

**THEORY OF SUPERCONDUCTING ALLOYS IN A STRONG MAGNETIC FIELD NEAR THE
CRITICAL TEMPERATURE**

L. P. GOR'KOV

Institute for Physical Problems, Academy of Sciences, U.S.S.R.

Submitted to JETP editor June 17, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 1407-1416 (November, 1959)

The equations of the phenomenological Ginzburg-Landau theory near T_C have been deduced from the Bardeen-Cooper-Schrieffer (BCS) theory. As in the case of pure superconductors, a double charge is encountered in the equations. The relation between the constant κ of the alloy with κ_0 of the pure superconductor has been found under the assumption that the shift in T_C is small. For a sufficiently "impure" alloy, κ depends only on the coefficient in the linear law for the electronic specific heat of the metal. Agreement between the theory and experiments has been found to be satisfactory.

It has been shown by us¹ that it follows from the superconductivity theory of Bardeen, Cooper and Schrieffer (BCS)² that, close to the critical temperature, the behavior of pure superconductors in a magnetic field is described by the equations of the phenomenological theory of Ginzburg-Landau.³ Below we shall carry out a similar consideration for superconducting alloys. As usual, we assume the concentration of the impurity to be so small that we can neglect any change in the critical temperature. This can be done, for while the shift of T_C is a comparatively slow effect, the magnetic properties of the superconducting alloys depend very strongly on the concentration of the impurity.

The role of the impurities in a superconductor, as is well known, lies in the fact that the scattering of the electrons by the atoms of the impurity leads to the disappearance of the correlation between electrons which is established in the transition to the superconducting state. For appreciable concentration of impurity, the role of the correlation length will be played by the mean free path. For sufficiently "impure" superconductors, the penetration depth is greater than the mean free path; consequently, its behavior in the magnetic field will be described by local "London" equations. As is known, one can distinguish two groups among pure conductors which differ, according to their properties, in a weak magnetic field.^{2,4} The so-called "London" superconductors are characterized by the fact that for them the penetration depth over the whole temperature range is larger than the characteristic correlation parameter of superconducting interaction. Such a parameter in the BCS theory is the size of the pair coupling

$\xi_0 \sim \hbar v / kT_C$. To this case there correspond the electrodynamics of London. A "Pippard" metal represents the opposite limiting case. For such superconductors, at almost all temperatures, the penetration depth is much smaller than ξ_0 . Most of the known superconductors belong to the Pippard type and to the intermediate type. The ordinary Pippard metal has a London interval of temperatures only in the immediate vicinity of the critical temperature. This region is very small. A superconducting alloy like a London or intermediate metal has a much larger region of temperatures close to T_C where their properties in a magnetic field (as is shown) can be described by equations corresponding completely to the equations of the Ginzburg-Landau theory.

As in reference 1, we shall in the derivation begin with the equations⁵ for the Fourier components of the thermodynamic Green's function:

$$\left\{ i\omega_n + \frac{1}{2m} \left(\frac{\partial}{\partial r} - ie\mathbf{A}(\mathbf{r}) \right)^2 + V(\mathbf{r}) + \mu \right\} \times \mathfrak{G}_\omega(\mathbf{r}, \mathbf{r}') + \Delta(\mathbf{r}) \mathfrak{F}_\omega^+(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (1)$$

$$\left\{ -i\omega_n + \frac{1}{2m} \left(\frac{\partial}{\partial r} + ie\mathbf{A}(\mathbf{r}) \right)^2 + V(\mathbf{r}) + \mu \right\} \times \mathfrak{F}_\omega^+(\mathbf{r}, \mathbf{r}') - \Delta^*(\mathbf{r}) \mathfrak{G}_\omega(\mathbf{r}, \mathbf{r}') = 0,$$

$$\Delta^*(\mathbf{r}) = gT \sum_n \mathfrak{F}_\omega^+(\mathbf{r}, \mathbf{r}), \quad \omega = \pi T (2n + 1),$$

$\mathbf{A}(\mathbf{r})$ is the vector potential (below we shall use everywhere the standard $\text{div } \mathbf{A} = 0$), $V(\mathbf{r})$ is the potential energy of interaction with all atoms of the impurity:

$$V(\mathbf{r}) = \sum_a u(\mathbf{r} - \mathbf{r}_a), \quad (2)$$

where the sum runs over all atoms of the impurity arbitrarily distributed over a lattice. The smallness of the concentration means that the mean distance between atoms of the impurity is large in comparison with the interatomic distances in the lattice.

Repeating the derivation of reference 1, expanding \mathfrak{G} and \mathfrak{F}^* near T_C in powers of $|\Delta|$, we obtain the following equation:

$$\begin{aligned} \Delta^*(\mathbf{r}) = & gT \sum_n \int \tilde{\mathfrak{G}}_{V\omega}(\mathbf{r}, \mathbf{r}') \tilde{\mathfrak{G}}_{V-\omega}(\mathbf{r}, \mathbf{r}') \Delta^*(\mathbf{r}') d^3\mathbf{r}' - gT \\ & \times \sum_n \iiint \tilde{\mathfrak{G}}_{V\omega}(\mathbf{s}, \mathbf{r}) \tilde{\mathfrak{G}}_{V-\omega}(\mathbf{s}, \mathbf{l}) \tilde{\mathfrak{G}}_{V\omega}(\mathbf{m}, \mathbf{l}) \tilde{\mathfrak{G}}_{V-\omega}(\mathbf{m}, \mathbf{r}) \\ & \times \Delta(\mathbf{s}) \Delta^*(\mathbf{l}) \Delta^*(\mathbf{m}) d^3\mathbf{s} d^3\mathbf{m} d^3\mathbf{l}. \end{aligned} \quad (3)$$

This equation differs from Eq. (8) of reference 1 by the fact that the Green's function $\tilde{\mathfrak{G}}_{V\omega}$ appearing in it represents the Green's function of electrons in a normal metal in the absence of impurity atoms. The corresponding quantity in the pure metal $\tilde{\mathfrak{G}}_{\omega}$, as was found in reference 1, depends on the magnetic field in simple fashion:

$$\tilde{\mathfrak{G}}_{\omega}(\mathbf{r}, \mathbf{r}') = \exp\{ie(\mathbf{A}(\mathbf{r}), \mathbf{r} - \mathbf{r}')/c\} \mathfrak{G}_{\omega}(\mathbf{r} - \mathbf{r}'), \quad (4)$$

where $\mathfrak{G}_{\omega}(\mathbf{r} - \mathbf{r}')$ is the Green's function of normal electrons in the absence of a field. The dependence of $\tilde{\mathfrak{G}}_{V\omega}$ on the field has precisely the same form

$$\tilde{\mathfrak{G}}_{V\omega}(\mathbf{r}, \mathbf{r}') = \exp\{ie(\mathbf{A}(\mathbf{r}), \mathbf{r} - \mathbf{r}')/c\} \mathfrak{G}_{V\omega}(\mathbf{r}, \mathbf{r}'), \quad (4')$$

where $\mathfrak{G}_{V\omega}(\mathbf{r}, \mathbf{r}')$ is the Green's function in the normal metal in the absence of impurities, but without a magnetic field. Actually, the twisting of electrons in a magnetic field is very small, the radius of twisting being much larger than the free path length. Therefore, scattering on impurities takes place independent of the motion in the magnetic field.

Equation (3) should be averaged over the position of each atom of the impurities at distances large in comparison with interatomic distances. The function $\Delta(\mathbf{r})$ is slowly changing since the functions $\mathfrak{G}_{V\omega}$ oscillate rapidly at atomic distances ($\sim 1/p_0$); therefore, averaging in (3) leads to an average of the corresponding combinations of functions $\tilde{\mathfrak{G}}_{V\omega}(\mathbf{r}, \mathbf{r}')$.

In accordance with what has been said about the dependence on a magnetic field, one can take the phase multipliers in (4') out from under the averaging sign. After averaging, Eq. (3) has a much simpler form. Getting ahead of ourselves, we can point out that the essential distances in the integration will be distances of the order of the free path length l if $l \ll \xi_0 = \hbar v/2\pi k T_C$, and of

the order ξ_0 if $l \gg \xi_0$. Further, close to T_C the change of Δ and of the field takes place at distances much larger than when the same quantity Δ is small. Repeating the discussions of reference 1, we can, for example, take all Δ and Δ^* in terms of third order in (3) out from under the integral at the point \mathbf{r} and neglect the dependence on the field (4'), after which it is shown to be necessary to compute the constant

$$\begin{aligned} B = T_c \sum_n \iiint \overline{\mathfrak{G}_{V\omega}(\mathbf{s}, \mathbf{r}) \mathfrak{G}_{V-\omega}(\mathbf{s}, \mathbf{l}) \mathfrak{G}_{V\omega}(\mathbf{m}, \mathbf{l}) \mathfrak{G}_{V-\omega}(\mathbf{m}, \mathbf{r})} \\ \times d^3\mathbf{s} d^3\mathbf{m} d^3\mathbf{l}, \end{aligned} \quad (5)$$

in this term, where the bar indicates averaging over the position of the impurity atoms.

If we introduce the notation

$$Q(\mathbf{r} - \mathbf{r}') = T \sum_n \overline{\mathfrak{G}_{V\omega}(\mathbf{r}, \mathbf{r}') \mathfrak{G}_{V-\omega}(\mathbf{r}, \mathbf{r}')}, \quad (6)$$

then, by consideration of (5) and (6), we can write Eq. (3) in the form

$$\begin{aligned} g^{-1}\Delta^*(\mathbf{r}) = & \int Q(\mathbf{r} - \mathbf{r}') \exp\{2ie(\mathbf{A}(\mathbf{r}), \mathbf{r} - \mathbf{r}')/c\} \\ & \times \Delta^*(\mathbf{r}') d^3\mathbf{r}' - B |\Delta(\mathbf{r})|^2 \Delta^*(\mathbf{r}) \end{aligned}$$

Expanding the expression under the integral in the second term on the right side in terms of second order in $\mathbf{r} - \mathbf{r}'$, we get

$$\begin{aligned} g^{-1}\Delta^*(\mathbf{r}) = & \Delta^*(\mathbf{r}) \int Q(R) d^3\mathbf{R} + \frac{1}{6} \left(\frac{\partial}{\partial \mathbf{r}} + 2ie\mathbf{A}(\mathbf{r}) \right)^2 \Delta^*(\mathbf{r}) \\ & \times \int Q(R) R^2 d^3\mathbf{R} - B |\Delta(\mathbf{r})|^2 \Delta^*(\mathbf{r}). \end{aligned} \quad (7)$$

To obtain the constants in (7) it is consequently necessary to calculate the average in (5) and (6).

A diagram technique has been developed in the work of Abrikosov and the author, and also in that of Edwards,⁶ by means of which it is convenient to perform the averaging over the location of the impurity atoms. Each scattering of an electron by an impurity atom leads to a factor $u(\mathbf{q}) \times \exp\{i\mathbf{q} \cdot \mathbf{r}_a\}$ in the expression for the Green's function in the momentum representation, where \mathbf{q} is the forward momentum, and $u(\mathbf{q})$ is the Fourier component of the potential of the impurity atom. In averaging over \mathbf{r}_a , inasmuch as $|\mathbf{q}| \sim p_0$ essentially, this factor vanishes only if there is no term in the averaged expression corresponding to scattering by this very atom, but with $\mathbf{q}' = -\mathbf{q}$; in this case there appears $|u(\mathbf{q})|^2$ — a quantity proportional in the Born approximation to the probability of scattering with exchange of momentum \mathbf{q} . Thus, in the averaging, it is necessary to throw away pairwise all scatterings by identical atoms. If we represent scattering by each atom by a cross on the diagram,

then in the averaging it is necessary to connect two crosses, corresponding to scattering by identical atoms, by a line. In the matrix element $n|u(\mathbf{q})|^2$ corresponds to such a line, where n is the number density of the impurity atoms. According to reference 6, if the dotted line on the diagram includes a vertex with a large transfer r of momentum, then the corresponding contribution from this diagram is small ($\sim 1/p_0 l$) and it can be discarded.

Averaging of the Green's function itself leads to the factor $\exp\{-|\mathbf{r}-\mathbf{r}'|/2l\}$, which must be multiplied by the Green's function of the pure metal. Here the mean free path $l = v\tau$ in the Born approximation is directly connected with the potential of the impurity atom:

$$\frac{1}{\tau} = \frac{nmv^2}{(2\pi)^2} \int |u(\mathbf{q})|^2 d\mathbf{q}.$$

Averaging of the product of several functions, which takes place in (5) and (6), is somewhat more complicated. Let us first consider the averages:

$$K_\omega(\mathbf{r}-\mathbf{s}, \mathbf{r}'-\mathbf{s}) = \overline{\mathcal{G}_{V\omega}(\mathbf{r}, \mathbf{s}) \mathcal{G}_{V-m}(\mathbf{r}', \mathbf{s})}. \quad (8)$$

We introduce the Fourier components

$$K_\omega(\mathbf{x}, \mathbf{y}) = (2\pi)^{-6} \iint e^{i\mathbf{p}_1\mathbf{x} + i\mathbf{p}_2\mathbf{y}} K_\omega(\mathbf{p}_1, \mathbf{p}_2) d^3\mathbf{p}_1 d^3\mathbf{p}_2.$$

The product of the two Green's functions in (8)



FIG. 1

is shown graphically in Fig. 1, where the crosses, as indicated above, express scattering by individual impurities. In the averaging, in correspondence with the program that we have outlined, we shall connect the crosses with one another pairwise in all processes by dotted lines, which corresponds to scattering by one and the same impurity atom.

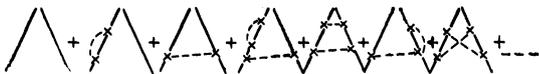


FIG. 2

In Fig. 2 diagrams are shown of lower orders which arise as a result of such averaging. As has been pointed out, diagrams with an intersection of dotted lines are not important in this group. All dotted lines joining points together that are located on a single electron line evidently give the following Green's function for the corresponding line:

$$\mathcal{G}_\omega^\tau(\mathbf{r}-\mathbf{r}') = \exp\{-|\mathbf{r}-\mathbf{r}'|/2l\} \mathcal{G}_\omega(\mathbf{r}-\mathbf{r}'). \quad (9)$$

Taking this into account, it is easy to write down the integral equation which $K_\omega(\mathbf{p}_1, \mathbf{p}_2)$ satisfies:

$$K_\omega(\mathbf{p}_1, \mathbf{p}_2) = \mathcal{G}_\omega^\tau(\mathbf{p}_1) \mathcal{G}_\omega^\tau(\mathbf{p}_2) \times \left[1 + \frac{n}{(2\pi)^3} \int |u(\mathbf{p}_1 - \mathbf{p}'_1)|^2 K_\omega(\mathbf{p}'_1, \mathbf{p}'_2) d^3\mathbf{p}'_1 \right], \quad (10)$$

where $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}'_1 + \mathbf{p}'_2 = \mathbf{q}$. This equation sums the set of diagrams shown in Fig. 3, where the electron line must now be compared with the average Green's function $\mathcal{G}_\omega^\tau(\mathbf{p})$.

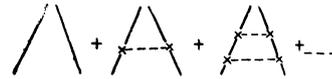


FIG. 3

The kernel $Q(\mathbf{r}-\mathbf{r}')$, in terms of which the first two coefficients in Eq. (7) are expressed, is evidently connected with (8). We write down the expression for the Fourier component $Q_\mathbf{q}$ in terms of $K_\omega(\mathbf{p}_1, \mathbf{p}_2)$:

$$Q_\mathbf{q} = T(2\pi)^{-3} \sum_\omega \int K_\omega(\mathbf{p}, \mathbf{p}-\mathbf{q}) d^3\mathbf{p};$$

Then, as is seen from (7), the constant which we must determine is simply

$$Q_0 = \int Q(R) d^3R, \quad -\partial^2 Q_\mathbf{q} / \partial q_i^2 |_{q=0} = \int Q(R) R^2 d^3R.$$

Therefore, in the calculation of $K_\omega(\mathbf{p}, \mathbf{p}-\mathbf{q})$ we must consider \mathbf{q} to be small and limit ourselves in the expression for $K_\omega(\mathbf{p}, \mathbf{p}-\mathbf{q})$ to terms of second order in \mathbf{q} .

For the solution of Eq. (10), it is useful to introduce the notation

$$L_{\omega, \mathbf{q}, \mathbf{p}} = n(2\pi)^{-3} \int |u(\mathbf{p}-\mathbf{p}')|^2 K_\omega(\mathbf{p}', \mathbf{p}'-\mathbf{q}) d^3\mathbf{p}'.$$

The equation for this quantity is obtained in obvious fashion after multiplying (10) by $n(2\pi)^{-3} \times |u(\mathbf{1}-\mathbf{p})|$ and integration over \mathbf{p} :

$$L_{\omega, \mathbf{q}, \mathbf{p}} = n(2\pi)^{-3} \int |u(\mathbf{p}-\mathbf{p}')|^2 \mathcal{G}_\omega^\tau(\mathbf{p}') \mathcal{G}_{-\omega}^\tau(\mathbf{q}-\mathbf{p}') \times [1 + L_{\omega, \mathbf{p}', \mathbf{q}-\mathbf{p}'}] d^3\mathbf{p}'. \quad (11)$$

In accordance with reference 6, we have the following simple expression for the components of the Green's function $\mathcal{G}_\omega^\tau(\mathbf{p})$:

$$\mathcal{G}_\omega^\tau(\mathbf{p}) = [i\eta\omega - \xi]^{-1}, \quad \xi = v(p - p_0), \quad \tau = 1 + 1/2\tau|\omega|$$

The product of the two Green's functions under the integral in (11) quickly falls off upon increasing distance from the Fermi surface. Integration over ξ can easily be completed. It is evident that because of the slow dependence of $u(\mathbf{p}-\mathbf{p}')$ on the quantity \mathbf{p} , only the dependence on the direc-

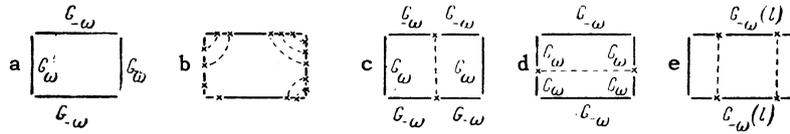


FIG. 4

tion of the vector \mathbf{p} ($|\mathbf{p}| \approx p_0$) enters into $L_{\omega, \mathbf{q}, \mathbf{p}}$. The equation takes the form

$$L_{\omega, \mathbf{q}, \mathbf{p}} = nv \int (d\sigma/d\sigma_{p'}) \nu_{\omega, \mathbf{q}, \mathbf{p}'} [1 + L_{\omega, \mathbf{q}, \mathbf{p}'}] d\sigma_{p'} \quad (12)$$

where

$$\nu_{\omega, \mathbf{q}, \mathbf{p}} = (2|\omega\eta|)^{-1} [1 - \mathbf{q}\mathbf{p}/2im\omega\eta]^{-1} \quad (12')$$

is the result of integration of the product of two \mathcal{G} functions in (11) over ξ .

We limit ourselves only to calculation of Q_0 . In (11), $\mathbf{q} = 0$, $\nu_{\omega, 0, \mathbf{p}'} = \frac{1}{2}|\omega\eta|$ and $L_{\omega, 0, \mathbf{p}} = \frac{1}{2}\tau|\omega|$. Then, in accord with (10),

$$K_{\omega}(\mathbf{p}, \mathbf{p}) = \gamma_l \mathcal{G}_{\omega}^{\tau}(\mathbf{p}) \mathcal{G}_{-\omega}^{\tau}(\mathbf{p}). \quad (13)$$

In the calculation of Q_0 , before summing over all frequencies in the expression

$$Q_0 = T \sum (2\pi)^{-3} \int d^3\mathbf{p} \mathcal{G}_{\omega}^{\tau}(\mathbf{p}) \mathcal{G}_{-\omega}^{\tau}(\mathbf{p}) \eta$$

we must satisfy ourselves, by integrating the difference $\mathcal{G}_{\omega}^{\tau}(\mathbf{p}) \mathcal{G}_{-\omega}^{\tau}(\mathbf{p}) \eta - \mathcal{G}_{\omega}(\mathbf{p}) \mathcal{G}_{-\omega}(\mathbf{p})$, that Q_0 for alloys coincides with the corresponding integral for the pure metal:

$$Q_0 = T \sum_{\omega} (2\pi)^{-3} \int d^3\mathbf{p} \mathcal{G}_{\omega}(\mathbf{p}) \mathcal{G}_{-\omega}(\mathbf{p}) = \frac{mT p_0}{2\pi^2} \sum_{\omega} \int_{\omega^2 + \xi^2}^{\infty} \frac{d\xi}{\omega^2 + \xi^2}$$

$$(\omega = \pi T(2n + 1)).$$

Completing the summation, we find

$$Q_0 = \frac{m p_0}{4\pi^2} \int d\xi \frac{\tanh(\xi/2T)}{\xi}$$

The divergence in this integral for large ξ must be eliminated by a cut corresponding to the fact that only electrons found in a layer within a distance $\tilde{\omega}$ from the Fermi surface interact in the model under consideration.² In accordance with reference 2.

$$Q_0 = \frac{m p_0}{2\pi^2} \ln \frac{2\tilde{\omega}}{\pi T} = \frac{m p_0}{2\pi^2} \left(\frac{1}{g} + \ln \frac{T_c}{T} \right).$$

No difficulty arises in finding $-\partial^2 Q_{\mathbf{q}} / \partial q_i^2 |_{\mathbf{q}=0}$; differentiating Eqs. (10) and (12) with respect to \mathbf{q} , we obtain, after some calculation,

$$-\partial^2 Q_{\mathbf{q}} / \partial q_i^2 |_{\mathbf{q}=0} = (m p_0 / 2\pi^2) (\pi v^2 T / 2) \sum_{\omega} \omega^{-3} \tau_{tr}^{-1},$$

where

$$\tau_{tr} = 1 + |2^{-tr}|\omega|, \quad \tau_{tr}^{-1} = nv \int (d\sigma/d\sigma) (1 - \cos\theta) d\sigma.$$

Finally, the average (5) is computed in the fol-

lowing fashion. The product of the four Green's functions in the integral (5) is drawn in Fig. 4. Their average over the position of the impurity atoms, in accord with the method employed, leads first to the diagram of Fig. 4a, in which now the average Green's functions appear. There will be other diagrams such as those in Fig. 4b, for example. These diagrams represent a modification of the corresponding vertices and are summed by means of Eqs. (10) and (12). In addition, there are two types of diagrams shown in Figs. 4c and 4d. It is not necessary to consider other diagrams, inasmuch as the diagrams with the intersection of dotted lines make a small contribution, according to the results of reference 6, while diagrams such as those shown in Fig. 4e contribute nothing in the integration over the internal momentum l , since the product of two \mathcal{G} functions appearing under the integral has a pole for ξ only in a single halfplane.



FIG. 5

Figure 5 shows the final form of the diagrams which are to be computed; the heavy dots at the vertices correspond to summation of all corrections to the vertex according to Fig. 3. Inasmuch as the momentum $\mathbf{q} = \mathbf{p}_1 + \mathbf{p}_2 = 0$ in Eq. (5) at all vertices, this means that the heavy dot corresponds to the factor η , as was found in (13). The integral and sums over the frequencies are easily computed:

$$B = (m p_0 / 2\pi^2) (7\zeta(3) / 8 (\pi T_c)^2),$$

where $\zeta(x)$ is the Riemann function, $\zeta(3) = 1.202$.

Substituting the results obtained above in Eq. (7), we find the following equation:

$$\left\{ \frac{1}{2m} \left(\frac{\partial}{\partial r} - 2ie\mathbf{A}(\mathbf{r}) \right)^2 + \frac{1}{\lambda_c} \left[\frac{T_c - T}{T_c} - \frac{7\zeta(3)}{8(\pi T_c)^2} |\Delta(\mathbf{r})|^2 \right] \right\} \Delta(\mathbf{r}) = 0,$$

which differs from the corresponding equation of reference 1 only in the expression for λ_{τ} :

$$\lambda_{\tau} = \lambda_0 \zeta(\rho), \quad (14)$$

where λ_0 is the corresponding coefficient for a pure superconductor, $\lambda_0 = 7\zeta(3) \epsilon_F / 12\pi^2 T_c^2$, and

$$\chi(\rho) = \frac{8}{7\zeta(3)} \sum_0^{\infty} \frac{1}{(2n+1)^2 [2n+1+\rho]}$$

$$= \frac{8}{7\zeta(3)} \left(\frac{1}{\rho} \right) \left[\frac{\pi^2}{8} + \frac{1}{2\rho} \left\{ \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \rho\right) \right\} \right] \quad (15)$$

(for $\rho \rightarrow 0$, $\chi(\rho) \rightarrow 1$, while for $\rho \rightarrow \infty$, $\chi(\rho) \approx \pi^2/7\zeta(3)\rho$). Here, $\rho = \frac{1}{2}\pi T_C \tau_{tr}$, $\psi(x)$ is the logarithmic derivative of the Γ function.

For the determination of the value of the electromagnetic current, it is necessary to know the contribution of second order in $|\Delta|$ to the Green's function. Repeating the derivation given in reference 1, we obtain

$$\delta \mathfrak{G}_\omega(\mathbf{r}, \mathbf{r}') = - \int \mathfrak{G}_{V\omega}(\mathbf{r}, \mathbf{s}) \Delta(\mathbf{s}) \tilde{\mathfrak{G}}_{V\omega}(\mathbf{l}, \mathbf{r}') \times \Delta^*(\mathbf{l}) \tilde{\mathfrak{G}}_{V-\omega}(\mathbf{l}, \mathbf{s}) d^3l d^3s. \quad (16)$$

As above, the averaging over the sites of the impurity atoms of a combination of three Green's functions of the normal metal does not show any dependence of the integrand on the magnetic field.

In the determination of the current,

$$\mathbf{j}(\mathbf{r}) = + \left[\frac{ie}{m} (\nabla_r - \nabla_{r'}) G(x, x') + \frac{2e^2}{mc} \mathbf{A}(\mathbf{r}) G(x, x') \right]_{r=r', t'=t+0} \quad (17)$$

a Green's function appears with coinciding arguments; therefore, in the averaging of (16), it is sufficient to limit ourselves to this case. Considering the field to be weak, and $\Delta(\mathbf{r})$ to be of the order of l , changing slowly with distance, we carry out the corresponding simplification in (16) and (17). We shall not linger over the calculations, limiting ourselves only to the scheme of averaging over the impurity in Fig. 6, which represents the corresponding diagram which must be summed.

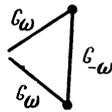


FIG. 6

We note that the corrections to the vertex into which enter two Green's functions with the same frequency are absent, inasmuch as such corrections lead to the integration of products of two \mathfrak{G} functions with poles located in one halfplane. After computation we obtain the following expression for the current:

$$\mathbf{j}(\mathbf{r}) = \left\{ \frac{ie}{m} \left(\Delta(\mathbf{r}) \frac{\partial \Delta^*(\mathbf{r})}{\partial \mathbf{r}} - \Delta^*(\mathbf{r}) \frac{\partial \Delta(\mathbf{r})}{\partial \mathbf{r}} \right) - \frac{4e^2}{mc} \mathbf{A}(\mathbf{r}) |\Delta(\mathbf{r})|^2 \right\} \chi(\rho) \frac{7\zeta(3)N}{16\pi^2 T_C^2}$$

If we introduce the "wave function" $\psi(\mathbf{r})$ connected with $\Delta(\mathbf{r})$ by the relation

$$\psi(\mathbf{r}) = [\chi(\rho) 7\zeta(3) N / 16\pi^2 T_C^2]^{1/2} \Delta(\mathbf{r}),$$

then the resultant system of equations can be written in a form corresponding to the equations of the phenomenological theory of Ginzburg-Landau:³

$$\left\{ \frac{1}{2m} \left(\frac{\partial}{\partial \mathbf{r}} - 2ie\mathbf{A}(\mathbf{r}) \right)^2 + \frac{1}{\lambda_\tau} \left[\frac{T_c - T}{T_c} - \frac{2}{N\chi(\rho)} |\psi(\mathbf{r})|^2 \right] \right\} \psi(\mathbf{r}) = 0,$$

$$\mathbf{j}(\mathbf{r}) = \frac{ie}{m} \left(\psi \frac{\partial \psi^*}{\partial \mathbf{r}} - \psi^* \frac{\partial \psi}{\partial \mathbf{r}} \right) - \frac{4e^2}{mc} \mathbf{A}(\mathbf{r}) |\psi|^2. \quad (18)$$

The most interesting part of this result is the two-fold charge which figures in the equations. The existence of bound pairs of electrons in the BCS theory makes this result entirely understandable. In this light, the meaning of the wave function $\psi(\mathbf{r})$ of the Ginzburg-Landau theory as a quantity proportional to the wave function of a bound pair (more precisely, the coordinates of its center of mass) becomes clarified. An identical result was obtained by us for a pure superconductor.¹

The difference of alloys from the case of a pure metal lies, as is well known, in the constant κ which appears in the phenomenological theory of Ginzburg-Landau, and which is determined (with consideration of the double charge) by the relation

$$\kappa = \sqrt{2} 2eH_{cT} \delta_T^2 / hc, \quad (19)$$

where H_{cT} is the critical field, and δ_T is the penetration depth close to T_C which is temperature dependent.

The equations (18) permit us to find the connection of the constant κ for alloys with κ_0 of a pure superconductor, and to express it in terms of the parameters of the BCS theory. The constant κ for alloys is equal to

$$\kappa = \kappa_0 / \chi(\rho),$$

where the function $\chi(\rho)$ has been introduced above in (15), while κ_0 for a pure metal is expressed (according to reference 1) in the following form:

$$\kappa_0 = (3\pi T_C m c e) \sqrt{2\pi m / 7\zeta(3) p_0^5} \quad (\zeta(3) = 1.202). \quad (20)$$

In the relations that have been obtained and in the determination of $\rho = \frac{1}{2}\pi T_C \tau_{tr}$, four parameters generally appear: T_C , m , p_0 for the pure metal, and the "transport" time of the free flight τ_{tr} . For pure superconductors of the London type and for intermediates, i.e., such for which throughout the whole temperature range the penetration depth is larger or comparable to the correlation length $\xi_0 = hv/2\pi T_C$, an appreciable temperature region close to T_C exists where one can use the relations (19) with the experimental values of H_C and δ_0 for the determination of κ_0 . The theoretical expression² for δ_0 close to T_C has the form

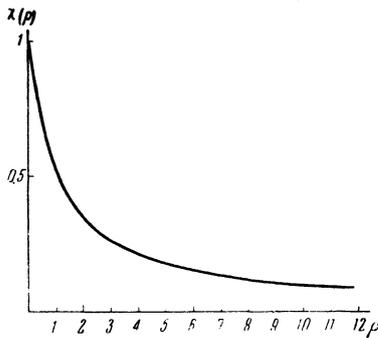


FIG. 7

$$\delta_0 = \delta_{0L} / \sqrt{2(1 - T/T_c)} \quad (\delta_{0L} = \sqrt{mc^2/4\pi Ne^2}) \quad (21)$$

In the isotropic model under consideration, the time between collisions τ_{tr} can be expressed in terms of δ_{0L} and the conductivity σ :

$$\tau_{tr} = 4\pi\sigma\delta_{0L}^2/c^2. \quad (22)$$

In terms of these quantities

$$\rho = \hbar c^2/8\pi^2 k T_c \sigma \delta_{0L}^2. \quad (23)$$

The function $\chi(\rho)$ is shown graphically in Fig. 7. We also give the formula which connects the penetration depth of a weak magnetic field for an alloy with the penetration depth for the pure superconductor

$$\delta = \delta_0 / \sqrt{\chi(\rho)}. \quad (24)$$

If the quantity of impurity is sufficiently large, so that $l \ll \xi_0$, the equations are simplified. In this case, κ for alloys can be expressed only in terms of the parameters of the normal metal: the coefficient in the linear law for the heat capacity γ and the conductivity σ :

$$\kappa = (ec\gamma^{1/2}/\sigma k \pi^3) \sqrt{21\zeta(3)/2\pi} = 0.065ec\gamma^{1/2}/\sigma k \quad (25)$$

For metals of the Pippard type, i.e., those for which at low temperatures the penetration depth is much smaller than the parameter ξ_0 , chief interest lies in the latter formula. This is connected with the fact that for a pure superconductor of the Pippard type, the London region of temperatures close to T_c , where the Ginzburg-Landau equations hold, is very narrow (for aluminum, $\Delta T/T_c \sim 10^{-4}$). The presence of impurities enlarges this region. In order that this region be experimentally accessible, it is necessary that $l \ll \xi_0$.

Equations (20) and (25) permit us to determine the critical concentration of impurities for which the superconductor in a strong field begins to reveal the characteristic peculiarities of alloys. As is well known, the surface energy on the boundary of separation between the normal and superconduct-

ing phases vanishes for $\kappa_{cr} = 1/\sqrt{2}$. At the same time, the critical field of supercooling is made equal to the critical thermodynamic field, while for large κ it exceeds it. Therefore, for $\kappa > 1/\sqrt{2}$, the transition in the magnetic field from the normal to the superconducting state begins at fields larger than the critical field H_{CT} of the pure superconductor, namely at the field $H_{C1} = \sqrt{2} \kappa H_{CT}$. This peculiarity of alloys was pointed out by Abrikosov.⁷ For Pippard superconductors, $\kappa = 1/\sqrt{2}$ is achieved at such concentrations of impurities that $l \ll \xi_0$; therefore, it follows from (25) that for the critical amount of impurities, the residual conductivity is connected only with the coefficient γ in the linear law for electronic specific heat $\sigma_{cr} = 0.092 ec\gamma^{1/2}/k$.

The linear dependence of H_{CT}/H_{C1} on the path length l for contaminated alloys on the basis of phenomenological considerations was first pointed out by Pippard.⁸ For $l \ll \xi_0$, the formula for the penetration depth was obtained in the work of Abrikosov and the author.⁶

We note that, as experiment has shown,^{8,9} the actual region $\kappa \sim 0.7$ corresponds, for example, in the case of an alloy of tin and indium, to $l \sim 5 \times 10^{-6}$ cm (or a concentration of indium ~ 3 percent). For such concentrations of impurities, there is a certain insignificant shift in the temperature⁹ ($\Delta T_c/T_c \sim 5$ percent). We shall neglect this effect, assuming the number of impurities to be so small that we do not have to take into consideration their effect on the fundamental lattice. At high concentrations, it appears that such an approximation is improper. However, it is possible to assume, that the relation (25) will still hold in such alloys, inasmuch as there appear in it only quantities which refer to the normal metal. The coefficient in the linear law for the specific heat γ must be introduced in this case for the alloy itself.

The available experimental data refer to tin.^{8,9} In the work of Dodge,⁹ the transition in a magnetic field from a normal phase to the superconducting phase was studied for an alloy of tin and indium. For concentrations of indium of about 3 percent, this transition has the usual character; for higher concentrations, the appearance of the superconducting phase occurs at fields larger than the critical thermodynamic field. The ratio H_{CT}/H_{C1} , i.e., the value $\chi(\rho)/\sqrt{2} \kappa_0$ is plotted in Fig. 8 as a function of the mean free path length l . (We have made use of the result of the research of Chambers¹⁰ for tin: $\sigma/l = 8.5 \times 10^{22}$, by means of which one can express ρ in terms of the mean free path length: $\rho = 2 \times 10^{-5}/l$.) For pure tin, $\kappa_0 = 0.158$,¹¹ the value of κ of the alloy (0.707) corresponds

in the calculations to $l \sim 5.5 \times 10^{-6}$ cm, while the experimental value is somewhat less than 8×10^{-6} cm.

Taking into account the known crudeness in the determination of δ_{0L} (we have used $\delta_{0L} = \delta_{00}/\sqrt{2}$, where δ_{00} is the coefficient in the law for the skin depth

$$\delta = \delta_{00}/\sqrt{1 - (T/T_c)^4},$$

is equal to $\delta_{00} = 5.1 \times 10^{-6}$ cm), one can consider the agreement to be excellent.

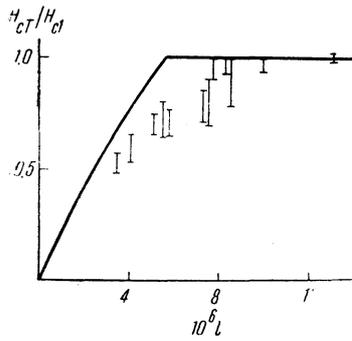


FIG. 8

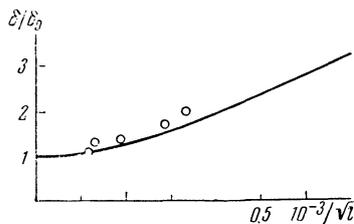


FIG. 9

Results are shown in Fig. 9 of a comparison of the formula (24) for the skin depth with the results of Pippard,⁸ which apply to an alloy of indium and tin with a concentration up to 3 percent of indium.

We note that the proportionality of the penetration depth $l^{-1/2}$ is still not attained for such concentrations.

In conclusion I express my deep gratitude to Acad. L. D. Landau for his interest in the work and for the comments that he has made.

¹L. P. Gor'kov, JETP **36**, 1918 (1959), Soviet Phys. JETP **9**, 1364 (1959).

²Bardeen, Cooper, and Schrieffer, Phys. Rev. **108**, 1175 (1957).

³V. L. Ginzburg and L. D. Landau, JETP **20**, 1064 (1950).

⁴A. A. Abrikosov and I. M. Khalatnikov, Usp. Fiz. Nauk **65**, 551 (1958).

⁵L. P. Gor'kov, JETP **34**, 735 (1958), Soviet Phys. JETP **8**, 505 (1958).

⁶A. A. Abrikosov and L. P. Gor'kov, JETP **35**, 1558 (1958); **36**, 319 (1959); Soviet Phys. JETP **8**, 1090 (1959); **9**, 220 (1959).

⁷A. A. Abrikosov, JETP **32**, 1442 (1957), Soviet Phys. JETP **5**, 1174 (1957);

⁸A. B. Pippard, Proc. Roy. Soc (London) **A216**, 547 (1953).

⁹P. R. Dodge, Phil. Trans. Roy. Soc. (London) **A248**, 553 (1956).

¹⁰R. G. Chambers, Proc. Roy. Soc. (London) **A215**, 481 (1952).

¹¹V. L. Ginzburg, JETP **36**, 1930 (1959); Soviet Phys. JETP **9**, 1372 (1959).

Translated by R. T. Beyer

278