ROTATION OF THE SPIN OF A RELATIVISTIC PARTICLE WITH A MAGNETIC MOMENT MOVING IN AN EXTERNAL FIELD

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The problem of the rotation of the spin is solved for a relativistic particle that has a magnetic dipole moment and moves in an external electromagnetic field. The angular velocity of the rotation of the spin is determined in the rest system of the particle, which is rigidly fixed to its trajectory (analog of the Frenet axis system for a four-dimensional curve). The results obtained are significant for experiments made to measure the magnetic and electric moments of elementary particles.

LET us consider the problem of the change of the polarization of a particle with arbitrary spin s and magnetic dipole moment $\mu = g(e\hbar/2mc)\mathbf{s}$ under the action of an external electromagnetic field. It is assumed that the field is a macroscopic field, i.e., that the motion of the particle in the field obeys the laws of classical mechanics:[1]

$$du_\mu/d\tau = (e/mc)\, F_{\mu\nu} u_\nu. \tag{1}$$

The spin operator is uniquely defined only with respect to spatial rotations in the rest system, and various relativistic generalizations of this operator are possible. We shall use the spin pseudovector $\hat{s}_\mu = -\frac{1}{2} i\epsilon_{\mu\nu\lambda\rho} M_{\nu\lambda} p_\rho$,[2] which coincides with the nonrelativistic spin operator in the rest system of the particle. For the average value of $\hat{s}_\mu$ one can obtain the equation (cf. reference 3)

$$ds_\mu/d\tau = (e/mc)\left[\tfrac{1}{2} g F_{\mu\nu} s_\nu + \left(\tfrac{1}{2} g - 1\right) u_\mu F_{\alpha\beta} u_\alpha s_\beta\right], \tag{2}$$

where g is the gyromagnetic ratio and u_μ is the four-velocity. The polarization is essentially a three-dimensional vector defined in the rest system of the particle. Therefore after solving the equation (2) one must translate the four-vector s_μ into the rest system. The direct solution of the equation (2) is cumbersome even in the simplest case of a uniform magnetic field. Therefore we shall find the angular velocity of the rotation of the spin directly in the rest system.

Let us introduce four axes η^α (α = 0, 1, 2, 3) rigidly connected with the trajectory (which is regarded as a curve in the four-dimensional Minkowski space). The four unit vectors η^α are the four-dimensional analog of the Frenet axes. They obey the equations

$$d\eta_\mu^\alpha/d\tau = \varkappa_{\alpha\beta}\eta_\mu^\beta, \qquad \eta_\mu^\alpha\eta_\mu^\beta = g_{\alpha\beta};$$
$$g_{\alpha\beta} = 0 \quad (\alpha \neq \beta), \qquad g_{00} = -1, \qquad g_{11} = g_{22} = g_{33} = 1. \tag{3}$$

We shall construct the vectors η^α in the following way. Suppose the vectors u_μ, $\dot{u}_\mu$, $\ddot{u}_\mu$, $\dddot{u}_\mu$ are linearly independent (the dot denotes differentiation with respect to the proper time τ). Then we set

$$\eta_\mu^\alpha = \sum_{\beta=0}^{\alpha} C_{\alpha\beta} d^\beta u_\mu / d\tau^\beta,$$

where the coefficients $C_{\alpha\beta}$ are found from the conditions of orthogonality and normalization. It follows from this that $\kappa_{\alpha\beta} \neq 0$ only for $\beta = \alpha \pm 1$, i.e., the curvature tensor has the form

$$\varkappa_0 = \varkappa_{01} = \varkappa_{10}, \qquad \varkappa_1 = \varkappa_{12} = -\varkappa_{21},$$
$$\varkappa_2 = \varkappa_{23} = -\varkappa_{32}$$

(the other components of the tensor $\kappa_{\alpha\beta}$ are all zero). For the unit vectors η_μ^α we get the following expressions:

$$\eta_\mu^0 = u_\mu, \quad \eta_\mu^1 = C_1\dot{u}_\mu, \quad \eta_\mu^2 = C_2(\ddot{u}_\mu - \tfrac{1}{2}\dot{p}p^{-1}\dot{u}_\mu - pu_\mu),$$
$$\eta_\mu^3 = C_3\varepsilon_{\mu\nu\lambda\rho}u_\nu\dot{u}_\lambda\ddot{u}_\rho, \tag{4}$$

where

$$C_1 = p^{-1/2}, \quad C_2 = (p^2 + q - \tfrac{1}{4}\dot{p}^2p^{-1})^{-1/2}, \quad C_3 = -iC_1C_2;$$
$$p = \dot{u}_\mu\dot{u}_\mu, \qquad q = \ddot{u}_\mu\ddot{u}_\mu. \tag{5}$$

Substituting Eq. (5) in Eq. (3), we find the connection of C_i with the curvatures $\kappa_{\alpha\beta}$:

$$C_1 = \varkappa_0^{-1}, \quad C_2 = \varkappa_0^{-1}\varkappa_1^{-1}, \quad C_3 = -i\varkappa_0^{-2}\varkappa_1^{-1}. \tag{6}$$

The spin four-vector s_μ can be expressed in terms of the vectors η_μ^i (i = 1, 2, 3), since $s_\mu u_\mu = 0$. Let

$$s_\mu = \sum_{i=1}^{3} \sigma_i \eta_\mu^i;$$

it follows from Eq. (2) that

$$d\boldsymbol{\sigma}/d\tau = [\boldsymbol{\Omega}\times\boldsymbol{\sigma}],$$

where

$$\Omega_{ij} = -\varkappa_{ij} - g\,(e/2mc)\,F_{\mu\nu}\eta_\mu^i\eta_\nu^j, \tag{7}$$

and $\boldsymbol{\Omega}$ is the vector of the angular velocity of the rotation of the polarization $\boldsymbol{\sigma}$ in the rest system we have chosen (the system of axes η^i). From Eqs. (4) and (6) it is clear that k_{ij} and η_μ^i can be expressed in terms of the four-velocity u_μ and its derivatives, which are determined by the field $F_{\mu\nu}$. Therefore to find $\boldsymbol{\Omega}$ we need first to determine the trajectory of the particle, i.e., to solve (1).

Let us consider the case in which $\mathbf{E} = 0$ but $\mathbf{H}$ is an arbitrary function of the coordinates (and the time). Using the equations of motion (1), we can obtain the following expression for the components of $\boldsymbol{\Omega}$:

$$\Omega_1 = u_0\Delta^{-2}\,[\dot{\varepsilon}\cos\theta + (1+\varepsilon\sin\theta)\,\dot{\theta}],$$
$$\Omega_2 = -\tfrac{1}{2}g\Omega_L\Delta^{-1}u_0\varepsilon\,\mathrm{tg}\,\theta,$$
$$\Omega_3 = \left(\tfrac{1}{2}g - 1\right)\Omega_L\Delta - \tfrac{1}{2}g\Omega_L\Delta^{-1}\varepsilon\,(\varepsilon+\sin\theta); \tag{8}$$

here

$$\varepsilon = \frac{mc}{e}\frac{[\dot{\mathbf{H}}\times\mathbf{H}]\,\boldsymbol{\nu}}{H^3\cos^2\theta}, \quad \Omega_L = \left|\frac{eH}{mc}\right|,$$
$$\Delta = (1 + u^2\cos^2\theta + 2\varepsilon\sin\theta + \varepsilon^2)^{1/2},$$

$\boldsymbol{\nu}$ is the direction of the velocity of the particle, and θ is the "pitch angle" of the trajectory, defined by the condition $\boldsymbol{\nu}\cdot\mathbf{H} = \sin\theta$. It is obvious that ϵ is a parameter that characterizes the degree of inhomogeneity of the magnetic field.

Starting from the formulas (5), we can determine the positions of the unit vectors η_μ^i in the laboratory coordinate system (l. s.) at each instant of time:

$$\eta_\mu^1 = \{\varkappa_0^{-1}\,[\boldsymbol{\Omega}_L\times\mathbf{u}];\ 0\},$$
$$\eta_\mu^2 = \varkappa_0^{-1}\varkappa_1^{-1}\{[\dot{\boldsymbol{\Omega}}_L\times\mathbf{u}] + (\boldsymbol{\Omega}_L\times\mathbf{u})\,\boldsymbol{\Omega}_L - (1+u^2\cos^2\theta)\,\Omega_L^2\mathbf{u}$$
$$-\,([\boldsymbol{\Omega}_L\times\mathbf{u}]\,[\dot{\boldsymbol{\Omega}}_L\times\mathbf{u}]/|[\boldsymbol{\Omega}_L\times\mathbf{u}]|^2)\,[\boldsymbol{\Omega}_L\times\mathbf{u}];\ -i\varkappa_0^2u_0\},$$
$$\eta_\mu^3 = \{-u_0\Omega_L\varkappa_1^{-1}(\xi\mathbf{H}/H + \varepsilon\mathbf{u}/u);\ -iu\Omega_L\varkappa_1^{-1}(\varepsilon+\sin\theta)\}. \tag{9}$$

The curvatures κ_0, κ_1 that appear here are given by

$$\varkappa_0 = |[\boldsymbol{\Omega}_L\times\mathbf{u}]|, \quad \varkappa_1 = \Omega_L\Delta; \quad \xi = e/|e|, \quad \boldsymbol{\Omega}_L = -(e/mc)\,\mathbf{H}.$$

In a uniform magnetic field we must set $\epsilon = 0$, $\theta = \text{const}$, so that we have

$$\boldsymbol{\Omega} = \{0,\ 0,\ (\tfrac{1}{2}g - 1)\,\Omega_L\sqrt{1+u^2\cos^2\theta}\}.$$

The electron then moves in a spiral with a constant pitch angle θ ($u_z = u\sin\theta$, if the z axis is the axis of the spiral). It follows from Eq. (9) that the vectors η^i are in fixed positions with respect to the accompanying Frenet vectors **t**, **n**, **b** of the electron trajectory. In the rest system defined by the axes η^i the polarization vector precesses around the axis η^3, and the angular velocity is zero for $g = 2$. For the vector η^3 in the laboratory system we have

$$\eta_\mu^3 = \left\{\frac{u_0}{\sqrt{1+u^2\cos^2\theta}}(-\mathbf{t}\sin\theta + \mathbf{b}\cos\theta);\ -i\frac{u\sin\theta}{\sqrt{1+u^2\cos^2\theta}}\right\}.$$

If the polarization was longitudinal initially (at t = 0), then after a certain time it becomes transverse. This time is given by

$$T = \frac{2u_0}{(g/2-1)\,\Omega_L\sqrt{1+u^2\cos^2\theta}} \times \arcsin\left(\sqrt{1+u^2\cos^2\theta}/\sqrt{2}\,u_0\cos\theta\right)$$

(from the point of view of clocks in the laboratory system).

The polarization will be longitudinal at the times

$$2\pi n\left[\left(\tfrac{1}{2}g-1\right)\Omega_L\sqrt{1+u^2\cos^2\theta}\right]^{-1} \qquad (n = 0, 1, 2, \ldots)$$

and transverse at the times

$$2\pi n\left[\left(\tfrac{1}{2}g-1\right)\Omega_L\sqrt{1+u^2\cos^2\theta}\right]^{-1} \pm T.$$

Assuming $u \ll 1$, $\theta \ll 1$ and neglecting quantities of the orders $u^2\theta^2$ and θ^4, we get

$$T \approx (\pi/(g-2)\,\Omega_L)\,(1 + 0.64\theta^2),$$

i.e., in the nonrelativistic region the time of transition from longitudinal to transverse polarization does not depend on the speed of the particle. Measurement of this time gives a possibility for determining the gyromagnetic ratio g.[4] If high accuracy is to be obtained the angle of rotation of the spin must be large. One achieves this by bounding the region of uniform field by magnetic traps at its two ends, so that the particle is reflected and passes back and forth between the traps many times, while being essentially in a uniform field. Then, however, one must examine the additional rotation of the spin on reflection from a trap, and the effects of unavoidable inhomogeneities of the field. This can be done conveniently by starting with the equations (8).

As is well known, the existence of an electric dipole moment of an elementary particle is in contradiction with CP or T invariance. With a slight modification, the experiment considered here gives a possibility for an experimental test of the vanishing of the electric moment.[4] Let us write down the equation of motion of the spin for a particle that has both magnetic and electric dipole moments:

$$\frac{ds_\mu}{d\tau} = \frac{e}{mc}\left\{\left(\frac{g}{2} F_{\mu\nu} + \frac{f}{2}\widetilde{F}_{\mu\nu}\right)s_\nu + u_\mu\left[\left(\frac{g}{2} - 1\right)F_{\alpha\beta} + \frac{f}{2}\widetilde{F}_{\alpha\beta}\right]u_\alpha s_\beta\right\}. \quad (10)$$

Equation (10) is the only possible equation linear in s_μ that together with Eq. (1) satisfies the conditions

$$d(s_\mu u_\mu)/d\tau = d(s_\mu^2)/d\tau = 0.$$

In the nonrelativistic case Eq. (10) has the form

$$ds/dt = -(e/2mc)\{g[\mathbf{H}\times\mathbf{s}] + f[\mathbf{E}\times\mathbf{s}]\},$$

from which it is clear that the magnetic and electric moments of the particle are $\boldsymbol{\mu} = g(e\hbar/2mc)\mathbf{s}$, $\mathbf{p} = f(e\hbar/2mc)\mathbf{s}$. The tensor $\widetilde{F}_{\mu\nu}$ that appears in Eq. (10) is the dual tensor of the electromagnetic field:

$$\widetilde{F}_{\mu\nu} = \tfrac{1}{2} i e_{\mu\nu\lambda\rho} F_{\lambda\rho}.$$

Let the unit vectors η_μ^α be determined as before from Eqs. (5) and (6). The angular velocity of the rotation of the spin in the axes η^i is given by

$$\Omega_{ij} = -[\varkappa_{ij} + (e/2mc)(gF_{\mu\nu} + f\widetilde{F}_{\mu\nu})\eta_\mu^i\eta_\nu^j]. \quad (11)$$

We get from this for the case of a uniform magnetic field

$$\Omega = \{-\tfrac{1}{2} f\Omega_L u\cos\theta,\ 0,\ (\tfrac{1}{2}g - 1)\Omega_L\sqrt{1 + u^2\cos^2\theta}\}.$$

The electric dipole moment leads to the appearance of a component Ω_1, i.e., to an additional rotation of the spin around the principal normal **n** of the Frenet axis system. If, however, $|f| \ll |g/2 - 1|$, the component Ω_3 that comes from the magnetic moment is much larger than Ω_1 and makes measurement of f difficult. A method in which the rotation of the spin because of the anomalous magnetic moment is compensated seems more attractive.[4]

For this purpose one has only to apply in addition to the uniform magnetic field (along the z axis) a radial electric field $E_r = a/r$ (in a cylindrical coordinate system). It can be shown that as before it is possible for the particle to move along a spiral, but with altered frequency and radius. The positions of the axes η^i in the cylindrical coordinate system are then given by the formulas

$$\eta^1 = \{-1,\ 0,\ 0;\ 0\},$$

$$\eta^2 = \{0,\ \xi\Delta,\ -\Delta^{-1}u^2\sin\theta\cos\theta;\ -i\Delta^{-1}u_0 u\cos\theta\},$$

$$\eta^3 = \{0,\ 0,\ -\xi u_0\Delta^{-1};\ -i\xi\Delta^{-1}u\sin\theta\},$$

$$\Delta = (1 + u^2\cos^2\theta)^{1/2}, \quad \xi = e/|e|.$$

Here we have taken the axes in the order r, φ, z, t, and the z axis is directed along the magnetic field H.

In these axes the angular velocity of the spin precession is

$$\Omega = \left\{-f\frac{\Omega_L u\cos\theta}{2(1+x)},\ g\Omega_L\xi\frac{x}{1+x}\frac{u^2\sin\theta\cos\theta}{2u_0\Delta},\ \times\frac{\Omega_L}{1+x}\left[\left(\frac{g}{2} - 1\right)\Delta + \frac{gx}{2\Delta}\right]\right\},$$

where $x = (ea/mc)(u_0/u^2\cos^2\theta)$. If a is measured in volts, then $x = 1.96 \cdot 10^{-6} a u_0/u^2\cos^2\theta$. By a suitable choice of the electric field we can make the component Ω_3 equal to zero; to do this we must choose $x = x_0 = -(g-2)g^{-1}(1 + u^2\cos^2\theta)$. On each reflection from a trap the angle θ changes sign. We note that Ω_1 is an even function of θ, and Ω_2 is an odd function of θ; therefore after one complete cycle of motion between the traps the rotation of the spin comes only from the component

$$\Omega_1 = -f\frac{\Omega_L u\cos\theta}{2(1+x_0)} = -\frac{fg}{4}\frac{\Omega_L u\cos\theta}{1 - (g/2 - 1)u^2\cos^2\theta}.$$

Measurement of the spin precession frequency Ω_1 offers a method for determining the gyroelectric ratio f.

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