SPACE AND TIME REFLECTIONS IN RELATIVISTIC THEORY

Yu. M. SHIROKOV

Nuclear Physics Institute, Moscow State University

Submitted to JETP editor June 10, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **38**, 140-150 (January, 1960)

A complete classification has been obtained for the representations of the inhomogeneous Lorentz group, including space reflections and time reflections of the Wigner type. The question of the values of the squares of the various reflection operations for particles of half-integral spin is investigated in detail. The results are compared with those which are obtained when additional requirements are imposed on the theory, in particular the requirement of locality of the field operators. It is shown that, in addition to mass, spin and parity, elementary particles have still another purely geometrical characteristic, which might be called the symmetry type. The question of the symmetry type for real particles is discussed.

1. INTRODUCTION

THE purpose of this paper is to obtain all possible laws of transformation of wave functions (state vectors) under space and time reflections, which are permitted by the requirements of relativistic invariance, without making use of the property of locality of the field operators. The following postulates are taken as the starting point.

A. Relativistic invariance. The mathematical formulation of this requirement is that[1] the theory must contain operators $M_{\mu\nu}$, p_λ, satisfying the commutation relations

$$[M_{\mu\nu},\ M_{\lambda\sigma}] = i(\delta_{\mu\sigma}M_{\lambda\nu} + \delta_{\mu\lambda}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\mu\lambda} + \delta_{\lambda\nu}M_{\sigma\mu}),$$

$$[M_{\mu\nu},\ p_\lambda] = i(p_\nu\delta_{\mu\lambda} - p_\mu\delta_{\nu\lambda}),\quad [p_\mu,\ p_\nu] = 0 \qquad (1)$$

B. The Wigner formulation of the law of time reflection[2] is assumed. According to this formulation, to the operation $t = -t'$ there corresponds the state vector transformation

$$t = -t',\ \Psi = I_t K\Psi', \qquad (2)$$

where I_t is a certain linear operator and K is the nonlinear operation of charge conjugation

$$K\Psi' = \Psi'^*. \qquad (3)$$

The Schwinger rule for transposition[3] is obtained by substituting (3) into (2). The formulation within the framework of the conventional theory of representations was given earlier.[4]

C. The reflection operations are not assigned explicitly, but are determined from their geometrical properties. Thus the inversion I_s is defined as the operator that transforms the state vector when we reflect the space coordinates

$$\mathbf{x} = -\mathbf{x}',\ \Psi = I_s\Psi'. \qquad (4)$$

The time reflection operator was defined in reference 2. It is convenient to introduce still another operator $I_{st}K$, corresponding to the reflection of all four axes:

$$x_\mu = -x'_\mu,\quad \Psi = I_{st}K\Psi'. \qquad (5)$$

It is obvious that

$$I_s I_t = I_{st}. \qquad (6)$$

Figuratively speaking, the inversion is defined as the operator that relates the state vector to the vector for the state of the same physical system as observed in a mirror.

From these definitions and the fact that the sequences of coordinate transformations

$$x_\mu = x'_\mu + \xi_\mu + \varepsilon_{\mu\nu}x'_\nu,$$

$$x'_\mu = -x''_\mu,\quad x_\mu = -x'''_\mu,\quad x'''_\mu = x''_\mu - \xi_\mu + \varepsilon_{\mu\nu}x''_\nu \qquad (7)$$

lead to the same final system, we obtain the commutation relations for the operator I_{st}:

$$p_\mu I_{st} - I_{st}p^*_\mu = 0, \qquad (8)$$

$$M_{\mu\nu}I_{st} + I_{st}M^*_{\mu\nu} = 0. \qquad (9)$$

The asterisk denotes complex (and not Hermitian) conjugation.* Similarly, we get for I_s the relations

$$[I_s,\ \mathbf{M}] = 0,\ [I_s,\ p_0] = 0,$$

$$I_s\mathbf{p} + \mathbf{p}I_s = 0,\ I_s\mathbf{N} + \mathbf{N}I_s = 0, \qquad (10)$$

*The operation of complex conjugation does not affect the imaginary unit in the fourth components of vectors and tensors, so that, for example, $p_4^* = ip_0^*$. This point is not essential, since the use of contravariant vectors with the usual definition of conjugation leads to the same results.

where $N_i = -iM_{i4}$. Each of the reflection operations defined by Eqs. (2) — (5) may or may not contain the charge conjugation. On the other hand, according to (8) and (10), all these operations must be conserved for all interactions.

D. For particles with integral spin, the squares of all reflections are equal to unity. For particles with half-integral spin, the square of each of the reflections I_s, I_tK, $I_{st}K$ may be either plus or minus unity, but they must have the same values for all the half-integral spin particles. In other words, the values of squares of the reflection operators determine the properties of space-time, and not the properties of individual particles. Therefore particles with different values for the square of one of the reflections cannot exist simultaneously,[5] since this would lead to the existence of systems with integer angular momentum and a negative value for the square of the reflection.

As was shown earlier,[4,6] these considerations can be given a geometrical interpretation if we extend the rotation group by adjoining an element $I_{2\pi}$ for the rotation through angle 2π about an arbitrary axis. (Such an extension occurs naturally in treating the topological properties of the parameter space of the rotation group.) Then the square of each of the reflections may be equal to either the identity operator I or to the rotation through 2π, which leads to the following eight groups $G_1 - G_8$:

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
I_s^2	I	I	$I_{2\pi}$	$I_{2\pi}$	$I_{2\pi}$	$I_{2\pi}$	I	I
$(I_tK)^2 = I_tI_t^*$	I	$I_{2\pi}$	I	$I_{2\pi}$	$I_{2\pi}$	I	$I_{2\pi}$	I
$(I_{st}K)^2 = I_{st}I_{st}^*$	I	$I_{2\pi}$	$I_{2\pi}$	I	$I_{2\pi}$	I	I	$I_{2\pi}$

(11)

Table (11) holds for both integer ($I_{2\pi} = 1$) and half-integral ($I_{2\pi} = -1$) spins. Real space-time transforms according to one of these eight groups, and this fixes once and for all the values of the squares of the reflections for all particles. Thus there exist eight nonequivalent space-time structures. Later we shall show that the difference between these structures is accessible to experimental test.

E. No assumptions whatsoever are made concerning the locality of the field operators, or, in general, concerning the form of the equations of motion. These initial assumptions differ in some ways from those usually made in investigations of this kind (cf., for example, the surveys by Wick and Solov'ev,[7] which give references to the main papers).

The adoption of postulates A and E leads to a great generality and enables us to obtain a number of reflection transformations which satisfy the requirements of relativistic invariance but which do not reduce to local transformations of the Dirac field operators. Postulate C is essentially not new, but it rather restores to the reflection operators their original geometrical meaning. However this point does involve a change in the basic point of view of the investigation of the reflection operations. Usually (cf., for example, reference 7), certain operators (P, CP, T, CT, etc) are defined to act on the field operators, and one investigates the conservation of these operators. Here we impose the requirement that there exist conserved operators I_s, I_t, I_{st}, which correspond to definite coordinate transformations and which satisfy relations (6), (8) — (10), and one of the columns of (11). (Failure to satisfy this requirement is tantamount to denying the Euclidean nature of space-time.) Our problem is to find the explicit form of these operators. The requirements of postulate D are not new (though they contain a somewhat unusual geometrical interpretation), but they are investigated in detail here for the first time. These requirements impose quite rigid limitations (which are different for each of the eight groups) on the possible form of the reflection operators.

2. PARTICLES WITH NONZERO REST MASS

The apparatus of field theory is not suited for investigations which are not based on locality of the theory. We shall therefore make use of the mathematical technique which was used earlier[8] for obtaining the explicit form of the representations of the inhomogeneous Lorentz group. (These representations were first obtained by Wigner.[9]) According to reference 8, the state vector of a free relativistic particle, with mass κ and spin s, can always be brought to the form

$$\Psi_{m_s}^{\varkappa s \alpha}(\mathbf{p}), \tag{12}$$

where the kinematical variables are the three-dimensional momentum $\mathbf{p}$ and the spin projection m_s. In addition to mass and spin, the particle may possess other variables (such as charge) which are invariant under four-dimensional rotations and translations; we denote these by the index α. Relativistic invariance is assured by the fact that, for the state vector (12), we can define[8] operators $M_{\mu\nu}$, p_λ which satisfy Eqs. (1):

$$\hat{\mathbf{p}} = \mathbf{p},\ p_0 = e_p \equiv \sqrt{p^2 + \varkappa^2},\ \mathbf{M} = -i\left[\mathbf{p} \times \frac{\partial}{\partial \mathbf{p}}\right] + \mathbf{S},$$

$$\mathbf{N} = ie_p \frac{\partial}{\partial \mathbf{p}} - \frac{[\mathbf{S} \times \mathbf{p}]}{e_p + \varkappa}; \qquad (13)$$

$$[S_1, S_2] = iS_3, \ldots, \ \mathbf{S}^2 = s(s+1). \qquad (14)$$

The completeness of the argument is guaranteed by the fact that, to a given pair of values of mass and spin, there corresponds a single irreducible representation of the inhomogeneous Lorentz group, which can always be reduced to the form (13).

For the state vector (12), we must now find operators I_s, I_t, I_{st}, which satisfy (6), (8) – (10), and one of the columns of (11). Let us represent these operators in the form

$$I_s = \lambda_s P,\ I_t = \lambda_t T,\ I_{st} = \lambda_{st} PT, \qquad (15)$$

where P is an operator which acts only on the variable $\mathbf{p}$ and changes the sign of the momentum:

$$P\Psi_{m_s}^{\varkappa s\alpha}(\mathbf{p}) = i^{2s}\,\Psi_{m_s}^{\varkappa s\alpha}(-\mathbf{p}), \qquad (16)$$

while T is an operator which acts on $\mathbf{p}$ and m_s (but not on α), and changes the sign of the momentum and the spin projection:

$$T\Psi_{m_s}^{\varkappa s\alpha}(\mathbf{p}) = (-1)^{s-m_s}\Psi_{-m_s}^{\varkappa s\alpha}(-\mathbf{p}). \qquad (17)$$

From (16), (17) it follows that

$$P^{-1}\mathbf{p}P = -\mathbf{p},\quad P^{-1}\mathbf{S}P = \mathbf{S}, \qquad (18)$$

$$T^{-1}\mathbf{p}T = -\mathbf{p},\quad T^{-1}\mathbf{S}T = -\mathbf{S}^* \qquad (19)$$

$$P^2 = I_{2\pi},\quad TT^* = I_{2\pi} = \begin{cases} 1 & \text{(integer spin)} \\ -1 & \text{(half-integer spin)}. \end{cases} \qquad (20)$$

The operations P and T defined by (16) and (17) are the natural generalization, to the case of arbitrary spin, of the corresponding quantities which are used in the theory of spinor fields. Substituting (13) and (15) in (6), (8) – (10), we obtain, using (18) and (19):

$$\lambda_s\lambda_t = \lambda_{st}; \qquad (21)$$

$$[\lambda_s, p_\mu] = 0,\ [\lambda_s, M_{\mu\nu}] = 0,$$

$$[\lambda_{st}, p_\mu] = 0,\ [\lambda_{st}, M_{\mu\nu}] = 0. \qquad (22)$$

According to (22), the operators λ_s, λ_t, and λ_{st} do not act on the kinematic variables $\mathbf{p}$ and m_s; i.e., they are either numbers, or they are matrices with respect to the invariant variable α. According to (11) and (20), for single-valued representations,

$$\lambda_s^2 = 1,\quad \lambda_t\lambda_t^* = 1,\quad \lambda_{st}\lambda_{st}^* = 1 \text{ (integer spin)} \qquad (23)$$

and all the factors λ may be numbers. To each value of mass and spin , there correspond two equivalent representations which differ in parity:

$$\lambda_s = 1,\ \lambda_t = 1;\ \lambda_s = -1,\ \lambda_t = 1 \text{ (integer spin)}. \qquad (24)$$

(Representations which differ by a phase factor in λ_t are not different, since according to (2) and (3) this factor can be eliminated by multiplying the state vector by an appropriate phase factor.) To obtain the two-valued representations, we must substitute (15) and (20) in (11), and set $I_{2\pi} = -1$. The result is

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
λ_s^2	–1	–1	1	1	1	1	–1	–1
$\lambda_t\lambda_t^*$	–1	1	–1	1	1	–1	1	–1
$\lambda_{st}\lambda_{st}^*$	–1	1	1	–1	1	–1	–1	1

(25)

If λ_t (or λ_{st}) is a number, the quantity $\lambda_t\lambda_t^*$ (or $\lambda_{st}\lambda_{st}^*$) must necessarily be positive. The factors λ may therefore be numbers only in groups G_2 and G_5. In the other six groups these factors must be two-by-two matrices acting on an additional independent variable, so that the dimensionality of the irreducible representation is double that of the corresponding proper group. One can verify directly that relations (21) and (25) are satisfied by the following sets of factors:

	G_1		G_2		G_3	G_4	G_5		G_6		G_7	G_8
s	i	$-i$	i	$-i$	ρ_1	ρ_1	1	–1	1	–1	$i\rho_1$	$i\rho_1$
λ_t	ρ_2	ρ_2	1	1	ρ_2	ρ_3	1	1	ρ_2	ρ_2	ρ_3	ρ_2
λ_{st}	$i\rho_2$	$-i\rho_2$	i	$-i$	$i\rho_3$	$-i\rho_2$	1	–1	ρ_2	$-\rho_2$	ρ_2	$-\rho_3$

26)

The matrices ρ_1, ρ_2, ρ_3 in (26) have the form of the corresponding Pauli matrices, and act on an additional (charge-type) variable of the state vector. The matrices ρ_1, ρ_2 correspond to an operation of the type of charge conjugation. Because of the presence of the operation of complex conjugation in (2) and (5), the matrices ρ_1 and ρ_2 cannot be transformed into one another by an equivalence transformation if they enter into λ_t or λ_{st}. Table (26) exhausts the irreducible representations for each of the groups $G_1 - G_8$ for the case of half-integral spin.

3. PARTICLES WITH ZERO REST MASS

For zero rest mass, for each value s of the absolute value of the spin, there are two irreducible representations of the proper group, which are one-dimensional in the spin variables, and which differ in the sign of the projection of the spin onto the momentum[9] (spirality). A change in the sign of the momentum changes the sign of the spirality, so that the representations of the group including space reflections must be two-rowed in the spin-like variable for the sign of the spirality. The operators p_μ, $M_{\mu\nu}$ for these representations can be written in the form

$$\hat{\mathbf{p}} = \mathbf{p}, \quad p_0 = |p|;$$

$$M_1 = -i\left(p_2 \frac{\partial}{\partial p_3} - p_3 \frac{\partial}{\partial p_2}\right) + s\sigma_3 \frac{\cos\varphi}{\sin\vartheta},$$

$$\times M_2 = -i\left(p_3 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial p_3}\right) + s\sigma_3 \frac{\sin\varphi}{\sin\vartheta},$$

$$M_3 = -i\left(p_1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial p_1}\right);$$

$$N_1 = ip \frac{\partial}{\partial p_1} - s\sigma_3 \sin\varphi \cot\vartheta,$$

$$N_2 = ip \frac{\partial}{\partial p_2} + s\sigma_3 \cos\varphi \cot\vartheta, \quad N_3 = ip \frac{\partial}{\partial p_3}. \quad (27)$$

Here σ_3 is the Pauli matrix which acts on the variable for the sign of the spirality, and φ, ϑ are the polar angles of the momentum vector.

For integer spin, the operators I_s, I_t, and I_{st} satisfying (6) and (8) — (11) are equal to

$$I_s = \sigma_1 P, \quad I_t = P, \quad I_{st} = \sigma_1 \text{ (integer spin)}. \quad (28)$$

For each absolute value of the spin there is just one irreducible representation.

For half-integral spin, we look for operators I_s, I_t, I_{st} of the form

$$I_s = \lambda_s' P', \quad I_t = \lambda_t' P', \quad I_{st} = \lambda_{st}'. \quad (29)$$

The operation P' acts on the wave function in the same way as P, i.e., it changes the sign of the momentum, and its square is also equal to -1:

$$P'\Psi(\mathbf{p}) = \Psi(-\mathbf{p}), \quad (P')^2 = -1, \quad (30)$$

since the eigenfunctions of the operators (27) contain a factor $\exp(\pm i\varphi/2)$, which is multiplied by i when φ is replaced by $\varphi + \pi$. Substituting (29) in (6) and (8) — (10), we find

$$\lambda_s' \lambda_t' = \lambda_{st}', \quad (31)$$

$$\lambda_s' \sigma_3 + \sigma_3 \lambda_s' = 0, \quad (32)$$

$$\lambda_t' \sigma_3 - \sigma_3 \lambda_t' = 0, \quad (33)$$

$$\lambda_{st}' \sigma_3 + \sigma_3 \lambda_{st}' = 0. \quad (34)$$

Relations (32) — (34) express the fact that the projection of the spin along the momentum does not change sign under time reflection, and changes sign under inversion.

Substituting (29) in (11) and using (30), we obtain for the values of the squares of the λ''s, the table

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
$(\lambda_s')^2$	-1	-1	1	1	1	1	-1	-1
$\lambda_t'\lambda_t'^*$	-1	1	-1	1	1	-1	1	-1
$\lambda_{st}'\lambda_{st}'^*$	1	-1	-1	1	-1	1	1	-1

(35)

According to (33), the factor λ_t' for the spin variable is either equal to unity or proportional to σ_3. Therefore, in the groups G_1, G_3, G_6 and G_8, where the quantities $\lambda_t'\lambda_t'^*$ are negative, we must introduce an additional discrete variable, so that the representation becomes four-rowed. This doubling is not necessary in the other groups. Relations (31) — (35) are satisfied by the following sets of factors λ', which exhaust the irreducible representations for the state vectors of particles with zero rest mass and half-integral spin:

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
λ_s'	$i\sigma_1\rho_3$	$i\sigma_1$	σ_1	σ_1	σ_1	$\sigma_1\rho_3$	$i\sigma_1$	$i\sigma_1$
λ_t'	ρ_2	σ_3	ρ_2	1	σ_3	ρ_2	1	ρ_2
λ_{st}'	$\rho_1\sigma_1$	σ_2	$\sigma_1\rho_2$	σ_1	$-i\sigma_2$	$-i\rho_1\sigma_1$	$i\sigma_1$	$i\sigma_1\rho_2$

(36)

4. SYMMETRY TYPES OF ELEMENTARY PARTICLES

In addition to mass, spin, and parity, elementary particles possess another purely geometric characteristic, which we may call the symmetry type of the particle. The concept of symmetry type is based on the macroscopically obvious fact that, under each of the reflections $\mathbf{x} \to -\mathbf{x}$, $t \to -t$, $x_\mu \to -x_\mu$, the particle can either go over into itself (symmetry), or into another (anti-particle type) state with the same mass and spin (asymmetry). In the latter case, the transformation of the state vector which is associated with the corresponding reflection must contain an operation C of the type of charge conjugation.

For nonzero rest mass, five types of symmetry are possible:

1. Complete Symmetry. The particle is transformed into itself under all reflections. In this case the operations P (change in signs of momenta) and T (change in signs of momenta and spins) are conserved.

2. T-symmetry. The particle transforms into the antiparticle under space reflections, and is unchanged by time reflection. PC and T are conserved, while P and PT are not. The macroscopic analog of this type of symmetry is the symmetry of a rotating screw-nut.

3. P-symmetry. The particle is unchanged by inversion and changes into the antiparticle under time reflection. P and CT are conserved, while T and PT are not. The macroscopic analog is a rotating weather-vane.

4. PT-symmetry. The particle changes to the antiparticle under inversion, and does not change under reversal of all four coordinates. PC, CT, and PT are conserved, while P, T, and PCT are not. This case contradicts the PCT-theorem, i.e., it is not possible in a local theory. The macroscopic analog is a rotating conical cogwheel.

5. Complete asymmetry. The particle does not change into itself under any of the reflections, i.e., the particle has four different states with the same mass and spin. In this case there are two independent operators C_1 and C_2 of the type of charge conjugation. PC_1, C_2T, and PC_3T (where $C_3 = C_1C_2$) are conserved.

For zero rest mass, we associate with a state of the particle a definite projection of the spin along the momentum, which is invariant with respect to proper transformations but which changes sign under inversion. Thus, for zero mass (and nonzero spin) only T-symmetry (type 2) and complete asymmetry (type 5) are possible.

In some cases (for example, for photons) the operation C does not change the sign of any charge. For this reason we have been careful to call it an operation of the type of charge conjugation.

There are different restrictions on the S-matrix for the various symmetry types. Thus, in cases 3 and 4, the S-matrix is not symmetric, as is usual, but is related to its transposed matrix S^T by $S^T = C^{-1}SC$. The considerations presented here can be presented somewhat differently, by discussing the transformation properties of the physical quantities characterizing the particle. From group-theoretical considerations it follows that the state of a free particle is described by its mass, spin, momentum, spin projection, as well as by other possible variables which must necessarily be invariant under proper transformations.[8] However, these additional variables may be either scalar or pseudoscalar with respect to reflections. If all the additional variables are scalar, the particle belongs to the first type (complete symmetry). If the additional variable is a pseudoscalar with respect to inversion and a scalar with respect to time reflection (for example, the projection of the spin along the momentum in the case of zero mass), the particle has T-symmetry, etc. Thus the investigation of symmetry types of particles is equivalent to the investigation of the behavior under reflections of the quantities characterizing the particle, such as the different charges.

5. THE DIRAC EQUATION

If we impose on the theory the requirement of locality of the field operators, there is an essential change in the final results, which can be followed on the example of the Dirac spinor. Differences occur for three reasons. First of all, the Dirac equation is always a four-component equation, whereas two-component spinors are possible for the groups G_2 and G_5 [cf. Eq. (26)] found in our investigation. Secondly, certain representations are forbidden in a local formulation. Thirdly, certain representations which are equivalent within the framework of the general theory of representations may prove to be inequivalent with respect to local transformations.

Before turning to an investigation of the Dirac equation, let us enumerate all the "physically" inequivalent four-component representations, for a particle with spin $\frac{1}{2}$, which are permissible according to table (26):

	G	λ_s	λ_t	λ_{st}		G	λ_s	λ_t	λ_{st}
1	G_1	i	ρ_2	$i\rho_2$	14	G_5	1	1	1
2		$-i$	ρ_2	$-i\rho_2$	15		-1	1	-1
3	G_2	i	1	i	16		ρ_3	ρ_3	1
4		$-i$	1	$-i$	17		1	ρ_1	ρ_1
5		i	ρ_1	$i\rho_1$	18		-1	ρ_1	$-\rho_1$
6		$-i$	ρ_1	$-i\rho_1$	19		ρ_1	ρ_1	1
7		$i\rho_3$	ρ_3	i	20		ρ_1	1	ρ_1
8		$i\rho_1$	1	$i\rho_1$	21	G_6	1	ρ_2	ρ_2
9		$i\rho_1$	ρ_1	i	22		-1	ρ_2	$-\rho_2$
10	G_3	ρ_1	ρ_2	$i\rho_3$	23	G_7	$i\rho_1$	ρ_3	ρ_2
11		ρ_3	ρ_2	$-i\rho_1$	24		$i\rho_3$	ρ_1	$-\rho_2$
12	G_4	ρ_1	ρ_3	$-i\rho_2$	25	G_8	$i\rho_1$	ρ_2	$-\rho_3$
13		ρ_3	ρ_1	$i\rho_2$	26		$i\rho_3$	ρ_2	ρ_1

(37)

We say that two representations, for example representations 7 and 8, are "physically" inequivalent, even though they are equivalent, if they correspond to different symmetry types. In representation 7, under inversion the particle goes over into itself (conservation of parity), while in 8 the particle under inversion goes over into its antiparticle (conservation of combined inversion).

Now let us consider the possible transformations for the Dirac field operators, which correspond to the addition of the requirement of locality to the invariance requirements (1) and (8) — (10). In the local formulation, these conditions are equivalent to the following requirements: 1) invariance of the equations for the field operators, 2) invariance of the commutation relations, 3) invariance of the definition of the vacuum. Taking account of these requirements, one can show that the following nonequivalent sets of transformations of the Dirac field operator $\psi(\mathbf{x}, t)$ can be associated with the reflection transformations:

	Group	Operators corresponding to coordinate reflections			Number of corresponding representation in (37)
		$x=-x'$	$t=-t'$	$x_\mu=-x'_\mu$	
—	G_1	—	—	—	—
1	G_2	iP	T	iPT	7
2	G_3	PC	iCT	iPT	10
3	G_4	PC	iT	$-iPCT$	12
4		PC	$-iT$	$iPCT$	12
5	G_5	P	T	PT	14
6		$-P$	T	$-PT$	15
7		P	CT	PCT	17
8		P	$-CT$	$-PCT$	17
9		$-P$	CT	$-PCT$	18
10		$-P$	$-CT$	PCT	18
11		PC	CT	PT	19
12		PC	T	PCT	20
13		PC	$-T$	$-PCT$	20
14	G_6	P	iCT	$iPCT$	21
15		$-P$	iCT	$-iPCT$	22
16	G_7	iP	CT	$iPCT$	24
17		iP	$-CT$	$-iPCT$	24
18	G_8	iP	iCT	$-PCT$	26
19		iP	$-iCT$	PCT	26

(38)

In (38) the operations P, T, C are defined in the standard way, with the phase factors

$$P\psi(\mathbf{x},t) = i\gamma_4\psi(-\mathbf{x},\ t). \tag{39}$$

$$T\psi(\mathbf{x},t) = \gamma_1\gamma_3\gamma_4\bar{\psi}(\mathbf{x},\ -t), \tag{40}$$

$$C\psi(\mathbf{x},t) = \gamma_2\gamma_4\bar{\psi}(\mathbf{x},\ t). \tag{41}$$

It can be shown that, with this choice of phase factors, the transformations (39) and (40) on the Dirac field operators are equivalent to the transformations (16) and (17) on the corresponding state vector. For this reason these operations are denoted by the same letters. This equivalence can be established by means of a Foldy-Wouthuysen transformation[10] of the solutions of the Dirac equation for the field operators, followed by a shift to the configuration representation as in reference 11.

Table (38) exhausts the possible laws for local reflection transformations of Dirac particles. We remind the reader that particles can exist simultaneously only if their operators transform according to representations which belong to the same group, and that all the reflection operations are conserved.

6. DISCUSSION OF RESULTS

Our investigation shows that the known properties of space-time must be fixed in accordance with one of the columns of (11), so that the question arises of determining the group among $G_1 - G_8$ according to which real space-time transforms. To solve this problem, we may use the following differences between the representations of these groups.

a) The pseudoscalar nature of the ground state of positronium. This argument does not depend on whether inversion is or is not accompanied by charge conjugation, since the two-photon system has even charge parity. This requirement eliminates from table (37) the representations 1 — 6, 8, 9, 11, 13, 16, 23, and 25, i.e., just those representations which are absent from the local variants (38). Thus the requirement of pseudoscalarity of positronium eliminates the possibility of group G_1, in which the squares of all reflections are equal to unity.

b) The two-component neutrino. If the neutrino is two-component, then, according to (36), the groups G_1, G_3, G_6 and G_8 are not possible.

c) The four-component nature of all known spinor particles with nonzero rest mass. This argument, together with the previous one, makes those representations particularly probable for which the wave function of the spin -$1/2$ particle may be two-component for zero rest mass, but necessarily four-component for particles of finite mass. Groups G_4 and G_7 satisfy this condition.

The following group of arguments is related to non-conservation of the operation of charge conjugation C [ρ_1, ρ_2 in Eq. (37)]. It is to be understood that we are talking about conservation under all interactions, since the problem under discussion involves the geometric properties of space-time itself. Nonconservation of C has the consequence that if, for example, the quantity PC is associated with the transformation $\mathbf{x} = -\mathbf{x}'$ and is therefore conserved, the quantity $PC \cdot C = P$ will not be conserved. On the basis of this remark, we may consider the following additional restrictions on the representations.

d) Conservation of PCT. This requirement is a hypothesis, if the locality condition is not imposed on the theory. Conservation of PCT results in non-conservation of PT, which eliminates representations 3, 4, 7, 9, 10, 14 — 16, 19 in (37). Taken together with a), this requirement eliminates groups G_2 and G_3.

e) The hypothesis of conservation of combined parity PC.[12-14] Conservation of PC excludes from (37) the representations 1 — 7, 11, 13, 14 — 18, 21, 22, 24, and 26. Taken together with point a), this eliminates all the representations of the groups G_1, G_2, G_6, G_7 and G_8.

f) The hypothesis of conservation of T. This excludes from (37) the representations 1, 2, 5, 6, 9 — 11, 13, 17 — 19, 21, 22, and 24 — 26, so that, together with a) it excludes all the representations of groups G_1, G_3, G_6, G_7 and G_8. The locality condition is equivalent to the combined conditions a) and d).

Only representation 12 in (37) satisfies all the requirements enumerated above. This representation belongs to group G_4. In (38) there correspond to it the two locally nonequivalent representations 3 and 4. If we drop requirement c), the remaining conditions will also be satisfied by representation 20 of the group G_5 in (37). In (38) there will correspond to it the two locally nonequivalent representations 12 and 13. In this case, however, it becomes possible to have other particles, whose wave functions transform according to various representations of group G_5, so that the conservation laws d), e) and f) lose their universal character.

Thus, existing experimental data lead to the conclusion that real space-time transforms either according to group G_4 or according to group G_5, the more probable one being group G_4, in which the squares of the inversion and of the reflection of all four axes are equal to unity. (We note that if we treat the reflection as a rotation through angle π about some additional coordinate axis, its square reduces to a rotation through 2π.) It is interesting that such a delicate property of space-time, which is associated with the topology of the parameter space of the rotation group, turns out to be accessible to experimental determination.

In group G_4, all particles must transform according to representation 12 of (37). When we go over to the local formulation, the two representations 3 and 4 in (38), which are non-equivalent under local transformations, will correspond to it. These representations are characterized by the fact that the transformation properties of the particles they describe are identical, while the field operators transform differently under reflection of all four axes. This difference between the particles will therefore not affect the conditions for invariance of the S-matrix, as determined, for example, according to reference 15. But they will have a marked effect on the composition of the Lagrangian for the interaction which transforms one particle into another.

With regard to actual particles, we may say that the π^0-meson belongs to the completely symmetric type, while the photon and charged particles have T-symmetry. The neutrino has T-symmetry if it is two-component, and is completely asymmetric if it is four-component. One must, however, observe a certain caution in drawing such conclusions. For example, the Wu experiment, strictly speaking, shows only that at least one of the particles participating in the process (for example, the neutrino) does not go over into itself under inversion. On the other hand, if say it should turn out that the quantities P, T and PT are not conserved in proton-proton collisions, this would imply that the proton has four different charge states, in accordance with the initial hypothesis of Lee and Yang.[12] It is of interest to set up a system of experiments necessary for the exact determination of the symmetry types of all known particles.

Usually, in investigating the reflections, one assigns these operations at the start for all the fields being studied, and then investigates the conservation of these operations. Such an approach is, on the one hand, incomplete, and on the other hand mixes up the purely geometrical properties of the reflection operations with the special properties of the operators of the as yet incomplete local theory of fields. The main result of the present paper is the strict separation of the geometrical properties of the reflection operations from the properties of specific equations of motion.

Summarizing, we may say that the investigation of the reflection operations reduces to the answer to three questions: 1) According to which of the groups $G_1 - G_8$ does real space-time transform? 2) According to which representation does each of the particles transform? 3) To which symmetry type does each particle belong? All other questions which arise in studying the reflections either reduce to the ones enumerated or depend essentially on the particular choice of the equations of motion.

[1] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U.S. **34**, 211 (1948). P. A. M. Dirac, Revs. Modern Phys. **21**, 392 (1949).

[2] E. P. Wigner, Nachr. Akad. Wiss. Göttingen, Math.-physik. Kl. p. 546-559 (1932).

[3] J. Schwinger, Phys. Rev. **82**, 914 (1951).

[4] Yu. M. Shirokov, JETP **36**, 879 (1959), Soviet Phys. JETP **9**, 620 (1959).

[5] Wick, Wightman, and Wigner, Phys. Rev. **88**, 101 (1952). I. S. Shapiro, Usp. Fiz. Nauk **53**, 7 (1954).

[6] Yu. M. Shirokov, JETP **34**, 717 (1958), Soviet Phys. JETP **1**, 493 (1958).

[7] V. G. Solov'ev, Preprint P-145, Joint Inst. for Nucl. Studies, 1958. G. C. Wick, Ann. Rev. Nuclear Sci. **8**, (1958).

[8] Yu. M. Shirokov, JETP **33**, 861, 1196, 1208 (1957), Soviet Phys. JETP **6**, 664, 919, 929 (1958).

[9] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[10] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

[11] Yu. M. Shirokov, JETP **24**, 129, 135 (1953).

[12] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1671 (1957).

[13] L. D. Landau, JETP **32**, 405 (1957), Soviet Phys. JETP **5**, 336 (1957).

[14] V. G. Solov'ev, Nucl. Phys. **6**, 618 (1958).

[15] Yu. M. Shirokov, JETP **35**, 1005 (1958), Soviet Phys. JETP **8**, 703 (1959).

Translated by M. Hamermesh
22