ON THE CALCULATION OF SCATTERING PHASE SHIFTS

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A method for the calculation of the phase shifts of particles in a centrally symmetric potential field is proposed. The case where the phase shift can be expressed in the form of a series in powers of a constant characterizing the dimensions of the potential well is considered.

As is well known, the problem of the scattering of particles in a centrally symmetric potential field reduces to solving an equation of the type

$$d^2u/dx^2 + [k^2 + W(x) + U(x)]\,u(x) = 0 \qquad (1)$$

with the conditions

$$u(0) = 0, \quad u_{as} = \sin(kx + \delta(x) + \eta).$$

Here we assume that $W(x)$ is a function of x for which [with $U(x) \equiv 0$] the two independent solutions with the asymptotic behavior[1]

$$v_{1\,as} = \sin(kx + \delta(x)), \qquad v_{2\,as} = \cos(kx + \delta(x))$$

are known. An example of such a function is

$$W(x) = -l(l+1)x^{-2} - 2\alpha k x^{-1};$$

in this case

$$\delta(x) = -\tfrac{1}{2}\pi l + \arg\Gamma(l + 1 + i\alpha) - \alpha\ln 2kx.$$

Problems of such a type with a complex potential occur, in particular, in the nuclear optical model calculations. For small k this involves complicated numerical computations.[2,3] A general method of solving these problems was developed by Nemirovskiĭ.[4] Below we shall propose a different method in which the phase η is sought in the form of a power series in U_0 [$U(x) = U_0 f(x)$]. The method can be applied successfully in the region of intermediate and high energies.

We shall seek the solution of Eq. (1) in the form

$$u(x) = [1 - D(x)]\left\{v_1(x)\cos\int_0^x \Phi(x')\,dx' + v_2(x)\sin\int_0^x \Phi(x')\,dx'\right\}, \qquad (2)$$

Assuming $D(\infty) = 0$ and $\Phi(\infty) = 0$, we obtain

$$\eta = \int_0^\infty \Phi(x)\,dx.$$

Substituting expression (2) in Eq. 1 and setting the coefficients of the sine and cosine equal to zero, we obtain, after some transformation, equations for the determination of D and Φ:

$$D'' - \beta'\beta^{-1}D' + \beta^2[(1-D)^{-3} + D - 1] = U(x)(1-D),$$
$$\Phi = \beta[(1-D)^{-2} - 1], \qquad (3)$$

where $\beta = k(v_1^2 + v_2^2)^{-1}$. With $W(x) = -l(l+1)x^{-2}$ we have for $l = 0, 1$

$$\beta_0 = k, \qquad \beta_1 = k^3x^2(1 + k^2x^2)^{-1}.$$

Equation (3) can be written in the integral form

$$D(x) = \int_\infty^x K(x, x')\{U(x')(1 - D(x')) - \beta^2(x')[(1 - D(x'))^{-3} - 3D(x') - 1]\}dx',$$

$$K(x, x') = \tfrac{1}{2}\sin\left(2\int_{x'}^x \beta(x_1)\,dx_1\right)\beta^{-1}(x'). \qquad (4)$$

If $|D(x)| \ll 1$, $(1-D)^{-3}$ can be expanded into a series in powers of D:

$$(1-D)^{-3} - 3D - 1 = \frac{1}{2}\sum_{n=2}^\infty (n+1)(n+2)D^n(x).$$

In first approximation we obtain

$$D_1(x) = \int_\infty^x K(x, x')\,U(x')\,dx'.$$

If $|D_1(x)| \ll 1$, we can seek $D(x)$ in the form of a power series in U_0:

$$D(x) = \sum_{n=1}^\infty D_n(x). \qquad (5)$$

Introducing the recurrence relations

$$D_{n1} = D_n, \qquad D_{nm} = \sum_{q=1}^{n-m+1} D_q D_{n-q,\,m-1}$$

(D_{nm} is proportional to U_0^n), we obtain equations for the determination of the $D_n(x)$ ($n > 1$):

$$D_n(x) = -\int_{\infty}^{x} K(x, x') \Big\{ U(x') D_{n-1}(x') + \frac{1}{2} \beta^2(x') \sum_{m=2}^{n} (m+1)(m+2) D_{nm}(x') \Big\} dx'.$$

$\Phi(x)$ can also be expressed in the form of a power series in U_0:

$$\Phi(x) = \beta(x) \sum_{n=1}^{\infty} \sum_{m=1}^{n} (m+1) D_{nm}(x). \qquad (6)$$

If the series (6) converges in the interval $(0, \infty)$ we have

$$\eta = \sum_{n=1}^{\infty} \int_{0}^{\infty} \beta(x) \sum_{m=1}^{n} (m+1) D_{nm}(x)\, dx.$$

For a potential well of the form $U(x) = U_0 e^{-x}$ we find

$$D_1(x) = \frac{U_0}{1+4k^2} e^{-x}, \qquad D_2(x) = \frac{-U_0(1+10k^2)}{(1+4k^2)^2(4+4k^2)} e^{-2x}, \ldots$$

Thus we obtain for $U_0 = 2$, $k^2 = 1$, which corresponds to the scattering of a neutron with energy 20.5 Mev by a potential well of depth 41 Mev which decreases by the factor e^{-1} at the distance 10^{-13} cm,

$$D(x) = 0.4\, e^{-x} - 0.22 e^{-2x} + 0.06585 e^{-3x} - 0.00331 e^{-4x} - 0.00244 e^{-5x} - 0.00053 e^{-6x} + 0.00046\, e^{-7x} + \ldots,$$

$$\eta = 0.8000 + 0.0200 - 0.0468 + 0.0005 + 0.0078 - 0.0003 - 0.0016 + \ldots \approx 0.780.$$

For larger k the convergence is faster; in the region of small k, on the other hand, $D(x)$ can be obtained in the form (5) only in some interval (x_0, ∞), if we use the above-mentioned form of the potential well. Beyond that the problem must be solved numerically.

If $U(x) = \text{const}$ for $x < a$, we can use the method of joining of the wave functions. In this case we seek $u(x)$ for $x > a$ in the form

$$u(x) = [1 - D(x)] \Big\{ v_1 \cos\Big(\eta_0 + \int_a^x \Phi(x')\, dx'\Big) + v_2 \sin\Big(\eta_0 + \int_a^x \Phi(x')\, dx'\Big) \Big\};$$

η_0 is determined by the boundary conditions at $x = a$.

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[2] P. É. Nemirovskiĭ, JETP **32**, 1143 (1957), Soviet Phys. JETP **5**, 932 (1957).

[3] Luk'yanov, Orlov, and Turovtsev, JETP **35**, 750 (1958), Soviet Phys. JETP **8**, 521 (1959).

[4] P. É. Nemirovskiĭ, JETP **30**, 551 (1956), Soviet Phys. JETP **3**, 484 (1956).

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