SCATTERING BY A SINGULAR POTENTIAL IN PERTURBATION THEORY AND IN THE MOMENTUM REPRESENTATION

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A method is developed for treatment of scattering by a singular potential in the momentum representation and in perturbation theory. Application of such renormalization techniques permits one to derive well-known results for cross sections, despite the fact that the integrals diverge and the matrix elements entering into the wave equation in the momentum representation vanish.

THE problem is considered below of scattering in the presence of a nearby resonant level — real, as a deuteron in the triplet state, or virtual, as in the singlet np scattering. In the limit, this problem can be regarded as scattering by a potential U, acting for $r < r_0$, where $U \to -\infty$, $r_0 \to 0$, such that

$$Ur_0^2 \to -\pi\hbar^2/8m + wr_0$$

(the coefficients refer to a rectangular well).

The usual solution of this problem is found in space coordinates by replacing the action of the potential by a boundary condition applied to the wave function $\varphi = r\psi$ at the origin:

$$d\ln\varphi/dr = mw/\hbar^2 = -1/a;$$

This boundary condition follows from a solution of the Schrödinger equation for $r \le r_0$. As is well known, the value of a completely determines the value of the scattering cross section:

$$\sigma = 4\pi a^2 [1 + (ap/\hbar)^2]^{-1}$$

for $pr_0 \ll 1$. If $a > 0$, there is also a single bound state with energy $E = -\hbar^2/2ma^2$.

The problem becomes singular if treated by perturbation theory. By considering the transition from a state with one momentum p into a state with another momentum p′ under the action of a potential U, it is easy to establish the fact that the corresponding matrix element

$$U(p, p') = (2\pi)^{-3} \int U(\mathbf{r}) e^{i(\mathbf{p}-\mathbf{p}')\mathbf{r}} d^3\mathbf{r} \tag{1}$$

does not exceed $U \cdot {}^4/_3 \pi r_0^3$ and, consequently, in the limit as $r_0 \to 0$, the matrix element also tends to zero as r_0, inasmuch as $U \sim r_0^{-2}$.

It is evident that in the limit not only the first approximation of perturbation theory disappears, but also the sum of any finite number of terms of the series; however, the sum of the entire infinite series of perturbation theory preserves completely a definite value of the transition probability, i.e., the scattering cross section. As we shall see below, in the transition to the limit $r_0 \to 0$, along with the transition to zero of each term of the series of perturbation theory, the number of terms of the series which must be taken into consideration tends to infinity. Since one ordinarily uses not the coordinate representation, as in field theory, but precisely perturbation theory in momentum representation, a complete explanation of such a peculiar situation in a simple example can be of interest.

Perturbation theory uses plane waves as a base, i.e., it is essentially close to a consideration of the problem in the momentum representation.* In this representation the matrix element U(p, p′) enters into the equation and the surmounting of the difficulties associated with $U(p, p') \to 0$ also opens up a path to the solution of the problem in perturbation theory. The method of obtaining the correct result in the momentum representation in the transition to the limit of the quantities entering into it is very instructive and has general similarities with the methods of renormalization, especially in that form in which they were applied by Heisenberg[3] to the Lee model.

1. THE EQUATION IN THE MOMENTUM REPRESENTATION

Substituting in the Schrödinger equation ($\hbar = m = 1$)

*Scattering and the bound state in the momentum representation were considered by Salpeter;[1] see also reference 3.

$$\psi = (2\pi)^{-3/2} \int \chi(p) e^{i\mathbf{p}\mathbf{x}} d^3\mathbf{p}, \tag{2}$$

we get an equation for $\chi(\mathbf{p})$

$$(p^2 - 2E)\chi(p) = -2\int d^3p'\chi(p')U(p', p) \tag{3}$$

[$U(p, p')$ is determined by Eq. (1)]. In the problem of the determination of the bound level, we set $E = -\kappa^2/2$; Eq. (3) is rewritten in the form

$$(\mathbf{p}^2 + \varkappa^2)\chi(p) = -2\int d^3p'\chi(p')U(p', p). \tag{4}$$

In the scattering problem, we set $E = k^2/2 > 0$; the momentum $\mathbf{k}$ and energy $E = k^2/2$ of the incident wave are given. Equation (3) is rewritten in the form

$$(\mathbf{p}^2 - k^2)\chi(p) = -2\int d^3p'\chi(p')U(p', p). \tag{5}$$

Its solution must contain the incident wave and the scattered wave of the same energy, i.e.,

$$\chi(p) = \delta(\mathbf{p} - \mathbf{k}) + f(\mathbf{p}),$$

such that the problem reduces to the determination of $f(\mathbf{p})$ for $\mathbf{p}^2 = k^2$.

Returning to the specifics of the singular potential, we observe that for $r_0 \to 0$, the function $U(p, p') \to 0$, but the dependence of U on $|p|$, $|p'|$ simultaneously vanishes up to $|p| \sim 1/r_0$, $|p'| \sim 1/r_0$. We set

$$U(p, p') = -C \quad \text{for } |p| < b, \quad |p'| < b;$$

$$U(p, p') = 0 \quad \text{for } |p| > b, \quad |p'| > b, \tag{6}$$

in order to make the subsequent transition to the limit $C \to 0$, $b \to \infty$.

For the bound state we get from the equation

$$(p^2 + \varkappa^2)\chi(p) = 2C\int_0^b \chi(p')d^3p' \tag{7}$$

$$\chi(p) = q/(p^2 + \varkappa^2), \qquad q = 8\pi C\int_0^b dp'qp'^2/(p'^2 + \varkappa^2). \tag{8}$$

For scattering, it follows from the equation

$$(p^2 - k^2)\chi(p) = 2C\int_0^b \chi(p')d^3p' \tag{9}$$

that

$$\chi(p) = \delta(\mathbf{p} - \mathbf{k}) + \frac{q}{p^2 - k^2},$$

$$q = 2C\left[\int_0^b dp' 4\pi q p'^2/(p'^2 - k^2) + 1\right]. \tag{10}$$

2. TRANSITION TO THE SINGULAR POTENTIAL

It is now seen why the equation can have a finite solution: in spite of the transition to zero $C = -U(p, p')$, the limiting value of the momentum b, for which decrease of U begins, increases along with decrease in r_0. The quantity C enters into Eqs. (8) and (10) multiplied by an integral which diverges as b for $b \to \infty$.

We set

$$2C\int_0^b \frac{4\pi p^2}{p^2}dp = 8\pi Cb = 1 + 4\pi^2 ZC. \tag{11}$$

This is the connection between the transition $C \to 0$ and $b \to \infty$, which is necessary in order that we obtain a finite answer, i.e., in order that resonance take place. It is easy to prove that the condition $Cb = \text{const}$ is equivalent to $Ur_0^2 = \text{const}$, $r_0 \to 0$. Here Cb must have a completely determined value for the existence of a solution. The coefficient Z, which shows just how Cb tends toward its limiting value, is also important for the result. The factor $4\pi^2$ is introduced in the term $4\pi^2 ZC$ only for convenience.

We further divide the diverging (as $b \to \infty$) integral (8), (10) into a constant diverging integral and a converging integral which depends on the quantities entering into the conditions of the problem. Making use of the identities

$$(p^2 + \varkappa^2)^{-1} = p^{-2} - \varkappa^2/p^2(p^2 + \varkappa^2), \tag{12a}$$

$$(p^2 - k^2)^{-1} = p^{-2} + k^2/p^2(p^2 - k^2), \tag{12b}$$

we find

$$\int_a^b \frac{p^2}{p^2 + \varkappa^2}dp = \int_0^b dp - \int_0^b \frac{\varkappa^2}{p^2 + \varkappa^2}dp$$

$$= b - \varkappa\tan^{-1}\frac{b}{\varkappa} \to b - \frac{\pi}{2}\varkappa \quad (b \to \infty) \tag{13}$$

and, by substituting in (8), we obtain

$$q = 8\pi Cq(b - \pi\varkappa/2), \qquad 4\pi^2(ZC - C\varkappa) = 0. \tag{14}$$

We now let $C \to 0$. The real level exists only for $Z > 0$, and its energy is given by the value $\kappa = Z$.

The scattering problem is solved in similar fashion. Substituting (12b) and (11) in (10), we obtain

$$q = q(1 + 4\pi^2 ZC) + 8\pi qCk^2\int_0^b \frac{dp}{p^2 - k^2} + 2C. \tag{15}$$

After division by q and C we allow the limiting transition $C \to 0$, $b \to \infty$ in Eq. (15). In taking the integral

$$\int_0^\infty dp/(p^2 - k^2)$$

it is necessary to take into account that $\kappa(p)$ must be written as a diverging wave. As is well known,[2] it is necessary to integrate in this case over the real axis of the variable p, assuming that p has a small imaginary part: $p \to p + i\epsilon$, $\epsilon > 0$, $\lim \epsilon = 0$. Carrying out the integration in this fashion, we get

$$\int_0^\infty \frac{dp}{p^2 - k^2 + i\varepsilon} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dp}{p^2 - k^2 + i\varepsilon} = \frac{\pi i}{2k}.$$

Substitution of the resulting integral in (15) gives

$$q = (2\pi^2)^{-1}(Z + ik)^{-1} \tag{16}$$

and consequently the wave function of the scattering problem in the momentum representation has the form

$$\chi(p) = \delta(\mathbf{p} - \mathbf{k}) + \frac{1}{2\pi^2(Z + ik)}\frac{1}{p^2 - k^2 + i\varepsilon} = \delta(\mathbf{p} - \mathbf{k}) + \chi_s(p). \tag{17}$$

We find the asymptotic form of the function, corresponding to the second term χ_s of (17) in the coordinate representation:

$$\psi(\mathbf{r}) = \int e^{i\mathbf{pr}}\chi_s(p)\,d^3p = \frac{2\pi}{ir}\int_0^\infty (e^{ipr} - e^{-ipr})\,p\chi_s(p)\,dp =$$

$$= \frac{2\pi}{ir}\int_{-\infty}^{+\infty} e^{ipr}\chi_s(p)\,p\,dp = 4\pi^2\frac{e^{ikr}}{r}\mathrm{Res}\{\chi_s(p)\} = \frac{1}{Z + ik}\frac{e^{ikr}}{r_0}. \tag{18}$$

By comparing the scattered current with the current in the incident wave, we find the scattering cross section in the form

$$\sigma = 4\pi\left|\frac{1}{Z + ik}\right|^2 = \frac{4\pi}{Z^2 + k^2}, \tag{19}$$

as was to have been expected. In the presence of a bound level ($Z = \kappa$) a quantity appears in the numerator of (19), which is proportional to the sum of the collision energy and the binding energy of the level; in the absence of a bound level ($Z < 0$), Z^2 is a quantity proportional to the energy of the virtual level.

3. PERTURBATION THEORY

The solution of the problem by the method of perturbation theory consists of iteration of the integral equation (5). In the zeroth approximation, we have only the incident wave

$$\chi_0 = \delta(\mathbf{p} - \mathbf{k}). \tag{20}$$

Substituting (5) on the right hand side, we get

$$\chi_1 = \chi_0 - \frac{2}{p^2 - k^2}\int_0^\infty d^3p'\chi(p')\,U(p, p') = \delta(\mathbf{p} - \mathbf{k}) + \frac{2C}{p^2 - k^2}. \tag{21}$$

The subsequent approximation gives

$$\chi_2 = \delta(\mathbf{p} - \mathbf{k}) + \frac{2C}{p^2 - k^2} + \frac{2}{p^2 - k^2}\int_0^\infty d^3p'U(p', p)\frac{2C}{p'^2 - k^2}. \tag{22}$$

The integral on the right hand side is transformed by the method applied in Sec. 2:

$$\int U(p, p')\frac{2C}{p'^2 - k^2}d^3p' = 8\pi C\left[Cb + C\frac{k^2\pi i}{2k}\right], \tag{23}$$

from which it follows that

$$\chi_2 = \delta(\mathbf{p} - \mathbf{k}) + \frac{2C}{p^2 - k^2} + \frac{2C}{p^2 - k^2}[8\pi Cb + i4\pi^2 k]. \tag{22a}$$

It is easy to prove that the perturbation-theory series is a geometric progression, the denominator of which is the expression in the square brackets in (22a), while the first term is $2C/(p^2 - k^2)$. We then immediately obtain the sum of the infinite number of terms

$$\chi_\infty = \delta(\mathbf{p} - \mathbf{k}) + \frac{1}{p^2 - k^2}\frac{2C}{1 - 8\pi Cb - iC4\pi^2 k}. \tag{24}$$

It is now evident that for the result to be finite it is necessary that the difference $(1 - 8\pi Cb)$ be of order C. Taking for Cb the limiting expression in Eq. (11), we get the expression $\chi(p)$ of Eq. (17) from (24). Thus, in the limit the denominator α of the geometric progression actually tends to unity (the difference $1 - \alpha \sim C$), while the effective number of terms of the series tends to infinity as $1/C$, which also compensates the trend to zero of each separate term of the series, which is proportional to C.

We note that Eq. (24) remains in force even when $Z > 0$, $|\alpha| > 1$ (the case of the existence of a bound level), in spite of the divergence of the series of perturbation theory for $|\alpha| > 1$.

4. MORE RIGOROUS METHOD

The transformations connected with the elimination of the zeros ($C \to 0$) and the infinities ($b \to \infty$) were carried out above schematically under the assumption that $U(p, p')$ is broken off sharply for $p > b$. Evidently, all the discussions can be carried out more rigorously if we write

$$U = Cf(|p - p'|/b), \qquad \chi = \frac{q}{p^2 + \varkappa^2}g(|p|/b),$$

where the characteristic functions f and g satisfy the condition $f(0) = g(0) = 1$ and coincide at large values of the arguments. The condition connecting C and b is written as follows:

$$2C\int_0^\infty f(|p - p'|/b)\,g(|p - p'|/b)\frac{1}{p^2}d^3p = 1 + 4\pi^2 ZC.$$

After subtraction similar to (12a) and (12b), we obtain integrals of the form

$$\varkappa^2 \int_0^\infty \frac{f(|p-p'|/b)\,g(|p-p'|/b)}{(p'^2+\varkappa^2)}\, d^3p',$$

$$k^2 \int_0^\infty \frac{fg}{p^2-k^2+i\varepsilon}\, d^3p'.$$

It is important that these integrals converge independently of the decrease of f and g, because of the high power of the denominator. Therefore, their values and dependence on κ can be immediately computed for $b \to \infty$, i.e., in the approximation $f = g = 1$.

Thus the connection between the scattering cross section and the location of the level (the real or virtual) is shown to be independent of the concrete form of U(r). Computations carried out in all their triviality and absence of new results are instructive. A mechanism is given for obtaining a finite scattering cross section of a bound level under the action of a potential which gives a vanishing matrix element in perturbation theory of arbitrary (n-th) order. One can easily show that only an infinitely short duration of such a perturbation is capable (at a given relation between the zero and the infinity) of giving a finite effect.

[1] E. E. Salpeter, Phys. Rev. **84**, 1226 (1951).

[2] M. Lippman and J. Schwinger, Phys. Rev. **79**, 469 (1950).

[3] W. Heisenberg, Nucl. Phys. **4**, 532 (1957).

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