ON THE INTRODUCTION OF AN "ELEMENTARY LENGTH" IN THE RELATIVISTIC THEORY OF ELEMENTARY PARTICLES

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A momentum space of constant curvature is introduced into the theory in place of the pseudo-Euclidean momentum space. The Feynman diagram technique is suitably generalized. Finite results are obtained in the lowest order perturbation theory approximation for the fermion and boson self-energy.

INTRODUCTION

THE formulation of a theory of elementary particles free of the "ultraviolet catastrophe" is, apparently, impossible without the introduction of an "elementary length" l_0 which defines the limit down to which the ordinary concepts of space are still valid. Another equivalent possibility is the introduction into the theory of a "limiting mass" $\mu \sim 1/l_0$, which establishes corresponding limitations in momentum space. The present paper is devoted to an attempt of formulation of such a theory. A brief formulation of the principal idea consists of the following. The four-dimensional momentum space is a space of constant curvature. The radius of curvature of this space is the limiting mass μ_0. The theory must be formulated in accordance with the geometry of a p-space of constant curvature.

The existence of a geometric principle introduces a considerable degree of uniqueness into the formulation of the theory. At the same time a consistent development of this principle leads to far-reaching consequences, the most important of which is the conclusion that in this theory the law of conservation of energy and momentum appears in an altered form. Apparently other physical concepts, for example, coordinate space and the condition of microcausality, will also have to undergo a similarly serious alteration.

In the present paper Feynman's diagram technique is generalized in the spirit of the geometry of a p-space of constant curvature (Sec. 3). Preliminary investigations (Sec. 4) show that "ultraviolet" divergences are not very likely to arise within such a scheme.

Diagram technique is one of the fundamental tools of the modern theory of elementary particles. Therefore the possibility of generalizing this technique to the case of a p-space of constant curvature may be regarded as a hopeful result. However, to formulate a consistent theory of elementary particles it is necessary to subject to a similar generalization many other aspects which play an important role in modern theory.

1. MOMENTUM SPACE OF CONSTANT CURVATURE

The principal role in the theory under discussion is played by the p-space of constant curvature. The radius of curvature of this space μ_0 has the dimensions of a mass. The usual theory corresponds to the case $\mu_0 = \infty$. We shall consider the magnitude of μ_0 to be finite, and shall assume only that the masses of the elementary particles satisfy the condition $m \ll \mu_0$. The numerical value of the constant μ_0 must be determined from experiment. In the following we adopt a system of units in which $\mu_0 = c = \hbar = 1$, so that all the relationships take on dimensionless form.

The general method of constructing spaces of constant curvature, based on the introduction of a projective metric into the space, was developed by F. Klein.[1] In following this method we isolate in the four dimensional p-space the hypersurface

$$p^2 = 1 \tag{1.1}$$

and we define the non-Euclidean distance between the points p and q in the form

$$D(p, q) = \ln(J + \sqrt{J^2 - 1}),$$
$$J = (1 - pq) / \sqrt{(1 - p^2)(1 - q^2)}. \tag{1.2}$$

We henceforth adopt the following notation: p, q, . . . denote four-vectors in p-space, for example $p = (p_1, p_2, p_3, p_4)$. The scalar product of the vectors p and q is given by $pq = p_4q_4 - p_1q_1 - p_2q_2 - p_3q_3$; $p^2 \equiv pp$.

If all the components of the vectors p and q are much less than unity, then up to terms of higher order

$$D(p, q) \approx \sqrt{(p-q)^2},$$

i.e., the metric is pseudo-Euclidean near the origin. The hypersurface (1.1) divides the whole p-space into two parts, interior ($p^2 < 1$) and exterior ($p^2 > 1$). According to (1.2) the "distance" D between points lying in the interior and exterior regions is complex. The momenta of real particles always lie in the interior region. However, there are no reasons for considering that the momenta of virtual particles cannot take on values lying in the exterior part.

From expression (1.2) we can easily obtain for our space the differential metric form which defines the square of the distance between two infinitely close points p and $q = p + dp$

$$ds^2 = (1-p^2)^{-1}\{dp^2 + (1-p^2)^{-1}(pdp)^2\},$$

from which we find in the usual manner the magnitude of the volume element

$$d\Omega = (1-p^2)^{-5/2}\, d^4p. \tag{1.3}$$

To compute the volume of the whole space it is necessary to indicate the manner of going around the singularities situated on the hypersurface $p^2 = 1$. We define the volume of the space in the following manner

$$\Omega = \lim_{\delta\to 0}\int (1 - i\delta - p^2)^{-5/2}\, d^4p. \tag{1.4}$$

The manner of evaluating the integral (1.4) is illustrated in Fig. 1. Cuts are introduced in the complex plane of p_4 from the branch points $p_4 = \pm\sqrt{1 + p^2 - i\delta}$. The path of integration 1 — 2 can be transformed without crossing the cuts into the path of integration 3 — 4 (the integrals along the arcs of the large circles tend to zero).

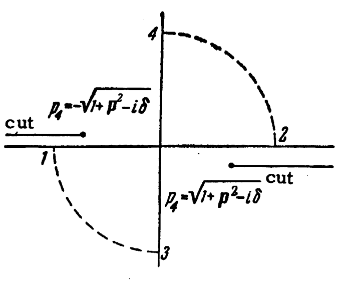

FIG. 1

As a result we obtain

$$\Omega = i\int (1 + p_1^2 + p_2^2 + p_3^2 + p_4^2)^{-5/2}\, d^4p = \tfrac{4}{3}\, i\pi^2.$$

The imaginary value of the volume Ω of the space does not lead to any difficulties. In view of the fact that the volume Ω is finite, the integrals over the momenta of the virtual particles of the self-energy type converge. In evaluating these integrals, the same procedure is utilized as for the evaluation of the volume Ω.

2. THE GROUP OF MOTIONS OF p-SPACE

The group of motions in p-space consists of all the point transformations that leave invariant the distance (1.2). According to the general theory due to Klein,[1] the motions in p-space are described by projective (i.e., linear or fractionally linear with respect to the components of the vector p) transformations which transform the hypersurface (1.1) into itself. Evidently the group of motions in p-space contains the group of Lorentz transformations which leave invariant the magnitude of p^2 and which transform the point $p = 0$ into itself.

In addition to the Lorentz transformations, the group of motions contains also the "displacement" transformations. Each displacement transforms the point $p = 0$ into some point k, with a one-to-one correspondence existing between the vectors k and the operations of displacement. This enables us to speak of "displacements by a vector k." Symbolically we shall represent the effect of displacements by equalities of the following type

$$q = p(+)k, \tag{2.1}$$

by which we understand that the vector q was formed as the result of displacing the vector p by the vector k. The relation (2.1) establishes a certain law of addition of vectors in p-space. The most characteristic feature of this "addition" is that it does not commute

$$p(+)k \neq k(+)p.$$

The operation which is the inverse of the displacement by a vector k is a displacement by a vector −k. The result of such a displacement we shall denote by an equality of the type

$$q = p(-)k.$$

These definitions may be generalized in an obvious manner to a "sum" of several vectors. For example, the equality

$$q = p(+)k(-)l(+)n, \tag{2.2}$$

denotes the consecutive application to the vector p of displacements by k, by $-l$ and by n. Of course the order of the "addends" in (2.2) is important.

We note that the operations of displacement do not form a group; the product of two displacements will no longer be a displacement. One may show that displacements generate the whole group of motions in p-space.

Explicitly the "sum" of two vectors is defined by the expression

$$q = p(+)k = \frac{p\sqrt{1-k^2} + k(1+pk)/(1+\sqrt{1-k^2})}{1+pk}. \qquad (2.3)$$

If both vectors p and k are small ($p_\mu, k_\mu \ll 1$) then

$$p(+)k \approx p + k,$$

i.e., near the origin of coordinates the "addition" of vectors defined by formula (2.3) coincides, up to higher order terms, with ordinary addition. The following equality follows immediately from (2.3):

$$1/\sqrt{1-q^2} = (1+pk)/\sqrt{(1-p^2)(1-k^2)}, \qquad (2.4)$$

from which it can be seen that the transformation (2.3) describes the motion in p-space, since it follows from $p^2 = 1$ that $q^2 = 1$.

In going over to a p-space of constant curvature a new feature arises, associated with the transformation of spinors under displacements. While in the case of a pseudo-Euclidean space spinors do not transform under a displacement, in the case of a "curved" space it is necessary to make a displacement by a vector k correspond to the following transformation of the bispinor ψ:

$$\psi \to d(k)\psi, \qquad d(k) = [(1-\hat{k})/(1+\hat{k})]^{1/4}, \qquad (2.5)$$

where

$$\hat{k} = k_\mu \gamma_\mu.$$

The relation (2.5) follows from the connection between the group of motions in p-space considered by us with the rotation group of a certain pseudo-Euclidean five-dimensional space.

3. PRINCIPLES OF THE DIAGRAM TECHNIQUE

To describe interactions between elementary particles, we make use of Feynman's diagram technique,[2] altered in the spirit of the geometry of p-space of constant curvature. We consider the interaction of a Fermi-field with a Bose-field. Let us trace the changes in the momentum of the fermion along a line (we shall consider both fermion and boson lines to be directed). In the usual theory, the law of conservation of energy-momentum holds at each vertex of the diagram. If the momentum of the fermion is p "before" the absorption of the boson, then "after" absorption it becomes equal to p + k, where k is the momentum of the absorbed boson. In going over to "curved" p-space, it is necessary to alter the rule for the addition of momenta at the vertices of the diagram: ordinary geometric addition of momenta is replaced by addition in the sense of Eq. (2.3). This rule is formulated more exactly in the following manner: if at the beginning of the line the fermion had a momentum p, then after the vertex of the diagram which corresponds to the absorption of a boson of momentum k, we must replace p by p(+)k, while after a vertex which corresponds to the emission of a boson with momentum k′, we should replace it by p(−)k′. In moving along a fermion line in this manner we obtain a unique result for the final momentum p′. The sequence of operations required for this is determined by the sequence of vertices along the fermion line. Figure 2 shows several of the simplest diagrams which illustrate the rule of addition of momenta. As can be seen in the examples of the diagrams of Fig. 2, the relation between the values of the energy-momentum vectors of the particles before and after scattering is no longer universal, but depends on the nature of the scattering process.

Thus we conclude that the introduction into the theory of a p-space of constant curvature instead of the pseudo-Euclidean space requires the alteration of the usual form of the law of conservation of energy-momentum. In the new scheme the latter is only approximate, and is valid in the case when the energy and the momentum of the interacting particles are small. This follows directly from the fact that for small values of energy and momentum the new law of addition goes over into the ordinary

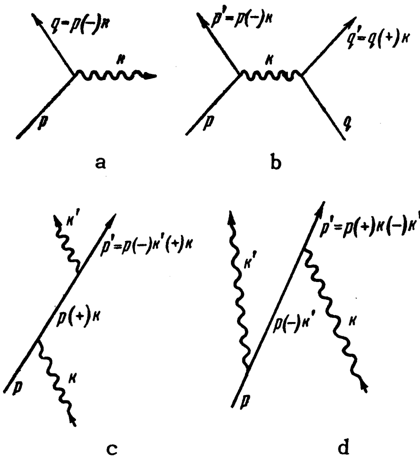

FIG. 2

law of geometric addition of vectors (Sec. 2).

After the nature of the change in the momenta along a fermion line has been established, all the Feynman rules for the calculation of matrix elements corresponding to a given diagram can be directly generalized. The propagation factor $D(k)$ corresponds to the boson line with momentum k, while the propagation factor $G(p)$ corresponds to the fermion line of momentum p. To the segment of an internal fermion line with momentum $p(+)k$ there corresponds the propagation factor

$$d^{-1}(k)\, G(p(+)k) d(k). \tag{3.1}$$

The appearance in the expression (3.1) of the matrix $d(k)$ is associated with the transformation of spinors (2.5) under displacements by vector k. For the sake of simplicity we set the vertex operator Γ equal to unity, i.e., we consider the case of scalar bosons with scalar coupling. Finally, it is necessary to integrate over the volume of the whole p-space over the momenta of all the virtual particles which are not defined uniquely by the law of addition. In carrying out this integration, it is necessary to take into account Feynman's rule for going around the poles and to rotate through quadrants the contour of integration over the momentum components as shown in Fig. 1.

4. SOME CONSEQUENCES OF NON-COMMUTATIVITY OF "ADDITION" OF MOMENTA

We shall illustrate by means of simplest examples results due to non-commutativity of "addition" of momenta. We first consider a first-order process (diagram a, Fig. 2). It can be easily shown that, just as in the usual theory, a free particle cannot emit a boson. This follows directly from relation (2.4), in which we must set $q^2 = p^2 = m^2$, $k^2 = \mu^2$. In going over to the fermion rest system, we obtain

$$1 + m\omega = \sqrt{1-\mu^2}. \tag{4.1}$$

Obviously Eq. (4.1) cannot hold for positive values of the boson energy ω.

As a second example, we consider the scattering of a boson by a fermion. This process corresponds to two second-order diagrams (Fig. 2, b, c), which differ in the order of emission and absorption of the boson. Whereas in the usual theory both these diagrams contribute to the same scattering process, in our case, because of the non-commutativity of the "addition" of momenta, they lead to different final states starting from the same initial momenta p and k of the colliding particles. In other words, these diagrams describe two different scattering processes. For the sake of simplicity we write for the boson mass $\mu = 0$ and establish certain generalizations of the well-known Compton formula. By assuming that the fermion is at rest in the initial state, while the momenta of the quantum before and after scattering are respectively equal to $k = (\mathbf{k}, \omega)$ and $k' = (\mathbf{k}', \omega')$, we easily obtain, with the aid of relations (2.3) and (2.4), expressions for the energy of the scattered quantum ω' as a function of the energy of the incident quantum ω and of the scattering angle θ. For the case of the diagram of Fig. 2c we obtain:

$$\omega_1' = 2\omega_c/[1 + \sqrt{1 - 4\omega_c^2 \sin^2(\theta/2)}]. \tag{4.2}$$

For the case of the diagram of Fig. 2d we have:

$$\omega_2' = \omega_c/[1 + \omega\omega_c \sin^2(\theta/2)], \tag{4.3}$$

where ω_C is the energy of the scattered quantum which corresponds to the Compton formula

$$\omega_c = \omega/[1 + (\omega/m)(1 - \cos\theta)].$$

It is clear that for ω or $\omega_C \ll 1$

$$\omega_1' \approx \omega_2' \approx \omega_c.$$

However, when $\omega \gtrsim 1$ considerable deviations from the Compton formula do occur. We consider scattering through an angle $\theta = 90°$. When $\omega \gg m$ we have $\omega_C = m \ll 1$. At the same time

$$\omega_1' = m, \quad \omega_2' = m/(1 + m\omega/2).$$

The "line splitting" arising in this case

$$\Delta\omega/\omega_2' = (\omega_1' - \omega_2')/\omega_2' = m\omega/2$$

increases proportionally to the energy of the incident quantum.

5. SIMPLEST SELF-ENERGY DIAGRAMS

Let us consider the expressions for the self-energy of the fermion and the boson in second-order perturbation theory, corresponding to the diagrams of Fig. 3. According to the general rules of Sec. 4 we obtain for the self-energy of the fermion

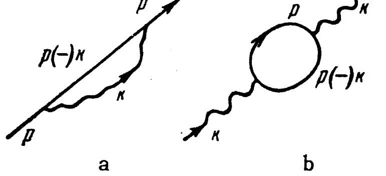

FIG. 3

$$\Sigma(p) = \frac{g^2}{(2\pi)^4 i} \int d(k)\, G(p(-)k)\, d^{-1}(k)\, D(k)\, d\Omega_k, \quad (5.1)$$

while for the self-energy of the boson we obtain

$$\Pi(k) = \frac{g^2}{(2\pi)^4 i} \int \mathrm{Sp}\{G(p)\, d(k)\, G(p(-)k)\, d^{-1}(k)\}\, d\Omega_p. \quad (5.2)$$

To estimate the integrals (5.1) and (5.2) it is necessary to know the explicit form of the propagation factors D(k) and G(p).

However, within the framework of the preliminary outline of the theory developed here, we cannot determine uniquely the forms of these factors. Therefore as an additional hypothesis we set

$$D(k) = (1 - k^2)/(k^2 - \mu^2),$$

$$G(p) = \sqrt{1 - p^2}/(\hat{p} - m). \quad (5.3)$$

The choice of expressions (5.3) is based, on the one hand, on the correspondence principle for small values of the momenta and, on the other hand, on the special nature of the hypersurface (1.1) which plays a fundamental role in the whole theory. We particularly emphasize the utterly tentative nature of expressions (5.3), the form of which may undergo significant changes in a consistent theory.

Fairly simple calculations yield

$$d(k)\, G(p(-)k)\, d^{-1}(k)$$

$$= \frac{\sqrt{(1 - k^2)(1 - p^2)}\,\{(1 - m)(1 - kp) - (1 - \hat{k})(1 + \hat{p})\}}{(1 + k^2)(1 - p^2) - (1 - m^2)(1 - kp)^2}. \quad (5.4)$$

Substituting (5.3) and (5.4) into the expressions for the self-energy we obtain

$$\Sigma(p) = \frac{g^2 \sqrt{1 - p^2}}{(2\pi)^4 i} \times \int \frac{(1 - m)(1 - kp) - (1 - \hat{k})(1 + \hat{p})}{[(1 - k^2)(1 - p^2) - (1 - m^2)(1 - kp)^2]\,(k^2 - \mu^2)(1 - k^2)}\, d^4k, \quad (5.5)$$

$$\Pi(k) = \frac{g^2 \sqrt{1 - k^2}}{4\pi^4 i} \times \int \frac{1 - p^2 - (1 + m^2)(1 - kp)}{[(1 - k^2)(1 - p^2) - (1 - m^2)(1 - kp)^2]\,(p^2 - m^2)(1 - p^2)^{3/2}}\, dp^4. \quad (5.6)$$

To evaluate the integrals (5.5) and (5.6) we must utilize the procedure indicated at the end of Sec. 3. In this way, convergent results are obtained. There is no point in carrying out a more detailed analysis of the singularities of the foregoing expressions, since such an investigation depends on the forms of the functions D and G. The most important result is the absence of the ultraviolet catastrophe, and this result most probably will hold also in a consistent theory.

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[1] F. Klein, Non-Euclidean Geometry (Russian transl.) Moscow-Leningrad, 1936.

[2] R. P. Feynman, Phys. Rev. 76, 796 (1949).

Translated by G. Volkoff
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