THE SINGULAR POINTS OF SOME FEYNMAN DIAGRAMS

V. A. KOLKUNOV, L. B. OKUN', and A. P. RUDIK

Submitted to JETP editor August 10, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **38**, 877-881 (March, 1960)

The positions of the singular points are determined for the two Feynman diagrams shown in Figs. 1 and 7.

BECAUSE of the recent interest in the singularities of scattering amplitudes it seems worthwhile to examine in detail the method of finding the singular points of complicated Feynman diagrams. We here use a method developed earlier[1,2] to calculate the singular points of the two Feynman diagrams shown in Figs. 1 and 7. We shall look for those singular points of these diagrams for which the values of all the integration parameters α_i in the corresponding Feynman integrals are different from zero (cf. reference 2).

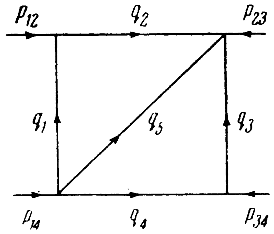

FIG. 1

1. Let us consider the Feynman diagram shown in Fig. 1. We shall assume that the masses of all the particles involved in this diagram are equal, and take them equal to unity. Then this diagram is a special case of the scattering of scalar mesons. By symmetry we have the following equalities, which greatly simplify the further calculations:

$$q_{54} = q_{25}, \quad q_{53} = q_{15}, \tag{1}$$

where $q_{ik} = q_i q_k$ is the scalar product of the four-vectors q_i and q_k.

The singular points of the Feynman diagram of Fig. 1 can be found by two methods: either by a trigonometric computation based on the properties of the scheme corresponding to the diagram of Fig. 1, or by an analysis of the determinants corresponding to the condition $\Sigma_i \alpha_i q_i = 0$ for each of the contours of the diagram of Fig. 1.

The Trigonometric Method. The scheme corresponding to the diagram of Fig. 1 is shown in Fig. 2, where we have introduced the usual notations $W^2 = (p_{12} + p_{14})^2$ and $Q^2 = (p_{12} + p_{23})^2$.

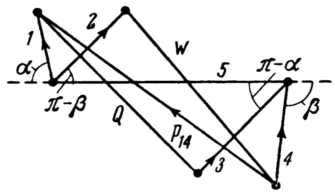

FIG. 2

As has been shown previously,[2] the lines 3, 4, and 5 must lie in one plane, and the lines 1, 2, and 5 in some other plane. We shall denote the angle between these two planes by φ. Then from the relation $p_{14}^2 = (q_1 + q_4 + q_5)^2 = 1$ we have

$$(1 + \cos\alpha)(1 + \cos\beta) = \sin\alpha \sin\beta \cos\varphi; \tag{2}$$

from the relation $W^2 = (q_2 + q_4 + q_5)^2$,

$$W^2 = 1 + 2(1 + \cos\alpha)^2 - 2\sin^2\alpha \cos\varphi; \tag{3}$$

and from the relation $Q^2 = (q_1 + q_3 + q_5)^2$,

$$Q^2 = 1 + 2(1 + \cos\beta)^2 - 2\sin^2\beta \cos\varphi. \tag{4}$$

Substituting the value of $\cos\varphi$ obtained from Eq. (2) in Eqs. (3) and (4), we get the following parametric representation of the function $F(W^2, Q^2) = 0$:

$$Q^2 = 1 + 2(1 + \cos\beta)^2 - 2(1 + \cos\alpha)(1 + \cos\beta)\sin\beta/\sin\alpha,$$
$$W^2 = 1 + 2(1 + \cos\alpha)^2 - 2(1 + \cos\alpha)(1 + \cos\beta)\sin\alpha/\sin\beta, \tag{5}$$

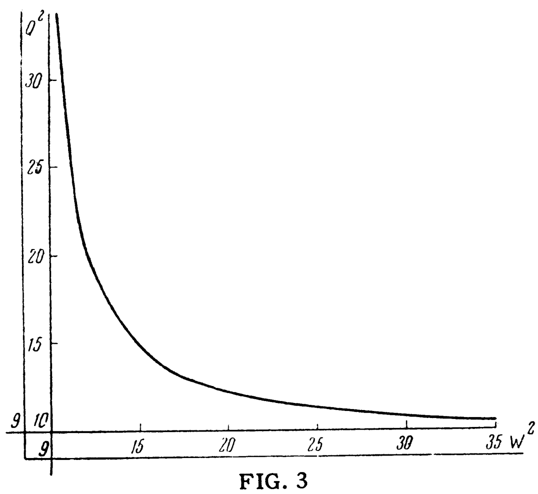

FIG. 3

where the angles α and β are connected by the relation

$$\alpha - \beta = -\pi/3. \tag{6}$$

The curve $F(W^2, Q^2) = 0$ is shown in Fig. 3.

The value $\alpha = 0$ corresponds to the asymptote $Q^2 \to \infty$, $W^2 \to 9$; $\beta = 0$ corresponds to the asymptote $W^2 \to \infty$, $Q^2 \to 9$. These asymptotes arise from the two dipole patterns of Figs. 4 and 5, re-

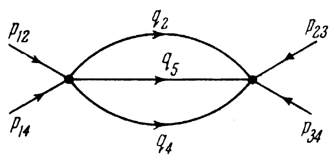

FIG. 4

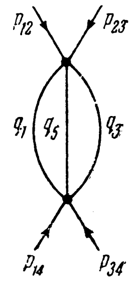

FIG. 5

spectively, which are obtained by reductions of the diagram of Fig. 1. To the point $W^2 = Q^2$ there correspond the values $\alpha = -\beta = -\pi/6$; at this point $W^2 = Q^2 = 4(2 + 3^{1/2})$.

<u>The Determinant Method</u>. From Eq. (1) and the conservation laws of the four-momenta one easily gets the following relations:

$$W^2 = 3 + 2q_{24} + 4q_{25}, \quad Q^2 = 3 + 2q_{13} + 4q_{15}, \tag{7}$$

$$q_{14} = q_{23} = -(1 + q_{15} + q_{25}). \tag{8}$$

We then use the equation $\alpha_1 q_1 + \alpha_2 q_2 - \alpha_5 q_5 = 0$. Multiplying it successively by the four-vectors (q_1, q_2, q_5), (q_1, q_2, q_3), and (q_1, q_2, q_4), we get the following three determinants:

$$\begin{vmatrix} 1 & q_{12} & q_{15} \\ q_{12} & 1 & q_{25} \\ q_{15} & q_{25} & 1 \end{vmatrix} = 0, \tag{9'}$$

$$\begin{vmatrix} 1 & q_{12} & q_{15} \\ q_{12} & 1 & q_{25} \\ q_{13} & q_{23} & q_{53} \end{vmatrix} = 0, \tag{9''}$$

$$\begin{vmatrix} 1 & q_{12} & q_{15} \\ q_{21} & 1 & q_{25} \\ q_{14} & q_{24} & q_{54} \end{vmatrix} = 0. \tag{9'''}$$

Since q_{12} is known ($q_{12} = 1/2$), the condition (9′) gives a connection between q_{15} and q_{25}. Figure 6 shows the curve $\varphi(q_{15}, q_{25}) = 0$, which is an

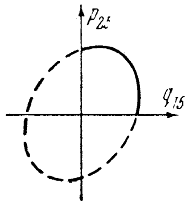

FIG. 6

ellipse with the semiaxes $(3/2)^{1/2}$ and $(1/2)^{1/2}$. In virtue of the condition that the Feynman parameters α_i are positive, only the part of the ellipse with $q_{25} > 0$, $q_{15} > 0$ is used.

Solving Eq. (9″) for q_{13} and Eq. (9‴) for q_{24} and using Eq. (8), after substitution in Eq. (7) we get

$$W^2 = 9 + 4(q_{25} - 1) - 2\left(\tfrac{3}{2} + q_{15} + q_{25}\right)\frac{q_{25}/2 - q_{15}}{q_{25} - q_{15}/2},$$

$$Q^2 = 9 + 4(q_{15} - 1) - 2\left(\tfrac{3}{2} + q_{15} + q_{25}\right)\frac{q_{25} - q_{15}/2}{q_{25}/2 - q_{15}}, \tag{10}$$

where q_{15} and q_{25} are connected by the condition (9′). The expressions (10) with the condition (9′) are equivalent to the expressions (5) with the condition (6).

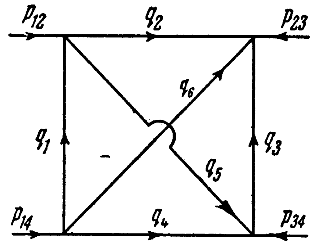

FIG. 7

2. As a second example we shall consider one of the Feynman diagrams for the scattering of pseudoscalar mesons (Fig. 7). The masses of all particles in the diagram of Fig. 7 are unity. This diagram is interesting because it is of the same order as the diagrams considered by Mandelstam for π-π scattering[3] (cf. Figs. 8 and 9).

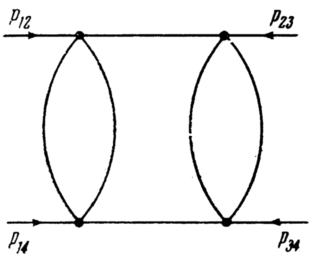

FIG. 8

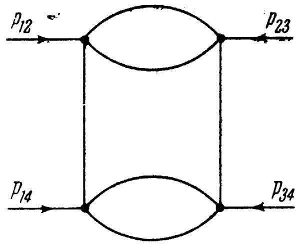

FIG. 9

To determine the singular point of the diagram of Fig. 7 it is more convenient to use the determinant method. From the conditions of symmetry it is easy to get the following equalities:

$$q_{15} = q_{35} = -q_{16} = -q_{36}, \quad q_{25} = q_{45} = q_{26} = q_{46},$$

$$q_{12} = q_{34} = -q_{14} = -q_{23}. \tag{11}$$

Using these relations, we get the following expressions for $W^2 = (p_{12} + p_{14})^2$ and $Q^2 = (p_{12} + p_{23})^2$:

$$W^2 = 4 + 2q_{24} + 8q_{25} + 2q_{56},$$

$$Q^2 = 4 + 2q_{13} - 8q_{15} - 2q_{56}. \tag{12}$$

Let us now determine the connections between the scalar products that occur in the right members of the expressions (12). From the relation $p_{12}^2 = (q_2 + q_5 - q_1)^2 = 1$ we have

$$q_{12} = 1 - q_{15} + q_{25}. \tag{13}$$

We then use the equation $\alpha_1 q_1 + \alpha_2 q_2 - \alpha_6 q_6 = 0$. By multiplying this equation successively by (q_1, q_2, q_6), (q_1, q_2, q_4), (q_1, q_2, q_3), and (q_1, q_2, q_5), we obtain determinants which we denote respectively by Δ_{126}, Δ_{124}, Δ_{123}, and Δ_{125}. Each of these determinants is equal to zero. Using Eqs. (13) and (11), we get from Δ_{126} a connection between q_{25} and q_{15}:

$$q_{25}^2 (1 + q_{15}) + q_{25} (1 - q_{15}^2) - q_{15} (1 - q_{15}) = 0. \tag{14}$$

In solving the equation (14) one must remember the condition $\alpha_i > 0$.

From $\Delta_{124} = 0$ we get an expression for q_{24}:

$$q_{24} = 1 - 2q_{12} (q_{12}q_{25} + q_{15})/(q_{25} + q_{12}q_{15}), \tag{15}$$

from $\Delta_{123} = 0$ an expression for q_{13}:

$$q_{13} = 1 - 2q_{12} (q_{25} + q_{12}q_{15})/(q_{12}q_{25} + q_{15}), \tag{16}$$

and from $\Delta_{125} = 0$ an expression for q_{56}:

$$q_{56} = (q_{25}^2 - q_{15}^2)/(1 - q_{12}^2). \tag{17}$$

In the expressions (15) — (17), q_{12} must be expressed in terms of q_{15} and q_{25} as shown in Eq. (13).

Thus by substituting Eqs. (15), (16), and (17) in Eq. (12), we get the following one-parameter form of the curve $F(W^2, Q^2) = 0$:

$$W^2 = 6 - 4q_{12}\frac{q_{12}q_{25} + q_{15}}{q_{25} + q_{12}q_{15}} + 8q_{25} + 2\frac{q_{25}^2 - q_{15}^2}{1 - q_{12}^2},$$

$$Q^2 = 6 - 4q_{12}\frac{q_{25} + q_{12}q_{15}}{q_{12}q_{25} + q_{15}} - 8q_{15} - 2\frac{q_{25}^2 - q_{15}^2}{1 - q_{12}^2}, \tag{18*}$$

where q_{12}, q_{15}, and q_{25} are connected by the relations (13) and (14).

*The expressions for W^2 and Q^2 can be transferred to the form

$$W^2 = 6 + 8q_{25} - 4\frac{1 + q_{25}}{1 + q_{15}} + 2\frac{q_{15} + q_{25}}{q_{15} - q_{25} - 2};$$

$$Q^2 = 6 - 8q_{15} - 4\frac{1 - q_{15}}{1 - q_{25}} - 2\frac{q_{15} + q_{25}}{q_{15} - q_{25} - 2}.$$

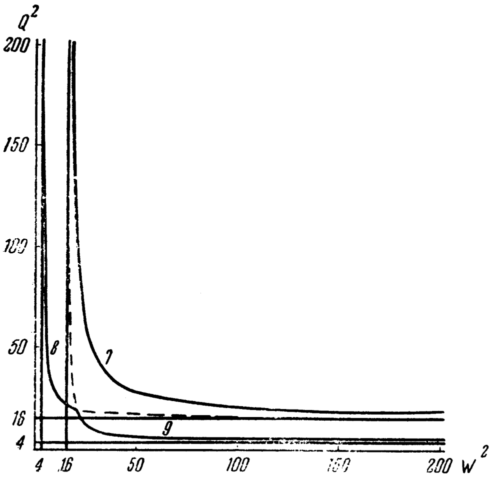

FIG. 10

Figure 10 shows the curve of $F(W^2, Q^2) = 0$. To the asymptote $Q^2 \to \infty$, $W^2 \to 16$ there correspond the values $q_{16} \to \infty$, $q_{24} = q_{25} = q_{56} = 1$, and to the asymptote $W^2 \to \infty$, $Q^2 \to 16$ the values $q_{25} \to \infty$ and $q_{13} = -q_{15} = -q_{56} = 1$. These asymptotes are obtained as the singular points of the dipole patterns reduced from the diagram of Fig. 7. To the point $W^2 = Q^2$ there correspond the values $q_{25} = -q_{15} = (1 + 5^{1/2})/2$, $q_2 = 2 + 5^{1/2}$. At this point $W^2 = Q^2 = 18 + 8 \times 5^{1/2}$.

In Fig. 10 curve 7 corresponds to the diagram of Fig. 7, and curves 8 and 9 correspond respectively to the diagrams of Figs. 8 and 9. As can be seen from Fig. 10, curve 7 lies much higher than curves 8 and 9.

The writers express their gratitude to Academician L. D. Landau for his interest in this work.

[1] L. D. Landau, JETP **37**, 62 (1959), Soviet Phys. JETP **10**, 45 (1960).

[2] L. B. Okun' and A. P. Rudik, Nuclear Phys. (in press).

[3] G. Chew and S. Mandelstam, Preprint.

Translated by W. H. Furry
167