DEPENDENCE OF THE PARAMETERS OF THE NUCLEAR POTENTIAL ON THE NUMBER OF PARTICLES

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The dependence of the oscillator potential parameters on the number of nucleons is determined starting from the assumption that the mean energy per nucleon in the nucleus is constant. The possible effect of three-particle interactions is considered.

THE present work is concerned with the problem of how well the shell model describes the integral properties of heavy nuclei, especially the separation energies Q of the last nucleon and the mean energy per nucleon E_m.

It is well known (see, for example, reference 1) that E_m is approximately constant for all nuclei. (E_m decreases slowly through the entire periodic system, from $E_m = 8.8$ Mev for $A = 55$, to $E_m \cong 7.6$ Mev for $A = 238$.) Therefore, to a rather high degree of accuracy, one can consider that*

$$\partial E_m / \partial A = 0. \tag{1}$$

For a system consisting of two-particle interactions, Eq. (1) is equivalent to

$$E_m = E_F, \tag{2}$$

where E_F is the Fermi energy of the system.

In order to demonstrate Eq. (2) in general, it is necessary to define E_F. Let the Hamiltonian of the system have the form

$$H_A = -\sum_{i=1}^{A} (\hbar^2/2m)\nabla_i^2 + \frac{1}{2}\sum_{i,k=1}^{A} U_{ik}, \tag{3}$$

$$H_A\psi = AE_m\psi. \tag{4}$$

We define E_F as

$$E_F = \Big(\psi\Big| -(\hbar^2/2m)\nabla_A^2 + \sum_{k=1}^{A-1} U_{Ak}\Big|\psi\Big),$$

where it is assumed that all

$$E_i = (\psi| -(\hbar^2/2m)\nabla_i^2 + \sum U_{ik}|\psi) \leqslant E_F.$$

Then, differentiating Eq. (4) with respect to A, multiplying by ψ^* and integrating over all A coordinates, we find that

$$E_m = E_F + \frac{1}{2}\Big(\psi\Big|\sum_{i,k}\partial U_{ik}/\partial A\Big|\psi\Big), \tag{5}$$

from which Eq. (2) follows if $\partial U_{ik}/\partial A = 0$.

We will illustrate Eq. (2) using the Hartree self-consistent field as an example:

$$E_m = \frac{1}{A}\Big(\sum_{i=1}^{A} T_i + \frac{1}{2}\sum_{i,k=1}^{A}\bar{U}_{ik}\Big),$$

$$T_i = -\int\psi_i^*(\hbar^2/2m)\nabla^2\psi_i\,d\tau_i,$$

$$\bar{U}_{ik} = \int\psi_i^*\psi_k^* U(\mathbf{r}_i - \mathbf{r}_k)\psi_i\psi_k\,d\tau_i\,d\tau_k.$$

From Eq. (1) it follows that

$$\frac{E_m}{A} = \frac{1}{A}\Big(T_A + \sum_{i=1}^{A-1}\bar{U}_{iA}\Big) + \frac{1}{A}\Big(\sum_i\frac{\partial T_i}{\partial A} + \frac{1}{2}\sum_{ik}\frac{\partial\bar{U}_{ik}}{\partial A}\Big),$$

but in so far as

$$\Big[-\frac{\hbar^2\nabla^2}{2m} + \sum_i\int\psi_i^* U\psi_i\,d\tau_i\Big]\psi_k = E_k\psi_k, \qquad \int\psi_k^*\psi_k\,d\tau_k = 1,$$

we have

$$\frac{1}{A}\Big(\sum_i\frac{\partial T_i}{\partial A} + \frac{1}{2}\sum_{i,k}\frac{\partial\bar{U}_{ik}}{\partial A}\Big) = 0,$$

which means that*

$$E_m = E_A = E_F.$$

Such a conclusion would be unjustified if tertiary collisions were present, since then U would depend upon A.

1. We assume that the potential well in the nucleus results from two-particle collisions only. Since we are interested in qualitative aspects of the problem, we consider, for simplicity, that the self-consistent potential is an oscillator well. Then the total number of particles in the system is equal to

$$A = \frac{1}{2}\sum_{n=0}^{N}(n+1)(n+2) + P_{N+1}, \tag{6}$$

*It can easily be seen that the conclusions below are not changed if $\partial E_m/\partial A \leq 0.01$.

*See reference 2, where it is shown that if $\partial E_m/\partial\rho = 0$, where ρ is the density of nuclear matter, then $E_m = E_F$ for arbitrary weak interaction.

where N is the principal quantum number of the last filled shell, P_{N+1} is the number of nucleons outside filled shells.

If

$$H = -(\hbar^2/2m)\nabla^2 - V_A + \tfrac{1}{2} m\omega_A^2 r^2,$$

then

$$E_m^{(A)} = \frac{3}{4A}\sum \hbar\omega_A\left(n+\frac{3}{2}\right) - \frac{1}{2}V_A \qquad (7)$$

and Eq. (1) can be written in the form*

$$\frac{3}{4}A\hbar\frac{\partial\omega_A}{\partial A}\left[\sum\left(n+\frac{3}{2}\right)+P_{N+1}\left(N+\frac{5}{2}\right)\right]$$

$$=\frac{A^2}{2}\frac{\partial V_A}{\partial A}+\frac{3}{4}\hbar\omega_A\left[\sum\left(n+\frac{3}{2}\right)\right.$$

$$\left.+P_{N+1}\left(N+\frac{5}{2}\right)-A\left(N+\frac{5}{2}\right)\right]. \qquad (8)$$

We then set

$$V_A = \hbar\omega_A\left[\left(N+\tfrac{3}{2}\right)+(1-\delta(P))\right]+\theta,$$

$$\delta(P) = \begin{cases}1 & \text{for } P=0;\\ 0 & \text{for } P\neq 0;\end{cases} \qquad (9)$$

where θ is the total energy of the last nucleon which, as will be shown, is not equal to its separation energy. Comparing V_A and V_{A+1}, we find

$$\partial V_A/\partial A = \hbar\left(N+\tfrac{5}{2}\right)\partial\omega_A/\partial A + \hbar\omega_A\delta(P) + \partial\theta/\partial A. \qquad (10)$$

We introduce the notation:

$$B_N = \tfrac{1}{6}(N+1)(N^2+5N+6),$$

$$C_N = \tfrac{1}{8}(N+1)(N^3+7N^2+16N+12),$$

with $A = P + B_N$. Then, from Eqs. (8) — (10) we find

$$\hbar\left[\tfrac{3}{4}C_N+\tfrac{3}{4}P\left(N+\tfrac{5}{2}\right)-\tfrac{1}{2}A\left(N+\tfrac{5}{2}\right)\right]\partial\omega_A/\partial A \approx \tfrac{1}{2}A\partial\theta/\partial A$$

$$+\hbar\left[\tfrac{3}{4}C_N/A+\tfrac{3}{4}P\left(N+\tfrac{5}{2}\right)/A-\tfrac{3}{4}\left(N+\tfrac{5}{2}\right)\right.$$

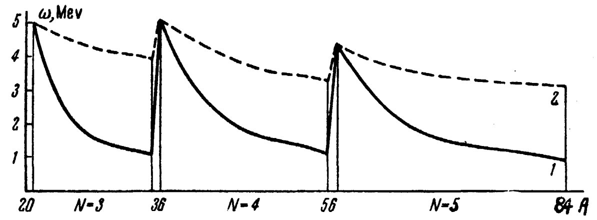
$$\left.+\tfrac{1}{2}A\delta(P)\right]\omega_A. \qquad (11)$$

Let $\partial\theta/\partial A = 0$. Assume that for a nucleus composed of closed shells +1 nucleon, $\omega_A \cong 5$ Mev. Starting from this assumption and Eq. (11) we will try to construct $\omega = \omega(A)$. For closed shells $\delta(P) = 1$ and $\partial\omega_A/\partial A > 0$, whereas for $P \neq 0$, $\partial\omega_A/\partial A > 0$. From Eq. (11) we obtain an equation determining the change in ω inside the shells in the process of filling:

$$\omega = \omega^{(1)}\frac{(B_N+P)\left[3C_N-2B_N\left(N+\frac{5}{2}\right)\right]}{B_N\left[3C_N+(P-2B_N)\left(N+\frac{5}{2}\right)\right]}, \qquad (12)$$

*If $\partial\omega/\partial A = \partial V_A/\partial A$ then Eq. (8) cannot be fulfilled.

where $\omega^{(1)}$ denotes the ω of the nucleus consisting of closed shells only. It is easy to see that the shape of the curve (12) is nearly independent of the principal quantum number. In the region of closed shells, the curve for ω undergoes a jump.*



Change in the oscillator frequency with A. Curve 1 is for $\alpha = 0$, 2 for $\alpha = 0.5$ (qualitative curve).

For $P = 0$, we obtain $\partial\omega/\partial A$ from Eq. (11). The curve constructed from Eqs. (11) and (12) with $\omega_{A=21} = 5$ Mev is given in the figure. It is of interest that in the case of two-particle interactions, $\omega \cong$ const for $\partial\theta/\partial A = 0$ for nuclei consisting of closed shells +1 nucleon. We note that the requirement of constant volume per nucleon in the nucleus leads to a change in ω qualitatively analogous to that displayed in the figure. This is quite understandable, in so far as the conditions of constancy of volume and Eq. (1) are essentially equivalent. It is of interest to see how much $\omega(A)$ is affected by taking into account several factors which were neglected earlier. It was noted above that taking the dependence of E_m on A into account does not change the curve shown in the figure.

We did not take into account the fact that there can be two nucleons in each state. As can be seen from Eqs. (11) and (12), this does not change $\partial\omega/\partial A$ for closed shells, and diminishes $\partial\omega/\partial A$ inside the shell by a factor of about two. The shape of the curve is unchanged.

Taking into account the difference of $\partial\theta/\partial A$ from zero can qualitatively change the dependence of $\omega = \omega(A)$ if, as seen from Eq. (11), $\partial\theta/\partial A < 0$ and $|\partial\theta/\partial A| > \omega_A$. A substantial change in θ from an additional particle is, however, improbable. We shall therefore assume that $|\partial\theta/\partial A| \ll \omega_A$. For $P \neq 0$, even a small value of $\partial\theta/\partial A$ can substantially change the trend of the curve. If one assumes[1] that for a value of the order of $\partial\theta/\partial A \cong \partial Q/\partial A$ and P near (but not equal to) zero, one can neglect all of the terms in Eq. (11) except $\partial\theta/\partial A$, then, as numerical evaluation shows, one obtains

$$\hbar\partial\omega/\partial A = A\left[\tfrac{3}{2}C_N+\tfrac{3}{2}P\left(N+\tfrac{5}{2}\right)\right.$$

$$\left.-A\left(N+\tfrac{5}{2}\right)\right]^{-1}\partial\theta/\partial A. \qquad (13)$$

*Use of a potential more realistic than the oscillator one would clearly lead to smoothing of the curve $\omega = \omega(A)$ in the region of shell closing.

If a substantial number of nucleons are outside closed shells, a term proportional to ω must be included in (11). In so far as ω is a discontinuous function, as we have seen earlier, one might wonder about the validity of expressions obtained by differentiating ω. However, it is clear that one can avoid differentiation by writing Eq. (1) as $E_m^{(A)} = E_m^{(A+1)}$. The result so obtained coincides with the above.

In all of the above results we neglected the presence of residual interaction. In order to include it, the total energy of the nucleus is written in the form

$$E = E_{s.c.} + \frac{1}{2}\sum_{ik} u_{ik}, \qquad (14)$$

where U_{ik} is the energy of a residual interaction between pairs, $E_{s.c.}$ is the total energy of the system described by the self-consistent field. Setting $u_{ik} \cong u$, we find a formula differing from Eq. (11) by the presence of terms on the right-hand side:

$$\frac{1}{2} A(A-1)\partial u/\partial A + \frac{1}{2}(A-1)u. \qquad (15)$$

Even a small change in the residual interaction can change the dependence of ω on A substantially. However, the applicability of the shell model in describing single-particle excitations indicates that $u \ll \hbar\omega$, at least. In this case, inclusion of the residual interaction does not change the jumps in ω in the region of closed shells; however, it might greatly change the dependence of ω on A inside the shells.

It is easy to relate $\partial V_i/\partial A$ and $\partial u_{ik}/\partial A$. To do this, one should consider that with a self-consistent field, the Hamiltonian (3) should be replaced by

$$H_A = -\sum_i \frac{\hbar^2\nabla_i^2}{2m} + \sum_i V_i + \frac{1}{2}\sum_{ik} u_{ik}. \qquad (16)$$

If the Hamiltonian (3) leads to (2), then an analogous relation follows also for (16). In so far as $\partial E_m/\partial A = 0$, then from (16) it follows that

$$E_m = E_F + \left(\psi \left| \sum_i \partial V_i/\partial A \right| \psi\right) + \frac{1}{2}\sum_{ik}(\psi|\partial u_{ik}/\partial A|\psi),$$

from which, taking Eq. (2) into account, we find

$$\left(\psi \left| \sum_i \partial V_i/\partial A \right| \psi\right) + \left(\psi \left| \frac{1}{2}\sum_{ik} \partial u_{ik}/\partial A \right| \psi\right)$$

$$= \left(\psi \left| \frac{1}{2}\sum_{i,k} \partial U_{ik}/\partial A \right| \psi\right). \qquad (17)$$

The potential V_i usually used in nuclear theory, which depends only on **r** and includes spin-orbit interaction, does not lead to $\partial E_m/\partial A = 0$. Therefore, $\partial V_i/\partial A \neq 0$ (this is confirmed by calculations of excited levels of several nuclei[3]). Since, as we shall show, in the nucleus $\theta \neq E_m$, i.e., $\partial U_{ik}/\partial A \neq 0$, it follows from Eq. (17) that $\partial V_i/\partial A$ and $\partial u_{ik}/\partial A \neq 0$ in general. We note that only indirect experimental data on u_{ik} exists, and this in insufficient quantity.

In so far as the calculation of residual interaction, the energy of which is $W = 2E_{A+1} - E_A - E_{A+2}$, is concerned, Eq. (1) leads to $W \equiv 0$. The experimentally observed values, $W = 1.5 - 3$ Mev, show that Eq. (1) is not exactly fulfilled in nuclei. Taking W into account, just as taking into account $\partial E_m/\partial A \ll 0.01$, does not change the curve 1 substantially.

We shall try now to take into account the effect of deformations of nuclei with a considerable number of nucleons outside closed shells on the dependence of ω on particle number. Employing the model of Inglis,[4] we write the total energy of the nucleus as (the nucleus is assumed to be stretched along the z axis)

$$E = E_1 + \beta E_2 + \beta^2 E_3, \qquad (18)$$

where β is the nuclear deformation, parameter

$$E_1 = \frac{3}{4}\sum \hbar\omega\left(n + \frac{3}{2}\right) - \frac{1}{2} V_A A,$$

$$E_2 = \frac{1}{2}\sum \hbar\omega(n - 3n_z),$$

$$E_3 = \frac{1}{2}\sum \hbar\omega(n + 3 + 3n_z), \qquad (19)$$

and the sum is carried out over all occupied states. For close shells, $\beta = 0$; therefore, the positions of the jumps in $\omega = \omega(A)$ do not change by including the effect of deformations. For $\beta \neq 0$, we can write the expression for V_A, remembering that V_A does not increase gradually, but in jumps, with the filling of subshells (a subshell consists of a group of nucleons with the same n and n_z):

$$V_A = \hbar\omega\Big[\left(N + \frac{3}{2}\right) + \frac{2}{3}\beta(N - 3N_z - 3)$$

$$+ \frac{2}{3}\beta^2(N + 3N_z + 6) + (1 - \delta(P))$$

$$+ 2\beta(1 - \delta(s)) - 2\beta^2(1 - \delta(s))\Big] + \theta, \qquad (20)$$

where s is the number of nucleons in the last unfilled subshell, N_z is the quantum number along the symmetry axis corresponding to the last filled subshell. Then, instead of Eq. (11), we have:

$$\hbar(\partial\omega/\partial A)\left\{\tfrac{3}{4}C_N+\tfrac{3}{4}P\left(N+\tfrac{5}{2}\right)-\tfrac{1}{2}A\left(N+\tfrac{5}{2}\right)+\beta\left[\tfrac{1}{2}\sum(n-3n_z)-\tfrac{1}{3}A(N-3N_z)\right]+\beta^2\left[\tfrac{1}{2}\sum(n+3n_z+3)-\tfrac{1}{3}A(N+3N_z+3)\right]\right\}=\tfrac{3}{4}\hbar\omega\left[C_N/A+P\left(N+\tfrac{5}{2}\right)/A-\left(N+\tfrac{5}{2}\right)+\tfrac{2}{3}A\delta(P)\right]+\beta\hbar\omega\left[A\delta(s)+\left(\tfrac{1}{2}A^{-1}-\tfrac{1}{2}\partial/\partial A\right)\sum(n-3n_z)\right]+\beta^2\hbar\omega\left[-A\delta(s)+\left(\tfrac{1}{2}A^{-1}-\tfrac{1}{2}\partial/\partial A\right)\sum(n+3+3n_z)\right]+\hbar\omega(\partial\beta/\partial A)\left[\tfrac{1}{3}A(N-3N_z)-\tfrac{1}{2}\sum(n-3n_z)\right]+\beta(\partial\beta/\partial A)\hbar\omega\left[\tfrac{2}{3}A(N+3N_z+3)-\sum(n+3n_z+3)\right]+\tfrac{1}{2}A\partial\theta/\partial A;$$

$$P=\tfrac{1}{2}(N-N_z)(N-N_z+3)+s_N+1. \qquad (21)$$

The formula obtained is analogous to Eq. (11). If, as earlier, it is assumed that the total energy of the last nucleon changes so that $\beta\hbar\omega > \frac{1}{2}\partial\theta/\partial A$, then jumps in ω (small in magnitude) appear also in the region of the filling of subshells. In order to compare Eqs. (21) and (11), we give the relations between $\partial\omega/\partial A$ and ω, obtained from Eqs. (21) and (11) with $\beta_{max} = 0.3$ and $\beta = 0$ at the limits of the shells (see Table I):

The increase in absolute value of the quantity $\partial\omega/\partial A$ is explained by the appearance of jumps in the filling of subshells ($P = 1, 10, 21$). For $P = 0$

$$\hbar\partial\omega/\partial A = 2.63\hbar\omega + 2.80\,\partial\theta/\partial A.$$

The separation energies of the last nucleon and the mean energy per nucleon are known from experiment; from these, one can obtain θ and $\partial\theta/\partial A$. It does not follow that $\theta = E_F$ can be identified with the separation energy of the last nucleon Q:

$$Q=E_{A+1}-E_A\approx\tfrac{3}{4}C_N\hbar\partial\omega/\partial A+\tfrac{3}{4}P\hbar\partial\omega/\partial A+\tfrac{3}{4}\hbar\omega\left(N+\tfrac{5}{2}\right)-\tfrac{1}{2}A\partial V/\partial A-\tfrac{1}{2}V_A+E_2\partial\beta/\partial A+\beta\partial E_2/\partial A+2\beta E_3\partial\beta/\partial A+\beta^2\partial E_3/\partial A. \qquad (22)$$

From Eqs. (7), (9), (19), and (22), θ and $\partial\theta/\partial A$ can be expressed in terms of Q and $E_m \equiv Q$ [if Eq. (1) is valid]. In particular, in the neighborhood of closed shells

$$\theta=2\hbar\omega\left[\tfrac{3}{4}A^{-1}C_N+\tfrac{3}{4}A^{-1}P\left(N+\tfrac{5}{2}\right)-\tfrac{1}{2}\left(N+\tfrac{5}{2}\right)+\tfrac{1}{2}\delta(P)-E_m/\hbar\omega\right],$$

$$\frac{\partial\theta}{\partial A}=\frac{2}{A}\left\{\left(\hbar\frac{\partial\omega}{\partial A}-\frac{\hbar\omega}{A}\right)\left[\tfrac{3}{4}C_N+\tfrac{3}{4}P\left(N+\tfrac{5}{2}\right)-\tfrac{1}{2}A\left(N+\tfrac{5}{2}\right)\right]+\hbar\omega\left[\frac{N}{4}+\frac{1}{8}-\frac{A}{2}\delta(P)\right]\right\}. \qquad (23)$$

From Eq. (23) it follows that, as assumed above, $\partial\theta/\partial A \ll \omega$. This conclusion remains valid if terms depending on β are taken into account. For closed shells, for example, $\theta \cong -2E_m$ and $\frac{1}{2}A\partial\theta/\partial A \geq 0$. Inside shells $\theta \leq -2E_m$ and $\frac{1}{2}A\partial\theta/\partial A \leq 0$. Consequently, within the framework of our assumptions

$$E_m \neq E_F. \qquad (24)$$

It was shown above that in systems with two-particle interactions, $E_m = E_F$ in Eq. (1) is valid. We calculated E_m for nuclei, assuming that the two-particle interaction led to a self-consistent oscillator potential, took account of (1) and came to Eq. (24), in contradiction to Eq. (2). In addition, it turned out that if $\partial\theta/\partial A \cong 0$, ω changes sharply in the region of closed shells. The difference between E_m and E_F can be explained either by the dependence of the two-particle interaction on A or by the inclusion of a three-particle force.

2. Within the Hartree approximation, a system with three-particle interactions is equivalent to one with two-particle interactions if

$$U_{ik}=\sum_l(\psi_l\,|\,v_{ikl}\,|\,\psi_l).$$

Here $\partial U_{ik}/\partial A \neq 0$. We assume that the inclusion of the three-particle interaction, together with the two-particle one, leads to an oscillator self consistent potential. Then, in the expression for E_m, the potential energy will be prefixed by the coefficient $\frac{1}{2} - \frac{1}{3}\alpha$, rather than $\frac{1}{2}$, where α characterizes the relative contribution of the three-particle interaction in the mean energy per nucleon.

We shall first show that inclusion of the three-

TABLE I. Values of $\hbar\partial\omega/\partial A = a\hbar\omega + b\partial\theta/\partial A$ for $N = 5$

	P	1	2	10	11	21	22
$\beta\neq0$	a	−0.094	−0.251	0.354	−0.095	0.212	−0.204
	b	+1.07	+1.07	+1.10	+1.10	+1.29	+1.29
$\beta=0$	a	−0.122	−0.122	−0.047	−0.047	−0.050	−0.050
	b	+2.18	+2.18	+1.12	+1.12	+0.765	+0.765

TABLE II. Values of $\hbar\partial\omega/\partial A = a\hbar\omega + b\partial\theta/\partial A$ for $N = 5$

	P	1	2	10	11	21	22
$\beta = 0$	a	≈ -0.023	≈ -0.023	≈ -0.015	-0.015	-0.020	-0.020
	b	$+0.317$	$+0.317$	$+0.266$	$+0.266$	$+0.234$	$+0.234$
$\beta \neq 0$	a	-0.031	-0.072	-0.066	-0.036	0.047	-0.037
	b	$+0.274$	$+0.274$	$+0.251$	$+0.251$	$+0.26$	$+0.26$

particle interaction does not lead to a qualitative change in the dependence of ω on A:

a) for closed shells it follows from Eq. (11) that the sign of $\partial\omega/\partial A$ changes for $\alpha \neq 0$ if

$$\left\{\tfrac{3}{4} C_N - \tfrac{1}{2} A \left(N + \tfrac{5}{2}\right)\right\} \rightarrow \left\{\left(\tfrac{3}{4} - \tfrac{1}{6}\alpha\right) C_N - \left(\tfrac{1}{2} - \tfrac{1}{3}\alpha\right) A \left(N + \tfrac{5}{2}\right)\right\}$$

changes sign with the growth of α from 0 to 1. In so far as $C_N > \frac{2}{3} A (N + \frac{5}{2})$ then $C_N > A(N + \frac{5}{2}) \times (\frac{1}{2} - \frac{1}{3}\alpha)/(\frac{3}{4} - \frac{1}{6}\alpha)$ if $\alpha \geq 0$;

b) inside shells the coefficient of $\partial\omega/\partial A$ does not change sign as α increases from 0 to 1, since the sign of the coefficient of ω in Eq. (11) does not depend on α (within the limits $0 \leq \alpha \leq 1$). Consequently, the character of the curve, i.e., the signs of $\partial\omega/\partial A$ and the positions of jumps in ω, is unchanged.

The situation with the mean energy is different; its magnitude depends essentially on α. This is connected with the fact that the coefficient of $\partial\omega/\partial A$ in Eq. (11) changes within very large limits with changing α. For example, for $N = 5$, $\alpha = 0$, the quantity $\{ \frac{3}{4} C_N - \frac{1}{2} A (N + \frac{5}{2}) \} = 10$ for $\alpha = 0.5$, whereas it is equal to 56 for $\alpha = 0.5$ and 100 for $\alpha = 1$. Already, for $\alpha = 0.1$, the coefficient of $\partial\omega/\partial A$ is doubled. Inclusion of the three-particle interaction, while not changing the character of the curve* 1, can diminish the jumps in ω several times and can make the dependence of ω on A smoother.

The parameters of the potential well obtained from calculations of low-lying excited states of nuclei are roughly equal[3] for nuclei consisting of closed shells ±1 nucleon. From this it is difficult to assert that the scheme of a two-particle type of self-consistent potential is inapplicable, and that it is necessary to introduce a three-particle interaction. However, if one succeeds in showing that $\partial\theta/\partial A \cong 0$, then the constancy of the potential would have to be explained by introducing a three-particle interaction.

The dependence of ω on A for $\alpha \neq 0$ is obtained analogously to Eq. (12)

*Equations (11) and (23) are not independent, but are connected by Eq. (1). Therefore, there is left to our disposal one parameter $\partial\theta/\partial A$ which can be and should be chosen so that for nuclei consisting of closed shells +1 nucleon, $\omega \cong$ const.

$$\omega = \omega^{(1)} \frac{(B_N + P)\left[(9 - 2\alpha) C_N - (6 - 4\alpha) B_N (N + 5/2)\right]}{B_N \left[(9 - 2\alpha) C_N - (6 - 4\alpha) B_N (N + 5/2) + (3 + 2\alpha) P (N + 5/2)\right]}. \tag{25}$$

For closed shells, $\partial\omega/\partial A$ is obtained from Eq. (11) by the substitutions $\frac{3}{4} \rightarrow \frac{3}{4} - \frac{1}{6}\alpha$; $\frac{1}{2} \rightarrow \frac{1}{2} - \frac{1}{3}\alpha$; The formulas (13) and (23) are modified, and the coefficient 2 in Eq. (14) is replaced by $6/(3 - 2\alpha)$. For example, $\theta \cong -3E$ for $\alpha = 0.5$.

We now take into account effects of nucleon deformation. Proceeding in the same way as in the derivation of Eq. (21), and taking into account the three-particle interaction, we obtain instead of Eq. (21)

$$\begin{aligned}
&\hbar (\partial\omega/\partial A) \Big\{\tfrac{1}{12}(9 - 2\alpha) C_N + \tfrac{1}{12}(9 - 2\alpha) P \left(N + \tfrac{5}{2}\right) \\
&\quad - \tfrac{1}{6}(3 - 2\alpha) A \left(N + \tfrac{5}{2}\right) + \beta\Big[\sum \tfrac{1}{18}(9 - 2\alpha)(n - 3n_z) \\
&\quad - \tfrac{1}{9}(3 - 2\alpha) A (N - 3N_z)\Big] + \beta^2\Big[\sum \tfrac{1}{18}(9 - 2\alpha)(n + 3 + 3n_z) \\
&\quad - \tfrac{1}{9}(3 - 2\alpha)(N + 3N_z + 3)\Big]\Big\} = \tfrac{1}{12}(9 - 2\alpha)\hbar\omega \big[C_N/A \\
&\quad + (P/A - 1)\left(N + \tfrac{5}{2}\right) + (6 - 4\alpha)/(9 - 2\alpha) A \delta(P)\big] \\
&\quad + \beta\hbar\omega \Big[\tfrac{1}{3}(3 - 2\alpha) A \delta(s) + \tfrac{1}{18}(9 - 2\alpha)(A^{-1} - \partial/\partial A) \\
&\qquad \sum (n - 3n_z)\Big] + \beta^2\hbar\omega \Big[-\tfrac{1}{3}(3 - 2\alpha) A \delta(s) \\
&\quad + \tfrac{1}{18}(9 - 2\alpha)(A^{-1} - \partial/\partial A) \sum (n + 3n_z + 3)\Big] \\
&\quad + \hbar\omega (\partial\beta/\partial A) \Big[\tfrac{1}{9}(3 - 2\alpha) A (N - 3N_z) \\
&\quad - \tfrac{1}{18}(9 - 2\alpha) \sum (n - 3n_z)\Big] + \hbar\omega\beta\,\partial\beta/\partial A \Big[\tfrac{2}{9}(3 - 2\alpha) \\
&\qquad A (N + 3N_z + 3) - \tfrac{1}{9}(9 - 2\alpha) \sum (n + 3n_z + 3)\Big] \\
&\quad + \tfrac{1}{6}(3 - 2\alpha) A\, \partial\theta/\partial A.
\end{aligned} \tag{26}$$

The effect of α in the quantity $\partial\omega/\partial A$ can most easily be shown by a concrete example. Values of $\hbar\partial\omega/\partial A$ for $\alpha = 0.5$, $N = 5$, $\beta_{max} = 0.3$ are given in Table II. It is easy to obtain expressions for $\theta = E_m$, analogous to Eqs. (22) and (23), including the three-particle interaction.

Recently a number of authors have discussed the use of a potential which depends on momentum (or, which is the same thing, the so-called effective mass approximation) in the shell model. A number of arguments can be adduced, according to which a system of strongly-interacting nucleons can be replaced by a model system of weakly-interacting quasiparticles (of number equal to the number of nucleons) so that the energies of the ground and low-excited states coincide in the two cases.[5]

Construction of a self-consistent potential for quasiparticles in nuclear matter leads to its dependence on the momentum of the quasiparticle, or to an effective mass, which, in turn, depends (but only weakly) on the momentum.[5] In the case of finite nuclei it is also possible to go over to a model system of quasiparticles, moving with effective mass in the field of a self-consistent potential (and the self-consistent potential and effective mass depend on the quantum numbers defining the quasiparticle state, see reference 6). Therefore, it might be hoped that replacement of the true mass by the effective mass would diminish the magnitude of the residual interaction [see Eq. (14)]. It is of interest to see how much the introduction of an effective mass changes the conclusions reached above. It is easy to see that introduction of the same effective mass for all nucleons is equivalent to increasing the frequency ω_A by the factor $(m/m_{\mathrm{eff}})^{1/2}$. Repeating the above development, we see that the qualitative character of curve 1 is not changed, but that the depth of the potential well is increased by a factor of about $(m/m_{\mathrm{eff}})^{1/2}$.

We note that the discontinuous changes in the parameters of the shell-model potential came, as seen from the above conclusions, from changes in the depth of the potential well with the addition of a single nucleon outside a closed shell. In fact, for arbitrary potential, the following relation is valid [compare Eq. (9)]:

$$V = -\sum_{i<j} \Delta E_{ij} - \theta,$$

where ΔE_{ij} is the distance between energy shells (the summation is carried out over the totality of filled or partially filled energy shells). Addition of a particle outside a closed shell adds one term to the sum over $i < j$ [see the term $\hbar\omega\,[1 - \delta(P)]$ in Eq. (9)]. Consequently, the discontinuous change in parameters also appears in use of a potential more realistic than the oscillator one.

It is known that the data on scattering of nucleons by nuclei are rather well described by an optical potential of constant depth and radius $R \sim A^{1/3}$. This does not, however, contradict the conclusions reached earlier about the shell-model potential. The situation is that for higher energies of the incident nucleon, the real part of the optical potential depends essentially on the energy of the incident nucleon and differs from the shell-model potential. We give below several considerations, according to which the real part of the optical potential should, for arbitrary energy of the incident nucleon, differ in general from the shell-model potential.

3. Up to now the problem of the change in the potential well of the nucleus necessary to ensure fulfilment of Eq. (1) was considered. In the problem of the interaction of an incident nucleon with the nucleus, the Hamiltonian is written as

$$(-\hbar^2\nabla^2/2m + U^{(A+1)}).$$

If the incident nucleon leads to a complete recomposition of the target nucleus, then

$$U^{(A+1)} = U^{(A)} + \partial U^{(A)}/\partial A, \qquad (27)$$

or if there is no recomposition, $U^{(A+1)} = U^{(A)}$.

In the present work, using Eq. (1), the magnitude of $\partial U^{(A)}/\partial A$ was found, i.e., the correction to the real part of the optical potential for low energies. From this it is clear that calculations of excited states of nuclei by several authors using the optical potential do not have a strong basis in view of Eq. (27). For an oscillator potential we have

$$V^{(A+1)} = -(N + \tfrac{5}{2})(\hbar\omega + \hbar\partial\omega/\partial A) + \tfrac{1}{2} m(\omega + \partial\omega/\partial A)^2 r^2 - \theta - \partial\theta/\partial A,$$

where $\partial\omega/\partial A$ is determined either from Eq. (11) or from Eqs. (21) and (26).

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