ON THE NONLINEAR THEORY OF ELEMENTARY PARTICLES

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The energies and momenta of spinor fields in theories with pseudovector and scalar nonlinear terms are calculated on the basis of a number of new exact solutions of the wave type. By a semiclassical quantization the mass of the nucleon is determined as $k_0 l = 2^{1/2}\pi^{3/2} \approx 7.84$. The dependence of the energy of the field on the degree of nonlinearity is established. The method of fusion is used to derive from the nonlinear spinor equation a nonlinear undor equation, which on certain assumptions reduces to a nonlinear meson equation of the Klein-Gordon type. The conformal invariance of the nonlinear equations of the meson and spinor fields is discussed.

ACCORDING to the unified nonlinear field theory the theory of the elementary particles is based on the spinor equation

$$\{\gamma_\mu (\partial/\partial x_\mu + l^2\gamma_5(\bar{\psi}\gamma_\mu\gamma_5\psi)) + A(\bar{\psi}, \psi)\}\psi = 0, \qquad (1)$$

where $x_\mu \equiv (x_n, it)$, $\hbar = c = 1$, and A is an arbitrary function of $\bar{\psi}$ and ψ. Equations of this type that have been discussed[1-5] are

$$D\psi \equiv \gamma_\mu (\partial/\partial x_\mu + l^2\gamma_5(\bar{\psi}\gamma_\mu\gamma_5\psi))\psi = 0, \qquad (2)$$

$$D\psi \equiv (\gamma_\mu\partial/\partial x_\mu + l^2(\bar{\psi}\psi))\psi = 0. \qquad (2a)$$

We shall call the nonlinear equation with the scalar (pseudovector) nonlinear term simply the "scalar" ("pseudovector") nonlinear equation.

Starting from Eqs. (2) and (2a), Heisenberg[1] determines the spectrum of masses and charges of the elementary particles, using for this purpose the complicated apparatus of quantum field theory and the Tamm-Dancoff approximate method. One can, however, also get close approximations to the masses of the elementary particles from Eqs. (2) and (2a) by a simpler semiclassical method, if one uses a certain approximate condition that is equivalent to quantization.

1. THE ENERGY, MASS, AND CHARGE OF THE NONLINEAR FIELDS

The Nonlinear Pseudovector Equation

1. Let us consider Eq. (2), in which the nonlinear pseudovector term has been chosen by Heisenberg and Pauli[2] from among the possible nonlinear terms suggested by Ivanenko and Brodskiĭ[3] by the use of all the known conservation laws both in ordinary space and in isotopic space. The equation adjoint to Eq. (2) is

$$\bar{\psi}\bar{D} \equiv \bar{\psi}(\partial/\partial x_\mu - l^2\gamma_5(\bar{\psi}\gamma_5\gamma_\mu\psi))\psi = 0. \qquad (2')$$

From Eqs. (2) and (2′) we find

$$\partial(\bar{\psi}\gamma_\mu\psi)/\partial x_\mu = 0, \qquad (1.1)$$

$$\partial(\bar{\psi}\gamma_\mu\gamma_5\psi)/\partial x_\mu = 0. \qquad (1.2)$$

Solutions of Eqs. (2), (2′) and (1.1), (1.2) in the form

$$\psi = a(s)\varphi(\sigma),$$

$$\bar{\psi} = \bar{a}(s)\varphi^*(\sigma), \quad \sigma = k_\mu x_\mu, \quad k_\mu \equiv (k_n, i\omega), \qquad (1.3)$$

where s is the spin coordinate and $\varphi(\sigma)$ does not depend on the matrices γ_μ, are found by the method used earlier.[4] As the result we get a complex solution unique for the given type of equation:

$$\varphi = \varphi_0 \exp\{ik_\mu x_\mu\}, \quad \varphi^* = \varphi_0^* \exp\{-ik_\mu x_\mu\}. \qquad (1.4)$$

The length of the four-vector k_μ and the eigenfunctions a(s), $\bar{a}(s)$ are determined from the equations

$$\gamma_\mu(k_\mu + \gamma_5 a_{\mu 5})a = 0, \qquad (1.5)$$

$$\bar{a}(k_\mu - \gamma_5 a_{\mu 5})\gamma_\mu = 0, \qquad (1.6)$$

where

$$a_{\mu 5} = -il^2(\bar{a}\gamma_\mu\cdot\gamma_5 a), \quad \bar{a} = a^*\gamma_4. \qquad (1.7)$$

Since the energy operator

$$H = \partial/\partial x_4 = -\gamma_4\gamma_n(\partial/\partial x_n + l^2\gamma_5 a_{n5}) - \gamma_5 a_{45} \qquad (1.8)$$

commutes with the spin operator

$$\sigma = \begin{pmatrix} 0 & \sigma' \\ \sigma' & 0 \end{pmatrix}, \qquad (1.9)$$

(σ' stands for the Pauli matrices), we can add to the list (1.3) the equation for the spin

$$(ks - \sigma k)a = 0, \quad \bar{a}(s)(ks - \sigma k) = 0. \qquad (1.10)$$

To calculate $a_{\mu 5}$ we take the matrices in the form

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_n = \begin{pmatrix} 0 & i\sigma_n' \\ -i\sigma_n' & 0 \end{pmatrix},$$

$$\sigma_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2' = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\alpha(s) = \begin{pmatrix} a_1(s) \\ a_2(s) \end{pmatrix}, \quad a_1(s) = \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \quad a_2(s) = \begin{pmatrix} b_3(s) \\ b_4(s) \end{pmatrix}. \qquad (1.11)$$

Then we find

$$a_{45} = -l^2(\overset{*}{a}_1 a_1 - \overset{*}{a}_2 a_2),$$

$$a_{35} = -sl^2(\overset{*}{a}_1 a_1 + \overset{*}{a}_2 a_2) = -sl^2(\bar{a}\gamma_4 a),$$

$$a_{25} = il^2(\overset{*}{b}_1 b_2 + \overset{*}{b}_3 b_4 - \overset{*}{b}_2 b_1 - \overset{*}{b}_4 b_3),$$

$$a_{15} = -l^2(\overset{*}{b}_1 b_2 + \overset{*}{b}_3 b_4 + \overset{*}{b}_2 b_1 + \overset{*}{b}_4 b_3). \qquad (1.12)$$

Let us impose on the amplitude constants the conditions

$$a_{15} = a_{25} = 0. \qquad (1.13)$$

If now in Eqs. (1.5), (1.6), (1.10) we go over to a primed coordinate system[6] in which k_3' is directed along $\boldsymbol{\sigma}$, then on using also the conditions (1.13), we get (dropping the primes)

$$(\omega - s\varepsilon k - i\varepsilon a_{45} - sa_{35})\, a = 0,$$
$$s = \pm 1, \quad \varepsilon = \pm 1. \qquad (1.14)$$

From this we get for the eigenvalue of ω

$$\omega = s\varepsilon k + i\varepsilon a_{45} + sa_{35}. \qquad (1.15)$$

The function $a(s)$ has been subjected only to the requirement (1.13) and the conditions (1.14). We can satisfy them if we prescribe $a(s)$ in the form

$$a_1(s) = \delta_{\varepsilon,1} \begin{pmatrix} c_1\sqrt{1+s} \\ c_2\sqrt{1-s} \end{pmatrix},$$

$$a_2(s) = \delta_{\varepsilon,-1} \begin{pmatrix} c_1\sqrt{1+s} \\ c_2\sqrt{1-s} \end{pmatrix}. \qquad (1.16)$$

Then we get

$$a_{45} = 0, \quad a_{\mu 5}^2 = a_{35}^2,$$

$$a_{35} = -sl^2(\delta_{\varepsilon,1} + \delta_{\varepsilon,-1})[(1+s)c_1\overset{*}{c}_1 + (1-s)\overset{*}{c}_2 c_2] \qquad (1.17)$$

and for ω we find $\omega = s\epsilon k + sa_{35}$.

2. Let us now determine the momentum, charge, and energy of the field of Eq. (2) that correspond to this solution. The Lagrangian density function of Eq. (2) is

$$\mathfrak{L} = \tfrac{1}{2}\{\bar{\psi}(D\psi) - (\bar{\psi}\bar{D})\psi - l^2(\bar{\psi}\gamma_\mu\gamma_5\psi)^2\}. \qquad (1.18)$$

From this we get for the momentum, charge, and energy of the field

$$G_n = i\int T_{n4} d^3x = -k_n(\bar{a}\gamma_4 a)L^3,$$

$$Q = \int \rho_e d^3x = -e(\bar{a}\gamma_4 a)L^3,$$

$$E = \int T_{44} d^3x = -\omega(\bar{a}\gamma_4 a)L^3 + \frac{1}{2}l^{-2}a_{\mu 5}^2 L^3. \qquad (1.19)$$

If we consider the solution (1.16), we get

$$\mathbf{G} = N\mathbf{k}, \qquad -Q = Ne, \qquad E = -\left(\omega + \tfrac{1}{2}Nl^2L^{-3}\right)N,$$

$$\omega = s\varepsilon k - l^2(\bar{a}\gamma_4 a) = s\varepsilon k - l^2NL^{-3},$$

$$N = (\bar{a}\gamma_4 a)L^3. \qquad (1.20)$$

From these formulas we have for $\mathbf{k} = 0$*

$$G = 0, \qquad -Q = Ne,$$

$$E = E_0 = \tfrac{1}{2}N\omega_0, \qquad \omega_0 = NL^{-3}l^2. \qquad (1.21)$$

Let us now calculate the energy contained in the basic periodicity volume given by

$$L^3 = (2\pi/\omega)^3. \qquad (1.22)$$

For $\mathbf{k} = 0$ we get $L = (2\pi/\omega_0) = l(2\pi/N)^{-1/2}$, and we further find

$$l\omega_0 = (2\pi)^{3/2}N^{-1/2}, \qquad lE_0 = \tfrac{1}{2}N(l\omega_0).$$

In the case $N = 1$ we arrive by this normalization, which is essentially equivalent to a sort of primitive quantization, at the rest mass of an elementary particle†

$$k_0 l = \sqrt{2}\,\pi^{3/2} \approx 7.84, \qquad Q = -e \qquad (1.23)$$

($E_0(N=1) = k_0$, $\omega_0(N=1) = k_0' = 2k_0$); this is close to the result of Heisenberg ($k_0 l \approx 6.5$ for the pseudoscalar nonlinear field and $k_0 l \approx 7.4$ for the scalar field[1]).

3. Equation (2) also has a solution of a different form. These solutions can be found by the method of integration of nonlinear equations proposed in reference 4. Equation (2) can be integrated if we introduce new functions φ and $\bar{\varphi}$ by the formulas

$$\psi = -D\varphi, \qquad \bar{\psi} = \bar{\varphi}\bar{D}, \qquad (1.24)$$

which can be rewritten in the form

$$\psi = -\gamma_\mu(\chi_\mu + \eta_{\mu 5}\gamma_5)a, \qquad \bar{\psi} = \bar{a}(\chi_\mu^* - \eta_{\mu 5}^*\gamma_5)\gamma_\mu, \qquad (1.25)$$

where a is a constant spinor. The functions χ_μ and η_μ can be regarded as two independent unknown functions. Substituting Eq. (1.25) in Eq. (2), we get a system of equations for the determination of χ_μ and η_μ. The solution of this system of equations is difficult, however, because of the presence of the summation over the index of the γ_μ. This difficulty can be evaded if we consider a special form of the function (1.25) without the matrices γ_μ:

$$\psi = (\varphi + \gamma_5\chi)a, \qquad \bar{\psi} = \bar{a}(\varphi^* - \gamma_5\chi^*). \qquad (1.26)$$

*Regarding the signs of the parameter l^2 and the Lagrangian (and energy) see Sec. 3.

†An elementary particle is taken to correspond to a field which has the momentum k, the charge e, and the volume given by Eq. (1.22).

Equation (2) then gives

$$\gamma_\mu(\partial\varphi/\partial x_\mu + B_{\mu5}\chi)\,a = 0,$$
$$\gamma_\mu\gamma_5(\partial\chi/\partial x_\mu + B_{\mu5}\varphi)\,\alpha = 0. \qquad (2'')$$

$$B_{\mu5} \equiv l^2(\bar\psi\gamma_\mu\gamma_5\psi) = l^2(\bar a\gamma_\mu\gamma_5 a)(\varphi^*\varphi + \chi^*\chi) + l^2(\bar a\gamma_\mu a \quad \chi + \chi^*\varphi). \qquad (1.27)$$

Along with Eq. (2″) we must also take the system of complex conjugate equations.

From Eq. (2″) we can get the equations

$$(\partial\varphi/\partial x_\mu + B_{\mu5}\chi)^2 a = 0, \qquad (1.28)$$
$$(\partial\chi/\partial x_\mu + B_{\mu5}\varphi)^2 a = 0. \qquad (1.29)$$

If we now seek a solution in the form

$$\varphi = \varphi(\sigma), \quad \chi = \chi(\sigma),$$
$$\varphi^* = \varphi, \quad \chi = i\eta, \quad \eta^* = \eta, \quad \sigma = k_\mu x_\mu, \qquad (1.30)$$

for the system of equations (1.28), we can satisfy Eq. (1.29) with

$$\varphi' = \eta, \quad \eta' = \varphi, \quad \varphi^2 + \eta^2 = \text{const},$$
$$B_{\mu5} = ia_{\mu5} = \text{const}, \qquad (1.31)$$

and we get for the solution

$$\psi = e^{i\gamma_5\sigma}a(s), \quad \bar\psi = \bar a(s)\,e^{i\gamma_5\sigma} \qquad (1.32)$$

The length k_μ^2 is determined from the equations

$$(k_\mu^2 - a_{\mu5}^2)\,a = 0, \quad k_\mu a_{\nu5} - k_\nu a_{\mu5} = 0 \qquad (1.33a)$$

or

$$(k_\mu^2 + a_{\mu5}^2 + 2k_\mu a_{\mu5})\,a = 0. \qquad (1.33b)$$

The eigenfunctions $a(s)$ themselves must be determined from the equation

$$\gamma_\mu(k_\mu + a_{\mu5})\,a = 0, \qquad (1.34)$$

which in the case (1.33a) gives $a = \gamma_\nu(k_\nu - a_{\nu5})a'$, where a' is an arbitrary constant spinor. In the case (1.33b), on the other hand, Eq. (1.34) is solved in just the same way as Eq. (1.5) with the supplementary condition (1.13). We find as the result

$$(\omega - ia_{45} - s\varepsilon(k + a_{35}))\,a = 0. \qquad (1.35)$$

For the eigenvalues ω we get*

$$\omega = s\varepsilon k + \varepsilon(i\varepsilon a_{45} + sa_{35}). \qquad (1.36)$$

If we now fix $a(s)$ in the form (1.16), we find

$$\omega = s\varepsilon(k + a_{35}). \qquad (1.37)$$

Let us determine the momentum, charge, and energy of the field that correspond to the solution (1.32) and (1.16). We have

$$\mathbf{G} = -\mathbf{k}(\bar a\gamma_4\gamma_5 a)L^3, \qquad Q = e(\bar a\gamma_4 a)L^3,$$
$$E = -\omega(\bar a\gamma_4\gamma_5 a)L^3 - \tfrac{1}{2}l^{-2}a_{\mu5}^2L^3 \qquad (1.38)$$

*An eigenvalue ω is also obtained from Eq. (1.33b), which with the condition (1.13) gives $(i\omega + a_{45})^2 + (k + a_{35})^2 = 0$, i.e., $\omega = k + ia_{45} + a_{35}$ ($s = \varepsilon = 1$).

and, using Eq. (1.17), we get

$$G = 0, \quad Q = Ne, \quad E = -\tfrac{1}{2}(Nl^2L^{-3})N, \qquad (1.39)$$
$$\omega = s\varepsilon(k - s\omega_0),$$

$$\omega_0 = -sa_{35} = l^2NL^{-3}, \quad (\bar a\gamma_4 a)L^3 = -sl^2a_{35}L^3 = N. \qquad (1.40)$$

From these results we find for $\mathbf{k} = 0$:

$$G = 0, \quad Q = Ne, \quad -E_0 = \tfrac{1}{2}N\omega_0, \quad \omega = \varepsilon\omega_0. \qquad (1.41)$$

Let us now again determine the energy contained in the fundamental periodicity volume (1.22). For $\mathbf{k} = 0$ Eq. (1.22) gives $L = 2\pi/\omega_0 = lN^{1/2}(2\pi)^{-1/2}$. We get

$$Q = Ne, \quad \omega_0 l = N^{-1/2}(2\pi)^{3/2}, \quad E_0 l = -\tfrac{1}{2}(l\omega_0)N. \qquad (1.42)$$

In the case $N = 1$ we again arrive at the charge and mass of an elementary particle:

$$k_0 l = \sqrt{2}\,\pi^{3/2} \approx 7.84, \quad Q = e. \qquad (1.43)$$

$$E_0(N = 1) = -k_0, \quad \omega_0(N = 1) = k_0' = 2k_0. \qquad (1.44)$$

The Nonlinear Scalar Equation

1. Let us consider the scalar equation (2a), or for the sake of generality the equation

$$(\gamma_\mu\partial/\partial x_\mu + A(\bar\psi\psi))\,\psi = 0. \qquad (2a')$$

The solution of the equation (2a′) in the form (1.3) for real $A(\bar\psi, \psi)$ is given by Eq. (1.4) with

$$(\gamma_\mu k_\mu - i\bar k_0)\,a = 0, \quad k_\mu^2 = -\bar k_0^2 = -A^2(\rho_0),$$
$$\rho_0 = \overset{*}{\varphi_0}\varphi_0. \qquad (1.45)$$

Equation (2a′) also has a solution of a different form, which can be obtained by the method of integration.[4] We must introduce new functions φ and $\bar\varphi$ by the formulas*

$$\psi = \varphi - \gamma_\mu\frac{\partial\varphi}{\partial x_\mu}\frac{1}{A(\bar\psi,\psi)},$$

$$\bar\psi = \bar\varphi + \frac{\partial\bar\varphi}{\partial x_\mu}\gamma_\mu\frac{1}{A(\bar\psi,\psi)}. \qquad (1.46)$$

*Here on the assumption $\varphi = a\Phi$, $\bar\varphi = \bar a\Phi$, $A(\bar\psi, \psi) = A(\rho)$, $\rho = (\bar\psi\psi)$ the integrated equation is of the form[4]

$$\left(\partial^2/\partial x_\mu^2 - A^2(\rho) + A^{-1}(\rho)\frac{dA}{d\rho}\frac{\partial\rho}{\partial x_\mu}\frac{\partial}{\partial x_\mu}\right)\Phi = 0,$$

where in the case of Eq. (2a) the quantity ρ is determined from the algebraic equation

$$\rho^3 - (\bar a a)\,\Phi^2\rho^2 + l^{-4}(\bar a a)(\partial\Phi/\partial x_\mu)^2 = 0,$$

which, as can be shown, has a single real solution for $(i\partial\Phi/\partial x_\mu)^2 > 0$.

If we now look for $\varphi(x_\mu)$ in the form* $\varphi = \varphi(\sigma)\,a$, $\bar{\varphi} = \bar{a}\varphi^*(\sigma)$, then Eq. (1.46) is written

$$\psi = (\varphi - \gamma_\mu k_\mu \bar{k}_0^{-1}\chi)\,a, \quad \bar{\psi} = \bar{a}\,(\varphi^* + \gamma_\mu k_\mu \bar{k}_0^{-1}\chi^*), \quad (1.47)$$

where φ and χ can be regarded as two independent unknown functions.† As solutions we have $\chi = 0$, $\varphi^*\varphi = \text{const}$, which reduces to Eq. (1.4), and also‡ $\varphi = \varphi_0 \cos(\sigma + c)$, $\chi = \varphi_0 \sin(\sigma + c)$, i.e.,

$$\psi = (\cos(\sigma + c) - \gamma_\mu k_\mu \bar{k}_0^{-1} \sin(\sigma + c))\,a, \quad (1.48)$$

where a is an arbitrary amplitude, and $k_\mu^2 = -\bar{k}_0^2 = A^2(\rho_0)$.

2. Let us determine the momentum, charge, and energy of the field (2a) with the Lagrangian

$$\mathfrak{L} = -\tfrac{1}{2}\{\bar{\psi}(D\psi) - (\bar{\psi}\bar{D})\psi - l^2(\bar{\psi}\psi)^2\}. \quad (1.49)$$

In the case of the solution (1.4), (1.45) we find

$$\mathbf{G} = \mathbf{k}\,(\bar{a}\gamma_4 a)\,L^3, \quad Q = e\,(\bar{a}\gamma_4 a)\,L^3,$$

$$E = \left[\omega - \tfrac{1}{2} l^2 (\bar{a}a)^2 / (\bar{a}\gamma_4 a)\right](\bar{a}\gamma_4 a)\,L^3, \quad (1.50)$$

and after the normalization $N = (\bar{a}\gamma_4 a)\,L^3$ we get $(\bar{a}a) = NL^{-3}(\omega_0/\omega) = \bar{k}_0 l^{-2}$,

$$\mathbf{G} = N\mathbf{k}, \quad Q = Ne, \quad E = \tfrac{1}{2}N\omega\left(1 - \tfrac{1}{2}(k^2/\omega^2)\right),$$
$$\omega = NL^{-3}l^2. \quad (1.51)$$

Finally we get for the charge and mass of the elementary particle

$$G = 0, \quad Q = e, \quad k_0 l = \sqrt{2}\,\pi^{3/2} \approx 7.84. \quad (1.52)$$

In the case of the solution (1.48) we have

$$\mathbf{G} = \mathbf{k}\,(\bar{a}a)\frac{\omega}{\omega_0}L^3, \quad Q = e\,(\bar{a}\gamma_4 a)\,L^3,$$

$$E = \left(\omega - \frac{l^2}{2}\left(\frac{\omega_0}{\omega}\right)(\bar{a}a)\right)(\bar{a}a)\left(\frac{\omega}{\omega_0}\right)L^{-3} \qquad (\omega_0 = \bar{k}_0) \quad (1.53)$$

and after the normalization $(\bar{a}a)(\omega/\omega_0)\,L^{-3} = N$ we find* $(\bar{a}\gamma_4 a) = NL^{-3}$,

$$\mathbf{G} = N\mathbf{k}, \quad Q = Ne, \quad E = \tfrac{1}{2}N\omega\left(1 + (k_0/\omega)^2\right),$$
$$\omega = NL^{-3}l^2. \quad (1.54)$$

Determining the energy in the volume defined by Eq. (1.21), we again arrive at the results (1.52).

Dependence of the Rest Mass of the Elementary Particle on the Degree of Nonlinearity

To find the dependence of the rest mass of the elementary particle on the degree of nonlinearity, we call attention to the fact that when $A(\bar{\psi}, \psi)$ is an arbitrary function of $\rho = (\bar{\psi}\psi)$ the functions (1.4), (1.45), and (1.48) are also solutions of the general equation (2a′). The Lagrangian density corresponding to Eq. (2a′) can be written in the form

$$\mathfrak{L} = -\tfrac{1}{2}\{\bar{\psi}(D\psi) - (\bar{\psi}\bar{D})\psi + [B(\hat{\rho}) - \hat{\rho}\,dB(\hat{\rho})/d\hat{\rho}]\}, \quad (1.55)$$

where $\hat{\rho} = l(\bar{\psi}\psi)$ and $B(\hat{\rho})$ is an arbitrary function.

From the Lagrangian (1.55) we at once find the field equation (2a′), in which

$$A(\hat{\rho}) = \tfrac{1}{2}l\,dB(\hat{\rho})/d\hat{\rho},$$

and the expressions for the momentum, charge, and mass take the forms

$$\mathbf{G} = N\mathbf{k}, \quad Q = Ne, \quad E = (\omega - [A(\hat{\rho}_0) - l\hat{\rho}_0^{-1}B(\hat{\rho}_0)])\,N,$$
$$N = (\bar{a}\gamma_4 a)\,L^3, \quad \omega^2 = k^2 + \omega_0^2,$$
$$\omega_0 = A(\hat{\rho}_0), \quad \hat{\rho}_0 = lNL^{-3}(\omega_0/\omega). \quad (1.56)$$

For $\mathbf{k} = 0$, $N = 1$ we obtain as the charge and mass of the elementary particle

$$Q = e, \quad k_0 = (l/2\hat{\rho}_0)\,B(\hat{\rho}_0) \quad (\omega_0(N = 1) = k_0'). \quad (1.57)$$

From the conditions

$$\rho_0 = L^{-3}, \quad k_0' = A(\hat{\rho}_0) = (l/2\hat{\rho}_0)\,B(\hat{\rho}_0)$$

with the additional requirement $L = 2\pi/k_0'$ we get

$$dB(\rho_0)/d\rho_0 = 4\pi\rho_0^{1/3}. \quad (1.58)$$

For a prescribed form of the function $B(\hat{\rho})$ we

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*If we look for solutions in the form $\varphi = \varphi(\xi)\,a$, $\varphi^* = \bar{a}\varphi(\varepsilon)$, $\xi = -x_\mu^2$, then Eq. (1.46) gives $\psi = (\varphi + \gamma_\mu x_\mu \chi)\,a$, $\bar{\psi} = \bar{a}(\varphi - \gamma_\mu x_\mu \chi)$, $\rho = (\bar{\psi}\psi) = (\bar{a}a)(\varphi^2 + \xi^2\chi^2)$, and from Eq. (2a′) we find

$$4\chi + 2\xi\chi' + A(\rho)\,\varphi = 0, \quad \text{(a)}$$
$$2\varphi' - A(\rho)\,\chi = 0. \quad \text{(b)}$$

In the case $A(\rho) = l^2(\bar{\psi}\psi)$ the equations (a) and (b) coincide with the equations obtained by Heisenberg.[5] If we now eliminate $A(\rho)$ from Eqs. (a) and (b), we get

$$\chi^2 = -\rho'/3(\bar{a}a) \quad \text{(c)}$$

and then substituting Eq. (c) in Eq. (a) we find

$$\xi\rho'' + 4\rho' - A(\rho)\sqrt{-\rho'(3\rho + \xi\rho')} = 0, \quad \rho' \equiv d\rho/d\xi.$$

With a special choice of $A(\rho)$ it may be possible in this way to find also the corresponding exact solution of Eq. (2a′).

†Then Eq. (2a′) gives

$$\gamma_\mu k_\mu(\varphi' + \bar{k}_0^{-1}A(\rho)\,\chi)\,a = 0, \quad \gamma_\mu k_\mu(\chi' + A(\rho)\,\varphi\bar{k}_0 k_\mu^{-2})\,a = 0,$$
$$\rho = \rho_0 + \rho_1, \quad \rho_1 = (\bar{a}a)(\varphi^*\varphi - k_\mu^2\bar{k}_0^{-1}\chi^*\chi), \quad \text{(A)}$$
$$\rho_2 = (\bar{a}\gamma_\mu k_\mu a)(\varphi^*\chi - \chi^*\varphi).$$

We must take along with Eq. (A) the system of complex conjugate equations. It is easy to show further that

$$\varphi^*\varphi + \chi^*\chi = \text{const}, \quad \varphi^*\chi - \chi^*\varphi = \text{const},$$
$$\rho = \rho_0 = \text{const}, \quad k_\mu^2 = -\bar{k}_0^2 = -A^2(\rho_0).$$

‡It is easy to verify that

$$\frac{\partial}{\partial x_\mu}(\bar{\psi}\gamma_\mu\psi) = \frac{d}{d\sigma}(\bar{\psi}\gamma_\mu k_\mu\psi) = \frac{d}{d\sigma}(\bar{a}\,(\varphi^2 - \chi^2 k_\mu^2/\bar{k}_0^2)\,a) = 0.$$

*Since the a(s) are arbitrary functions, they can be taken to be the solution of Eq. (1.45).

can use Eq. (1.58) to express $\rho_0 = (\bar{a}a)$ in terms of absolute constants. For example, taking $B(\hat{\rho})$ in the form

$$B(\hat{\rho}) = l^n (\bar{\psi}\psi)^n. \tag{1.59}$$

we get

$$\hat{\rho}_0 = (4\pi / n)^{3/(3n-4)}\, l^{-4/(3n-4)},$$
$$k_0' = A(\hat{\rho}_0) = \tfrac{1}{2} n l \rho_0^{n-1}$$
$$= \tfrac{1}{2}(4\pi)^{(3n-3)/(3n-4)}\, l^{-n/(3n-4)} n^{-1/(3n-4)}, \tag{1.60}$$

and the mass of the elementary particle is

$$k_0 = \hat{\rho}_0^n / 2\rho_0 = \tfrac{1}{2} l \hat{\rho}_0^{n-1} = k_0' / n,$$

that is,

$$k_0 l^{n/(3n-4)} = \tfrac{1}{2}(4\pi / n)^{(3n-3)/(3n-4)}. \tag{1.61}$$

Let us note some features of the function (1.61). For n = 1 (linear theory) Eq. (1.61) gives $k_0 l^{-1} = 1/2$, in complete agreement with reality. For n = 2 [the case of the nonlinear equation (2a)] we get $k_0 l = 2^{1/2}\pi^{3/2} \approx 7.84$, as was to be expected. For $n \to \infty$ we find

$$l^{1/3} k_0 (n \to \infty) \to \tfrac{1}{2}(4\pi / n)_{n\to\infty} \to 0.$$

For $n \to 0$ we have

$$k_0 (n \to 0) \to \tfrac{1}{2}(4\pi / n)_{n\to 0} \to \infty.$$

Thus in this interpretation we can get any value of the mass by varying the degree of nonlinearity n. For example, we can use for the neutrino the case of an infinite degree of nonlinearity, and for the electron the case of a finite but very large value of n.

The function (1.61) is meaningless for one case, n = 4/3 (the Gürsey case[7]), i.e., for

$$B(\hat{\rho}) = \hat{\rho}^{4/3} = l^{4/3} (\bar{\psi}\psi)^{4/3}. \tag{1.62}$$

The peculiarity of this case is that there is now no need for the requirement (1.22). The expression analogous to Eq. (1.22) is obtained automatically. Besides this, the parameter l of the nonlinear term is a dimensionless quantity in this case, and the length L plays the role of the dimensional parameter in the theory. In fact, in the case (1.62) Eq. (1.56) gives the relations

$$\mathbf{G} = N\mathbf{k},\quad Q = Ne,\quad E = \left(\omega + \tfrac{1}{2} l \hat{\rho}_0^{1/3}\right) N,\quad N = (\bar{a}\gamma_4 a) L^3,$$

$$\hat{\rho}_0 = l(\bar{a}a) = lNL^{-3}(\omega_0/\omega),\quad \omega_0 = \tfrac{2}{3} l^{4/3} (N(\omega_0/\omega))^{1/3}.$$

From these we find for $\mathbf{k} = 0$, N = 1

$$G = 0,\quad Q = e,\quad k_0 L = \tfrac{1}{2} l^{4/3},\quad k_0' = \tfrac{4}{3} k_0$$

and the condition analogous to Eq. (1.22) is

$$L k_0' = \tfrac{2}{3} l^{4/3}. \tag{1.22'}$$

Equation (1.22′) becomes identical with Eq. (1.22) for a suitable choice of the constant* $l^{4/3}$:

$$l^{4/3} = 3\pi,\qquad k_0 L = \tfrac{3}{2}\pi \approx 4.71. \tag{1.63}$$

2. ON THE THEORY OF THE FUSION OF NONLINEAR FIELDS

The introduction of a single fundamental spinor field as the basis of the theory of elementary particles requires the definition of some procedure for getting other fields from the fundamental field.

As has been pointed out in reference 8, the de Broglie fusion method, together with the use of group theory, indicates a possibility for such a procedure. Group theory is too general, however, and in particular does not fix any connection between the nondifferential parts of the equations of the various fields. For example, if we start the fusion with Eq. (2a), group theory will give no information about the concrete form of the nonlinear terms in the equations obtained after the fusion. For this reason in the present paper, in dealing with the problem of the fusion of nonlinear fields, we apply the second, fundamental, fusion method of de Broglie, which can be called the "method of fusion of equations."[6]

If we take two spinor fields $\bar{\psi}_i^{(1)}$ and $\psi_k^{(2)}$ that obey the linear Dirac equation, we can form a function $\psi_{ik} = \bar{\psi}_i^{(1)}\psi_k^{(2)}$, which is a component of an undor of the second rank $(\psi\Gamma)$, so that $(\psi\Gamma)$ obeys the Dirac equation or the corresponding Klein-Gordon equation. The undor $(\psi\Gamma)$ can be expanded in the following way:

$$(\psi\Gamma) = \gamma_0\varphi_0 + \gamma_5\varphi_5 + \gamma_\mu\varphi_\mu + \gamma_\mu\gamma_5\varphi_{\mu 5} + \sigma_{\mu\nu}\varphi_{\mu\nu} = \sum_{\alpha=1}^{16} \theta_\alpha \varphi_\alpha,$$
$$\gamma_0 \equiv \gamma_4, \tag{2.1}$$

where θ_α is one of the sixteen independent Dirac matrices.

Let us now consider fusion, starting with the nonlinear Dirac equation

$$(\gamma_\mu \partial / \partial x_\mu + k_0(\bar{\psi}, \psi))\psi = 0. \tag{2.2}$$

As in the case of the linear theory, we consider two fields $\bar{\psi}_i^{(1)}$ and $\psi_k^{(2)}$ that satisfy Eq. (2.2) and the adjoint equation. We form the function $\psi_{ik} = \bar{\psi}_i^{(1)}\psi_k^{(2)}$ and go over to the undor equation

$$(\gamma_\mu \partial / \partial x_\mu + \bar{k}_0(\bar{\psi}, \psi))(\psi\Gamma) = 0,$$
$$\bar{k}_0(\bar{\psi}, \psi) = \tfrac{1}{2}[k_0(\bar{\psi}^{(1)}, \psi^{(1)}) + k_0(\bar{\psi}^{(2)}, \psi^{(2)})]. \tag{2.3}$$

If we now impose on $\bar{k}_0(\bar{\psi}, \psi)$ the requirement

*It is easily verified that Eq. (1.62) with the value (1.63) inserted, i.e., the function $B(\hat{\rho}) = 3\pi\hat{\rho}^{4/3}$, is a solution of Eq. (1.58).

$$\bar{k}_0(\bar{\psi}, \psi) = k_0(\bar{\psi}^{(1)}, \psi^{(2)}) = k_0(\bar{\psi}^{(2)}, \psi^{(1)}) = \bar{k}_0(\bar{\psi}^{(1)}, \psi^{(2)}) \quad (2.4)$$

and use the fact that when we confine ourselves to the lowest powers of the invariants $\bar{k}_0(\bar{\psi}^{(1)}, \psi^{(2)})$ has the form[3]

$$\bar{k}_0(\bar{\psi}^{(1)}, \psi^{(2)}) = \sum_{\alpha=1}^{16} \lambda_\alpha \theta_\alpha (\bar{\psi}^{(1)}\theta_\alpha \psi^{(2)}), \quad (2.5)$$

where λ_α are arbitrary constants, we arrive at the conclusion that $\bar{k}_0(\bar{\psi}^{(1)}, \psi^{(2)})$ is also an undor. Let us now choose the constants λ_α so that

$$\lambda_\alpha (\bar{\psi}^{(1)}\theta_\alpha \psi^{(2)}) = \lambda_0 \varphi_\alpha. \quad (2.6)$$

Then Eq. (2.5) can be written in the form

$$\bar{k}_0(\bar{\psi}^{(1)}\psi^{(2)}) = \lambda_0 (\psi\Gamma). \quad (2.7)$$

Substituting Eq. (2.7) in Eq. (2.3), we get the nonlinear undor equation

$$(\gamma_\mu \partial/\partial x_\mu + \lambda_0(\psi\Gamma))(\psi\Gamma) = 0. \quad (2.8)$$

Applying the operator $\gamma_\nu \partial/\partial x_\nu$ to Eq. (2.8), we find

$$(\partial^2/\partial x_\mu^2 - 2\lambda_0^2(\psi\Gamma)^2)(\psi\Gamma) = 0. \quad (2.9)$$

Let us now consider the expression

$$(\psi\Gamma)^2 = \varphi_0^2 + \varphi_5^2 + \varphi_\mu^2 - \varphi_{\mu 5}^2 + (\sigma_{\mu\nu}\varphi_{\mu\nu})^2 + f(\ldots), \quad (2.10)$$

where $f(\ldots)$ denotes terms containing mixed derivatives of the various fields. In a similar way we get for $(\psi\Gamma)^3$

$$(\psi\Gamma)^3 = \gamma_0\varphi_0^3 + \gamma_5\varphi_5^3 + \gamma_\mu\varphi_\mu\varphi_\nu^2 - \gamma_\mu\gamma_5\varphi_{\mu\nu}\varphi_{\nu 5}^2 + \sigma_{\mu\nu}\varphi_{\mu\nu}(\sigma_{\alpha\beta}\varphi_{\alpha\beta})^2 + U(\ldots). \quad (2.11)$$

The mixed derivatives of various fields lead to a nonlinear interaction of these fields. In the case in which we consider only the self-interactions of the fields, the quantities $f(\ldots)$ and $U(\ldots)$ are to be neglected. We then get

$$(\partial^2/\partial x_\mu^2 - 2\lambda_0^2\varphi_\alpha^2)\varphi_\alpha = 0, \quad (2.12)$$

where φ_α is one of the components of the undor.*

We have previously considered[4] an expression of the type of Eq. (2.12) and have shown that it leads to a spectrum of meson masses.

*If we take into account the fact that according to field theory there are three relations[1] between the invariants φ_0^2, φ_5^2, $\varphi_{\mu 5}^2$, φ_μ^2, $\varphi_{\mu\nu}^2$, we can put the expression (2.10) in the form

$$(\psi\Gamma)^2 = \alpha_1\varphi_0^2 + \alpha_2\varphi_5^2 + f(\ldots).$$

Then Eq. (2.11) will have the corresponding form

$$(\psi\Gamma)^3 = \alpha_1\gamma_0\varphi_0^3 + \alpha_2\gamma_5\varphi_5^3 + U(\ldots),$$

and instead of Eq. (2.12) we get

$$\left(\frac{\partial^2}{\partial x_\mu^2} - 2\lambda_0^2\begin{pmatrix}\alpha_1\varphi_0^2\\ \alpha_2\varphi_5^2\end{pmatrix}\right)\begin{pmatrix}\varphi_0\\ \varphi_5\end{pmatrix} = 0, \quad \frac{\partial^2}{\partial x_{\mu^2}}\begin{pmatrix}\varphi_\mu\\ \varphi_{\mu 5}\\ \varphi_\mu\end{pmatrix} = 0, \quad \frac{\partial\varphi_\mu}{\partial x_\mu} = 0.$$

3: THE CONFORMAL INVARIANCE OF THE NONLINEAR EQUATIONS

As a rule nonlinear equations possess the property of so-called conformal invariance, which makes it possible to go from one particular solution to another by a simple change of scale of the coordinates. We have already called attention to the property of conformal invariance of the nonlinear meson-field equation

$$(\Box - \lambda\varphi^2)\varphi = 0 \quad (3.1)$$

in an earlier paper,[9] where in particular we showed that if $\psi(x_\mu, \lambda)$ is a solution of Eq. (3.1) then the function

$$\psi'(x_\mu, \lambda) = AB^{1/2}\psi(Ax_\mu, B\lambda) \quad (3.2)$$

is also a solution of that equation.*

Let us introduce coordinate transformations (regarding the nonlinear parameter as a fifth coordinate):

$$x'_\mu = Ax_\mu, \quad \lambda' = B\lambda. \quad (3.3)$$

The invariance of Eq. (3.1) under (3.3) gives

$$\psi'(x', \lambda') = S(AB)\psi(x_\mu, \lambda), \quad (3.4)$$

where $S(AB)$ satisfies the conditions

$$SS^{-1} = 1, \quad S^2A^2B = 1, \quad S^{-1} = AB^{1/2}, \quad (3.5)$$

which also leads to the expression (3.2).

Let us use the property of conformal invariance of the field equation to derive conservation laws. According to Noether's theorem, for this we need invariance of the Lagrangian function under the transformations in question.

Under the transformation (3.3), (3.4) and the condition (3.5) the Lagrangian of Eq. (3.1),

$$\mathfrak{L} = \frac{1}{2}\int\left\{\left(\frac{\partial\varphi}{\partial x_\mu}\right)^2 + \frac{1}{2}\lambda\varphi^4\right\}(d^4x) \quad (3.6)$$

transforms according to the formula

$$\mathfrak{L}'[\psi'(Ax_\mu, B\lambda)] = g(AB)\,\mathfrak{L}[\psi(x_\mu, \lambda)].$$

*From this it follows in particular that the functions $A\psi(Ax_\mu, \lambda)$ and $B^{1/2}\psi(x_\mu, B\lambda)$ are solutions of Eq. (3.1). For example, the second of these forms expresses the fact that the solution of Eq. (3.1) can contain the nonlinear parameter only in the combination $\lambda\varphi_0^2$, where φ_0 is the amplitude of the solution.

From this we get*

$$\mathfrak{L}(\lambda) = B\mathfrak{L}(B\lambda), \ g(AB) = g(B) = B^{-1}. \qquad (3.7)$$

Confining ourselves to the case $B = +1$, $S^{-1}(AB) = S^{-1}(A) = A$, we find

$$\psi'(x'_\mu, \lambda) = S(A)\psi(x_\mu, \lambda),$$
$$\psi'(x_\mu, \lambda) = S(A)\psi(A^{-1}x_\mu, \lambda). \qquad (3.8)$$

Let us now introduce the representative operator $T(A)$ and write

$$\psi'(x'_\mu, \lambda) = T(A)\psi(x_\mu, \lambda) = S(A)\psi(A^{-1}x_\mu, \lambda). \qquad (3.9)$$

Confining ourselves to a consideration of the continuous group, let us introduce a small parameter α by the relation $A = 1 + \alpha$, and also the infinitesimal operators† $\hat{J}$ and $\hat{S}$; we then find from Eq. (3.9)

$$\{\hat{J} - (\hat{S} - x_\mu \partial/\partial x_\mu)\}\psi(x_\mu, \lambda) = 0. \qquad (3.10)$$

Let us now consider the case of the nonlinear spinor equation

$$\{\gamma_\mu(\partial/\partial x_\mu + l_1^2\gamma_5(\bar{\psi}\gamma_\mu\gamma_5\psi)) + l_2^2(\bar{\psi}\psi)\}\psi = 0. \qquad (3.11)$$

The conformal invariance of Eq. (3.11) was pointed out recently by Heisenberg and his coworkers.[1] They also made an attempt to connect the conformal invariance of Eq. (3.11) with a definite conservation law. Here we shall consider the conformal invariance of Eq. (3.11) in a different aspect, in particular in connection with the meson field equation (3.1).

The Lagrangian function corresponding to Eq. (3.11) is

$$\mathfrak{L} = \frac{1}{2}\int\left\{\bar{\psi}\gamma_\mu\frac{\partial\psi}{\partial x_\mu} - \frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_\mu\psi + l_1^2(\bar{\psi}\gamma_\mu\gamma_5\psi)^2 + l_2^2(\bar{\psi}\psi)^2\right\}(d^4x). \qquad (3.12)$$

Under the transformation (3.3) the quantities $\psi(x_\mu, \lambda_i)$ and $\mathfrak{L}[\psi(x_\mu, \lambda_i)]$ transform according to the formulas‡

$$\psi'(Ax_\mu, B\lambda_i) = S(AB)\psi(x_\mu, \lambda_i),$$
$$S^{-1}(AB) = \sqrt{AB}, \ \lambda_i \equiv l_i^2;$$
$$\mathfrak{L}'[\psi'(Ax_\mu, B\lambda_i)] = g(AB)\mathfrak{L}[\psi(x_\mu, \lambda_i)], \ g(AB) = A^2/B. \qquad (3.13)$$

Taking $g(AB) = 1$, $A^2 = B$, i.e., $S(AB) = B^{-3/4} = A^{-3/2}$, and introducing the representative operator $T(A)$, we get

$$\psi'(x'_\mu, \lambda'_i) = T(A)\psi(x_\mu, \lambda_i) = S(A)\psi(A^{-1}x_\mu, A^{-2}\lambda_i). \qquad (3.14)$$

Introducing the small parameter $\alpha = A - 1$ and the infinitesimal operators J and S, we find*

$$\{\hat{J} - (\hat{S} - x_\mu\partial/\partial x_\mu - l_i\partial/\partial l_i)\}\psi(x_\mu, \lambda_i) = 0, \ \lambda_i \equiv l_i^2. \qquad (3.15)$$

As the eigenvalues of the operator $\hat{S}$ we get $+1$ for bosons and $-3/2$ for spinors.

We now turn out attention to the sign of the nonlinear parameter l_1^2. It follows from Eqs. (3.7) and (3.13) that if $B = -1$ the Lagrangian merely changes sign, and therefore as long as the sign of the Lagrangian does not impose any conditions on physical processes both signs are permissible in nonlinear equations.

In conclusion I express my gratitude to D. Ivanenko for a discussion on the nonlinear theory.

Note added in proof (December 12, 1959). Introducing the operator $\hat{p}\psi \equiv (\hat{J} - \hat{S})\psi = p\psi$ and the new coordinates $\xi_j(x_\mu, l_i)$ [in the case of Eq. (3.10)], or $\xi_j(x_\mu)$ [in the case of Eq. (3.15)], we can write Eqs. (3.10) and (3.15) in the form $(\xi_j\partial/\partial\xi_j + p)\psi = 0$, with the solution

$$\psi = c_1\sigma_1^{-p} + c_2\sigma_2^{-p/2} + c_3\sigma_1^{\lambda}\sigma_2^{-(p+\lambda)/2}, \qquad (a)$$

where c_1, c_2, c_3, λ are arbitrary constants, $\sigma_1 = k_j\xi_j$, and $\sigma_2 = -\xi_j\xi_j$. Now comparing Eq. (a) with the particular solution $\psi = \lambda^{-1/2}\sigma_2^{-1/2}$ of Eq.

*It may be helpful to indicate the analogy with the linear theory, in which the field equation is invariant with respect to the transformation

$$\psi'(x_\mu, k_0) = N^{1/2}\psi(x_\mu, k_0),$$

where N is a number. Under this the Lagrangian function transforms in the following way:

$$\mathfrak{L}'[N^{1/2}\psi'(x_\mu, k_0)] = N\mathfrak{L}[\psi(x_\mu, k_0)],$$

which gives the possibility of introducing the number of particles. In reference 9 we have used the analogy of these transformations with Eqs. (3.4) and (3.7) to introduce the number of particles $N = B$ in the nonlinear field theory.

†$\hat{J} = \partial T(\alpha)/\partial\alpha|_{\alpha=0}$, $\hat{S} = \partial S(\alpha)/\partial\alpha|_{\alpha=0}$, and, as follows from Eq. (3.9), the eigenvalue of $\hat{S}$ is +1.

‡As we see, the transformation (13) also gives the possibility of introducing the number of particles $N = BA^{-2}$

*The compatibility of the quantum number and the conformal invariance require that the conformal transformation operator commute with the translation operator. Generally speaking these operators do not commute, but Heisenberg[1] prescribes the transformation of translation in the form

$$x'_\mu = x_\mu + \alpha l_i,$$

and since in the case of Eq. (3.11), according to Eq. (3.15), l_i transforms in just the same way as x_μ, the operators do commute. In the case of Eq. (3.1), according to Eq. (3.9), prescription of the translations in the form (a) does not lead to commuting operators: instead of (a) we must write $x'_\mu = x_\mu + \alpha\varphi_0$, where φ_0 is the amplitude of the solution, which, according to Eq. (3.9), transforms just like x_μ.
In the case of Eq. (3.11) the translation can be prescribed in the form $x'_\mu = x_\mu + \alpha\varphi_0^{2/3}$, where φ_0 is the amplitude of the solution, which, according to Eq. (3.15), transforms just like $x^{2/3}$ and secures commutativity of the operators of translation and conformal transformation.

(3.1) and the particular solution $\psi = c\sigma_2^{-1/4} \times \{1 + \gamma_\mu x_\mu \sigma_2^{-1/2}\}$ of Eq. (3.11) (with $\sigma_2 = -x_\mu^2 + l_1^2 \approx -x_\mu^2$), we find p = 1 for bosons and p = ½ for spinors. From this we get for the eigenvalues

$$J = 2\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}, \qquad p = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \qquad s = \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \begin{matrix} \text{spinors} \\ \text{bosons} \end{matrix}.$$

If in the derivation of (4.16) (sic) we start not from (4.15) (sic) but from

$$\psi'(x'_\mu, \lambda'_i) = T(B)\psi(x_\mu, \lambda_i) = S(B)\psi(B^{-1/2}x_\mu, B^{-1}\lambda_i),$$

we again arrive at Eq. (3.15), but now with $\hat{J}$ replaced by $2\hat{J}$.

The results obtained also remain valid if instead of from Eq. (3.12) we start from the nonlinear Lagrangian (1.59), where, however, it is expedient to introduce instead of l the parameter

$$l_1 = (l^n / 3\pi)^{1/(3n-4)},$$

which has the dimensions of k_0^{-1} ($k_0 l_1$ = const, $B(\hat{\rho}) = 3\pi l_1^{3n-4}(\bar{\psi}\psi)^n$).

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