STATISTICAL THEORY OF MULTIPLE PARTICLE PRODUCTION

A. I. NIKISHOV

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor August 8, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **38**, 509-512 (February, 1960)

If equilibrium does not set in during a collision, one can employ the statistical theory for estimating the mean values of the squares of the matrix elements. In this case, one can hope to obtain satisfactory agreement with experiments for the multiplicity, charge state, and momentum distributions irrespective of the charge of the particle.

1. In the work of Witten and Block the energy spectra of π^+- and π^--mesons were given for the reaction $\pi^- + p \to n + \pi^- + \pi^+$ at an energy of 1.8 Bev. Corresponding calculations carried out by Maksimenko[2] according to statistical theory, with account of isobaric states, did not give agreement for these spectra which were considered separately for each sign of the charge of the pions. However, the total spectrum of the charged mesons agrees satisfactorily with the calculation.[2] Better agreement of the results of the statistical theory with experimental data on multiplicity, total spectrum and distribution over charge states (see references 2 — 4) leads to the thought that the circumstance just mentioned is not accidental.

Let us consider in detail the reaction $\pi^- + p \to n + \pi^- + \pi^+$.[2] By first using only isotopic invariance, we find the general expression for the matrix element and investigate its perturbation symmetry. We shall use the method of Belinfante.[5] We denote by $f(1, 2, 3) \equiv f(p_1, p_2, p_3)$ the part of the wave function of the system $n\pi^+\pi^-$ which is independent of the isotopic variables. We shall assume that variables 1 and 2 refer to mesons, while the variable 3 refers to the nucleon. As usual we denote the isotopic function of the proton by $[\frac{1}{2}, \frac{1}{2}]_3$, and that of the neutron by $[\frac{1}{2}, -\frac{1}{2}]_3$. The analogous functions for the π^+-meson will be $[1, 1]_1$ or $[1, 1]_2$, depending on whether it refers to the first or second meson (i.e., it refers to a particle whose momentum is denoted by p_1 or p_2). The total wave function of the system $n\pi^+\pi^-$ is obtained by symmetrization of the function $f(1, 2, 3)[\frac{1}{2}, -\frac{1}{2}]_3[1, 1]_1[1, -1]_2$ over the momentum and isotopic variables of the mesons:

$$\Psi_f = \Psi_{n+-} = \sqrt{\tfrac{1}{2}}\{f(1, 2, 3)[1,1]_1[1,-1]_2 + f(2, 1, 3)[1,1]_2[1,-1]_1\}[\tfrac{1}{2},-\tfrac{1}{2}]_3 \qquad (1)$$

Expanding the isotopic functions in definite total isotopic spins, we transform Eq. (1) to the form

$$\Psi_{n+-} = f_s\{\sqrt{\tfrac{1}{10}}[\tfrac{5}{2},-\tfrac{1}{2}]_{312} - \sqrt{\tfrac{1}{15}}[\tfrac{3}{2},-\tfrac{1}{2}]_{312}\} - f_a\{\sqrt{\tfrac{1}{3}}[\tfrac{3}{2},-\tfrac{1}{2}]_{312} - \sqrt{\tfrac{1}{6}}[\tfrac{1}{2},-\tfrac{1}{2}]_{312}\} + f_s\sqrt{\tfrac{1}{3}}[\tfrac{1}{2},-\tfrac{1}{2}]_{312};$$

$$f_s = \sqrt{\tfrac{1}{2}}\{f(1, 2, 3) + f(2, 1, 3)\},$$

$$f_a = \sqrt{\tfrac{1}{2}}\{f(1, 2, 3) - f(2, 1, 3)\}$$

(see the similar calculations of Belinfante[5]). If now for simplicity we limit ourselves to a definite total spin, for example $T = \frac{1}{2}$, then

$$M_{n+-} = \sqrt{\tfrac{1}{6}}\, M_a + \sqrt{\tfrac{1}{3}} M_s,$$

$$|M_{n+-}|^2 = \tfrac{1}{6}|M_a|^2 + \tfrac{1}{3}|M_s|^2 + \sqrt{\tfrac{2}{9}}\,\mathrm{Re}\, M_a M_s^*. \qquad (2)$$

If summation is carried out over the isotopic variables of the mesons, then the term $\mathrm{Re}\, M_a M_s^*$ drops out, since it is antisymmetric relative to a permutation of the isotopic variables of the mesons. Summation shows that we are interested in the momentum distribution irrespective of the sign of the charge of the mesons. We now make use of the assumption

$$\overline{\sum |M_a|^2} = \overline{\sum |M_s|^2} = \Omega^3.$$

It then follows from Eq. (2) that

$$\overline{\sum |M_{n+-}|^2} = \tfrac{1}{2}\Omega^3.$$

Thus even if $\mathrm{Re}\, M_a M_s^* \neq 0$, the statistical theory can give the correct results relative to the total energy spectrum.

2. We now consider the annihilation processes

$$\tilde{n} + p \to \pi^+ + \pi^+ + \pi^-, \qquad \tilde{n} + p \to \pi^+ + \pi^0 + \pi^0.$$

We symmetrize the function $\varphi(1, 2, 3)[1, 1]_1 \times [1, 1]_2[1, -1]_3$ in similar fashion:

$$\Psi_{++-}=\sqrt{\tfrac{1}{3}}\{\varphi(1, 2, 3)[1, 1]_1[1,1]_2[1,-1]_3$$
$$+\varphi(1, 3, 2)[1,1]_1[1,1]_3[1,-1]_2$$
$$+\varphi(2, 3, 1)[1,1]_2[1,1]_3[1,-1]_1\}. \quad (3)$$

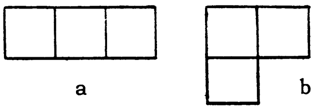
Since the mesons 1 and 2 are identical, we can assume

$$\varphi(1, 2, 3) = \varphi(2, 1, 3).$$

Expansion of the isotopic functions entering into Eq. (3) in terms of functions with a definite total isotopic spin T is given in the table. For example, according to the table we have

$$[1,1]_1[1,1]_3[1,-1]_2 \equiv [1_+3_+2_-]$$
$$=\sqrt{\tfrac{4}{15}}e_0-2\sqrt{\tfrac{1}{12}}(e_1-e_2)+\sqrt{\tfrac{1}{15}}e_3.$$

The Young diagrams[6] corresponding to irreducible representations are shown in the drawing. The diagram a corresponds to the functions e_0 and



e_3, the diagram b to the functions e_1, e_1', e_2, and e_2'. The quantity $e_0 = [1, 1]$ is the isotopic function of three mesons forming a system with a total isotopic spin T = 1. The Young diagram in the figures shows that in a permutation of the isotopic variables of the mesons 1, 2, 3, the function e_0 is transformed into itself.[7] Similarly, $e_2 = [2, 1]$ is the isotopic function with T = 2, $T_3 = 1$. In the permutation of the isotopic variables, e_2 and e_2' are transformed into linear combinations of themselves without involving the other e. All of the e functions are orthogonal.

By making use of the table, we transform Eq. (3) to the form

$$\Psi_{++-}=\sqrt{\tfrac{4}{15}}\varphi_s e_0+\sqrt{\tfrac{1}{6}}(\varphi_1 e_1+\varphi_1' e_1')$$
$$+\sqrt{\tfrac{1}{6}}(\varphi_1 e_2+\varphi_1' e_2')+\sqrt{\tfrac{1}{15}}\varphi_s e_3;$$
$$\varphi_s=\sqrt{\tfrac{1}{3}}\{\varphi(1, 2, 3)+\varphi(1, 3, 2)+\varphi(2, 3, 1)\},$$
$$\varphi_1=\sqrt{\tfrac{1}{6}}\{\varphi(1, 2, 3)-2\varphi(1, 3, 2)+\varphi(2, 3, 1)\},$$
$$\varphi_1'=\sqrt{\tfrac{1}{2}}\{\varphi(1, 2, 3)-\varphi(2, 3, 1)\}. \quad (4)$$

We immediately note the structure of Eq. (4). Each term of it represents the product of a momentum part by an isotopic part. The factors in the product have the same permutation symmetry, as follows from the table and the form of the functions φ_S, φ_1, and φ_1' in (4). This property is preserved even for systems with an arbitrary

	e_0	e_1	e_1'	e_2	e_2'	e_3
$1_+2_+3_-$	2	1	1	−1	−1	1
$1_+3_+2_-$	2	−2	0	2	0	1
$2_+3_+1_-$	2	1	−1	−1	1	1
$1_0 2_0 3_+$	−1	1	1	1	1	2
$1_0 3_0 2_+$	−1	−2	0	−2	0	2
$2_0 3_0 1_+$	−1	1	−1	1	−1	2
Square of the normalizing factor	$\frac{1}{15}$	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{1}{15}$
T	1	1	1	2	2	3

number of mesons, as will be shown somewhat later. In this case it is essential that the isotopic functions are orthonormalized (see reference 7), while the functions e are not orthonormalized. In the language of group theory, this means that the product φe contains only a single representation of the product of two groups of permutations when the permutation symmetry of φ and e is identical.

Returning now to Eq. (4), we have the matrix element corresponding to it:

$$M_{++-}=\sqrt{\tfrac{4}{15}}M_0+\sqrt{\tfrac{1}{6}}(M_1+M_1').$$

Once again it is easy to see that the sum over the isotopic variables is equal to

$$\sum|M_{++-}|^2=\tfrac{4}{15}|M_0|^2+\tfrac{1}{6}(|M_1|^2+|M_1'|^2)$$

By making further use of the assumption of Fermi, we obtain

$$\overline{\sum|M_{++-}|^2}=\tfrac{3}{10}\Omega^3.$$

In this formula the identity of the two π^+ mesons is also taken into account.

Similarly one can find

$$\overline{\sum|M_{+00}|^2}=\tfrac{1}{5}\Omega^3$$

Unfortunately, such a complete cancellation of the term $M_iM_k^*$ $(i \neq k)$ does not always take place. For example, it is not difficult to prove that

$$\Psi_{n+--}=\sqrt{\tfrac{1}{3}}\{\sqrt{\tfrac{4}{5}}\varphi_s e_0+\sqrt{\tfrac{1}{2}}(\varphi_1 e_1+\varphi_1' e_1')$$
$$-\sqrt{\tfrac{1}{2}}(\varphi_1 e_2+\varphi_1' e_2')+\sqrt{\tfrac{1}{5}}\varphi_s e_3\}[\tfrac{1}{2},-\tfrac{1}{2}].$$

It can also be demonstrated that

$$M_{n+--}=\sqrt{\tfrac{4}{15}}M_0+\sqrt{\tfrac{1}{6}}(M_1+M_1')-\sqrt{\tfrac{1}{30}}(M_2+M_2').$$

After summation over the isotopic variables, the terms Re $M_1M_2^*$ and Re $M_1'M_2'^*$ do not generally vanish. The fact that the results of the calculation by means of statistical theory with account of isobaric states agree with experiment points to the fact that the remaining cross terms are small.

3. In the case of a system consisting only of mesons and having a definite total spin T, all of the cross terms cancel each other identically upon summation over the isotopic variables. This can be established in the following way. The total wave function of the final state in which we are interested is obtained by symmetrization of the functions $f(1, 2, \ldots n)[1, 2, \ldots n]$, i.e.,

$$\Psi = cf(\ldots j \ldots)[\ldots j \ldots], \tag{5}$$

c is the normalizing coefficient and $[\ldots j \ldots]$ is the isotopic function of the system obtained from the initial $[1, 2, \ldots n]$ as the result of the j-th permutation (compare the obtaining of Eq. (3) for the system $\pi^+\pi^+\pi^-$, where $[1, 2, 3] = [1, 1]_1[1, 1]_2[1, -1]_3$).

We further represent $[\ldots j \ldots]$ in the form

$$[\ldots j \ldots] = a_{ij}e_i, \tag{6}$$

where the e_i are orthonormalized functions forming the bases of an irreducible representation of the permutation group (so that summation over i is a summation over types of irreducible representations and over the numbers of basic functions of the representation of the given type); for the particular n = 3, and total charge 1, see the table. Hence,

$$\Psi = cf(\ldots j \ldots)\, a_{ij}e_i = c\varphi_i e_i. \tag{7}$$

It is important that $\varphi_i = a_{ij}f(\ldots j \ldots)$ have exactly the same permutation structure as e_i, if all the e_i are orthonormal. Actually,

$$e_i = bc_{ij}[\ldots j \ldots] + b'c_{ij}[\ldots j \ldots]' + \ldots . \tag{8}$$

The terms $[\ldots j \ldots]'$ etc refer to other charge states with the same total charge as $[\ldots j \ldots]$; c_{ij} are determined by the type of permutation symmetry; b, b′ etc (actually these are the Clebsch-Gordan coefficients) depend on what total spin T corresponds to e_i. Since $e_ie_k = \delta_{ik}$, then it follows from (6) and (8) that $c_{ij} = a_{ij}$. In this connection the cross terms $\mathrm{Re}\, M_iM_k^*$ $(i \neq k)$ vanish in the square of the matrix element upon summation over the isotopic variable because, for $i \neq k$, there is no unique representation[8] in the product $M_iM_k^*$.

In conclusion, I express my gratitude to I. L. Rozental' and V. M. Maksimenko for interested discussions, and also thank F. Cerulus and R. Hagedorn for supplying preprints of their work.

[1] R. C. Witten and M. M. Block, Phys. Rev. **111**, 1676 (1958).

[2] Maksimenko, Nikishov, and Rozental', Матерялы конференции по физике высоких энергий в Киеве (Materials of the Conference on High-energy Physics, Kiev, 1959) (paper of V. I. Veksler).

[3] Belen'kiĭ, Maksimenko, Nikishov, and Rozental', Usp. Fiz. Nauk **62**, 1 (1957).

[4] F. Cerulus and R. Hagedorn, Particle Production in 6.2-Bev p-p Collisions by a Statistical Model (preprint). R. Hagedorn, Calculation of High-energy Particle Production on an Electronic Computer (preprint).

[5] F. J. Belinfante, Phys. Rev. **92**, 145 (1953).

[6] L. Landau and E. Lifshitz, Квантовая механика (Quantum Mechanics) Gostekhizdat, 1948; English translation, Addison Wesley, 1958.

[7] V. B. Berestetskiĭ, Dokl. Akad. Nauk SSSR **92**, 519 (1953).

[8] L. Landau and E. Lifshitz, op. cit., Par. 94.

Translated by R. T. Beyer
103