > mc^2$, when the electron gas is non-degenerate (κ is Boltzmann's constant),

$$1 - n^2 \approx \frac{2.4\pi e^2}{m\omega^2}\left(\frac{mc^2}{\kappa T}\right)^2 N,$$

where N is the electron density in the medium (here in the usual units). When, however, the electron gas is degenerate ($\kappa T < mc^2$),

$$1 - n^2 \approx 3c^2 N^{2/3}/137\omega^2.$$

Using these formulae for n and taking it into account that for the process under consideration we have practically $N \sim 8\lambda_C^{-3}$, we verify easily that the last term within the square brackets in (11) is appreciably larger than the first two. The pair-creation probability is thus of the order of magnitude

$$W \approx \frac{m}{137\omega}\left(1 - \frac{2m}{\omega}\right)\lambda_C^{-1}, \tag{11'}$$

where we have omitted the factors μ, α, and n^2, which are of the order of unity.

The pair-creation probability for γ quanta (by the usual mechanism, i.e., in the field of a nucleus) is equal to $W_0 = Z^2\Phi N$, where N is the density of nuclei and $Z^2\Phi$ the total pair-creation cross section. For the electron densities considered here, the nuclei break up into separate nucleons (neutron stars) so that we must put $Z = 1$. For photon energies $\omega = 3m$, $5m$, $10m$, $20m$, and $100m$ we have[9] $\Phi/(5.8\times 10^{-28}) \approx 0.49$, 3.5, 11, 21, and 46. Taking it into account that $N \sim (\omega/c)^3$ and comparing these data with (11′), we find $W/W_0 \approx 2700$, 146, 7.66, 0.63, and 0.0023, respectively.

In the energy range $2m < \omega < 10m$ the probability W is thus appreciably larger than the probability W_0.

3. We turn now to a consideration of the single-photon electron-positron annihilation process. We have, by analogy with (4), for the matrix element of this process

$$S_{if}^{(1)} = -e\left(\frac{\mu}{2\alpha\omega n^2 V}\right)^{1/2}\int \bar{\psi}_2\gamma_\nu e^{-i\tilde{k}\tilde{x}}\psi_1\, d^4x. \tag{12}$$

Integrating (12) over the four-dimensional volume we find

$$S_{if}^{(1)} = -(2\pi)^4 eV^{-3/2}(\mu/2\alpha\omega n^2)^{1/2}\bar{u}_2 \times (-\mathbf{p}_2)\gamma_\nu u_1(\mathbf{p}_1)\,\delta(\tilde{p}_1 + \tilde{p}_2 - \tilde{k}), \tag{13}$$

where $\tilde{p}_1$ and $\tilde{p}_2$ are the four-momenta of the electron and the positron.

The matrix elements (4) and (12) are Hermitian conjugates. The summation of the square of the modulus of the matrix elements over the polarization of the photon and of the electrons leads thus to the same result for the pair creation and annihilation processes (principle of detailed balancing). The probabilities for these processes are, however, not the same since the statistical weights of the final states are different. Taking this into account we get for the transition probability per unit time the formula

$$dW' = \frac{\pi\mu e^2}{V\alpha\omega n^2}\frac{E_1E_2 + m^2 - p_1p_2\cos\theta_1\cos\theta_2}{E_1E_2}\,\delta(\tilde{k} - \tilde{p}_1 - \tilde{p}_2)\,d^3k, \tag{14}$$

where θ_1 and θ_2 have the same meaning as before and are determined by Eqs. (10). Integrating (14) over the photon momentum we get

$$dW' = \frac{\pi\mu e^2}{V\alpha\omega n^2}\frac{E_1E_2 + m^2 - p_1p_2\cos\theta_1\cos\theta_2}{E_1E_2}\,\delta(\omega - E_1 - E_2). \tag{15}$$

The formula obtained here leads to one pair of particles in the volume V under consideration, where we have assumed that the electron and the positron have strictly well defined momenta.

We determine now the probability for the annihilation of a positron of given energy E_2 by all electrons in the medium with energies within the range $(E_1, E_1 + dE_1)$ and with arbitrary directions of motion. To do this we must multiply (15) by the electron distribution function

$$VN_1(E_1, \theta, \varphi)\,dE_1\sin\theta\,d\theta\,d\varphi \tag{16}$$

and integrate over the angles. θ is here the angle between the directions of motion of the colliding particles. When integrating one must take the dependence of ω on the angle θ into account.

We assume that the angular distribution of the electrons in the medium is isotropic, which in actual fact is always true. Using (10) and (16) we get then from (15)

$$dW' = \frac{\pi\mu e^2}{2n^2}\Big[n^2m^2 + \frac{1-n^4}{4}\omega^2 - (1-n^2)E_1E_2\Big]\frac{N_1(E_1)\,dE_1}{E_1E_2p_1p_2}, \tag{17}$$

where $N_1(E_1)dE_1$ is the number of electrons with energies within the range $(E_1, E_1 + dE_1)$ per unit volume. The total probability that a positron with energy E_2 is annihilated per unit time is equal to

$$W' = \frac{\pi\mu e^2}{2E_2p_2}\int\Big[n^2m^2 + \frac{1-n^4}{4}\omega^2 - (1-n^2)E_1E_2\Big]\frac{N_1(E_1)\,dE_1}{E_1p_1n^2}. \tag{18}$$

The lower limit of the integral is equal to m and the upper one to ∞ if the electron gas in the medium is non-degenerate (case of very high temperatures) or to the Fermi energy if the electron gas is strongly degenerate (case of low temperatures).

At sufficiently high temperatures, $T > 10^{10}$ °K, there will be a large number of positrons in the medium.* It is thus of interest to determine the total probability for the annihilation of an electron of given energy E_1. The corresponding equations for electrons can be obtained from (17) and (18) by replacing in them the electron distribution function by the positron distribution function $N_2(E_2)dE_2$.

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*See reference 2, p. 335.