ON THE CALCULATION OF THE INTERACTION CONSTANT IN A NONLINEAR THEORY

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The coupling constant is expressed in terms of the proton mass in the second approximation of the Tamm-Dancoff method. The formulas obtained can be used in the case in which the Lagrangian is an arbitrary linear combination of the five basic invariants.

CALCULATION of various physical quantities in the theory with the Lagrangian

$$L = \bar{\psi}\gamma_\mu\nabla_\mu\psi + \frac{1}{2} l^2 \sum_n C_n N(\bar{\psi}O_n\psi\cdot\bar{\psi}O_n\psi) \quad (1)$$

requires a knowledge of the coupling constant l. Various methods exist for its determination: from the meson-nucleon interaction,[1] from the mass of the π meson,[1,2] and so on. In the present paper we shall determine it from a calculation of the mass of the nucleon.[3]

The equation for a Fourier component of the nucleon wave function,

$$\psi(p) = \int \langle 0 | \psi(x) | \Phi\rangle e^{-ipx} d^4x \quad (2)$$

can be derived in the same way as in the paper of Heisenberg, Kortel, and Mitter.[3] It has the form

$$\psi(p) = -\frac{l^4}{3(2\pi)^8} G(p) \sum_{m,n} C_m C_n O_m \int d^4q d^4r$$
$$\times \{G(p+q-r) O_n S(q) O_m S(r)$$
$$- G(p+q-r)\,\mathrm{Sp}\, O_n S(q) O_m S(r)$$
$$+ S(r) O_n S(q) O_m G(p+q-r)$$
$$- S(r)\,\mathrm{Sp}\, O_n S(q) O_m G(p+q-r)$$
$$- S(r) O_n G(p+q-r) O_m S(-q)$$
$$+ S(r)\,\mathrm{Sp}\, O_n G(p+q-r) O_m S(r)\} O_n \psi(p). \quad (3)$$

In this integral the path is taken around the poles of the Green's function

$$G(p+q-r) = \gamma_\mu \frac{(p+q-r)_\mu}{(p+q-r)^2} \equiv \frac{\hat{S}}{(p+q-r)^2} \quad (4)$$

and the propagation function

$$S(q) = \frac{\varkappa^3}{q^2(q^2+\varkappa^2)}\left(\varkappa\frac{\gamma_\nu q_\nu}{q^2} - i\right)$$
$$\equiv \frac{\varkappa^3}{q^2(q^2+\varkappa^2)}(\hat{Q} - i) \quad (5)$$

as prescribed by Feynman's rules.

Using the identity[4,5]

$$(O_n M O_n)_{\alpha\beta} = O^n_{\alpha\gamma} M_{\gamma\delta} O^n_{\delta\beta}$$
$$= \frac{1}{4} M_{\gamma\delta} \sum_k A_{nk} O^k_{\alpha\beta} O^k_{\delta\gamma} = \frac{1}{4}\sum_k A_{nk} O^k_{\alpha\beta}\,\mathrm{Sp}\, M O^k, \quad (6)$$

we can reduce all the terms of Eq. (3) to calculations of traces:

$$\psi(p) = -\frac{l^4\varkappa^6}{3(2\pi)^8} G(p) \sum_{m,n,k} C_m C_n \left(\frac{1}{4}A_{nk} - \delta_{nk}\right)$$
$$\times \int \frac{d^4q d^4r}{N_1} \{O_m \hat{S} O_k\,\mathrm{Sp}\,(\hat{Q} - i) O_m (\hat{R} - i) O_k$$
$$+ O_m(\hat{R} - i) O_k\,\mathrm{Sp}\,(\hat{Q} - i) O_m \hat{S} O_k$$
$$+ O_m(\hat{R} - i) O_k\,\mathrm{Sp}\,\hat{S} O_m(\hat{Q} + i) O_k\} \psi(p)$$
$$= -\frac{l^4\varkappa^6}{3(2\pi)^8} G(p) \sum_{m,n} C_m C_n \int \frac{d^4q d^4r}{N_1} \sum_k \left(\frac{1}{4}A_{nk} - \delta_{nk}\right)$$
$$\times \{O_m \hat{S} O_k [\mathrm{Sp}\, O_k \hat{Q} O_m \hat{R} - i\,\mathrm{Sp}\, O_k O_m \hat{R} - i\,\mathrm{Sp}\, O_k \hat{Q} O_m$$
$$- \mathrm{Sp}\, O_k O_m] + O_m(\hat{R} - i) O_k [\mathrm{Sp}\, O_k \hat{Q} O_m \hat{S} - i\,\mathrm{Sp}\, O_k O_m \hat{S}$$
$$+ i\,\mathrm{Sp}\, O_k \hat{S} O_m + \mathrm{Sp}\, O_k \hat{S} O_m \hat{Q}]\} \psi(p). \quad (7)$$

Here we have for brevity introduced the notation

$$N_1 = (p+q-r)^2 q^2 (q^2 + \varkappa^2) r^2 (r^2 + \varkappa^2). \quad (8)$$

The calculation of the traces is based on the following observation: since the trace of any of the 16 Dirac matrices, except the unit matrix, is zero, $\mathrm{Sp}\,O_k\gamma_\mu O_m\gamma_\nu$ is different from zero (and is then equal to 4) only if $O_k = \gamma_\nu O_m \gamma_\mu = O_{m_1}$. Thus we have

$$\mathrm{Sp}\, O_k \gamma_\mu O_m \gamma_\nu = 4\delta_{km_1}. \quad (9)$$

Owing to the factor δ_{km_1} the sum over k can be found; in doing so we must keep in mind that $A_{nm_1} = A_{nm}$. Thus we have

$$\sum_k \left(\frac{1}{4}A_{nk} - \delta_{nk}\right) O_k\,\mathrm{Sp}\, O_k \hat{Q} O_m \hat{R} = \left(\frac{1}{4}A_{nm} - \delta_{nm}\right) 4\gamma_\nu O_m \gamma_\mu Q_\mu R_\nu$$
$$= (A_{nm} - 4\delta_{nm}) \hat{R} O_m \hat{Q}. \quad (10)$$

Similarly,

$$\sum_k \left(\frac{1}{4}A_{nk} - \delta_{nk}\right) O_k\,\mathrm{Sp}\, O_k \hat{Q} O_m = -(A_{nm'} - 4\delta_{nm'}) O_m \hat{Q}, \quad (11)$$

where the index m′ comes from the matrix $O^{m'} = O^m\gamma_\nu$. Multiplication of the matrix O^m by γ_ν transfers it from one class to another, the correspondence being

$$\begin{array}{llllll} O^m: & S & V & T & A & P \\ O^{m'}: & V & T & A & P & A \end{array}$$

In Eq. (7) the first and fourth terms in each set of square brackets give traces of the first type; all the other terms give traces of the second type. Applying Eqs. (10) and (11), we reduce the sum over k to the form

$$(A_{nm} - 4\delta_{nm})[O_m\hat{S}\hat{R}O_m\hat{Q} - O_m\hat{S}O_m + O_m(\hat{R} - i)\hat{S}O_m\hat{Q} + O_m(\hat{R} - i)\hat{Q}O_m\hat{S}] - (A_{nm'} - 4\delta_{nm'})[-iO_m\hat{S}\hat{R}O_m - iO_m\hat{S}O_m\hat{Q} - iO_m(\hat{R} - i)\hat{S}O_m + iO_m(\hat{R} - i)O_m\hat{S}]. \quad (12)$$

The expressions in square brackets can be simplified if we use the formulas

$$O_m\hat{R}\hat{S}O_m + O_m\hat{R}\hat{S}O_m = 2(RS)O_mO_m = 2(RS)A_{mS}, \quad O_m\hat{S}O_m = \hat{S}A_{mV} \quad (13)$$

and note that the momentum integrals in Eq. (7) are invariant under the interchange q ⇄ − r, which makes $\hat{Q}\hat{R} \to \hat{R}\hat{Q}$. This enables us to use here the equation

$$\hat{Q}\hat{R} = (QR). \quad (14)$$

Thus Eq. (12) can be reduced to

$$(A_{nm} - 4\delta_{nm})[2\hat{Q}(RS)A_{mS} + \hat{S}(QR)A_{mS} - \hat{S}A_{mV} - 2i(QS)A_{mV}] - (A_{nm'} - 4\delta_{nm'})[-2i(RS)A_{mS} - 2i(QS)A_{mV} - \hat{S}A_{mV} + \hat{S}A_{mS}] = a_1\hat{S} + 2a_1 i(QS) + b_1\hat{S}(QR), \quad (15)$$

where

$$a_1 = -A_{mV}(A_{nm} - 4\delta_{nm}) + (A_{mV} - A_{mS})(A_{nm'} - 4\delta_{nm'}), \quad b_1 = 3A_{mS}(A_{nm} - 4\delta_{nm}). \quad (16)$$

Here we have used the equality of the momentum integrals with integrands $\hat{S}(QR)$ and $\hat{Q}(RS)$, and in some terms have made the interchange q ⇄ − r.

Substitution of Eq. (15) in Eq. (7) gives

$$\psi(p) = -\frac{l^4\varkappa^6}{3(2\pi)^8}G(p) \times \int \frac{d^4q d^4r}{N_1}[a\hat{S} + b\hat{S}(QR) + 2ai(QS)]\psi(p), \quad (17)$$

with

$$a = \sum_{m,n} C_mC_n[-A_{mV}(A_{nm} - 4\delta_{nm}) + (A_{mV} - A_{mS})(A_{nm'} - 4\delta_{nm'})] = \sum_m C_m[-B_mA_{mV} + B_{m'}(A_{mV} - A_{mS})], \quad (18a)$$

$$b = 3\sum_{m,n} C_mC_nA_{mS}(A_{nm} - 4\delta_{nm}) = 3\sum_m C_mB_mA_{mS}. \quad (18b)$$

As is clear from the derivation, the terms in δ_{nm} have come from the terms in Eq. (3) that contain traces. A characteristic peculiarity of Eq. (17) is the presence in it of only two (and not three) independent quadratic combinations of the coefficients C_n.

It is well-known that the way the Lagrangian is written in Eq. (1) is not the only possibility.[4] Interchange of two operators ψ in the expression under the sign of the normal product, followed by application of Eq. (6) brings the nonlinear term to the previous form, but with different values of the constants:

$$C'_k = -\frac{1}{4}\sum_n C_nA_{nk}. \quad (19)$$

A consequence of this is the fact that of the five possible invariants only three are linearly independent.

It can be verified that the expressions (18) for the quadratic forms a and b are invariant with respect to the transformation (19). This property is possessed by the linear combinations

$$B'_l = \sum_k C'_k(A_{kl} - 4\delta_{kl}) = -\frac{1}{4}\sum_{k,n} C_nA_{nk}(A_{kl} - 4\delta_{kl}) = \sum_n C_n(A_{nl} - 4\delta_{nl}) = B_l \quad (20)$$

(the orthogonality of the coefficients A_{nk} has been used). Writing Eq. (19) in the form

$$C'_k = -\frac{1}{4}B_k - C_k, \quad (19a)$$

we see that this transformation brings the quadratic form (18b) to the form

$$b' = -\frac{3}{4}\sum_m B_m^2A_{mS} - 3\sum_m C_mB_mA_{mS} = -\frac{3}{4}\sum_m B_m^2A_{mS} - b. \quad (21)$$

It can easily be shown that the first term in this formula is twice the second, so that the invariance b′ = b holds. Along with this, Eq. (21) makes it possible to give for b the obviously invariant expression

$$b = -\frac{3}{8}\sum_m B_m^2A_{mS} = -\frac{3}{8}(B_S^2 + 4B_V^2 + 6B_T^2 + 4B_A^2 + B_P). \quad (22)$$

The quadratic form (18a) can be handled in a similar way; for this purpose it is convenient to rewrite it in the form

$$a = \sum_m C_m(B_{m'} - B_m)(A_{mV} - A_{mS}) - b/3. \quad (23)$$

The first term of this expression is also changed by the transformation (19a) into two terms, one of which is twice as large as the other. We get finally

$$a = -\frac{1}{8}\sum_m B_m(B_{m'} - B_m)(A_{mV} - A_{mS}) - b/3$$
$$= \frac{1}{4}(3B_V B_V' + 3B_T B_T' + B_A B_A' + B_P B_P') - b/3 \quad (24)$$

Here

$$B_m' = B_{m'} - B_m. \quad (24a)$$

The values of the constants a and b for the pure interaction types are given in the table.

Interaction type	a	b	$(4\pi/\varkappa l)^2$
S	3	—9	2.897
V	—12	—72	5.078
T	0	—108	4.738
A	20	—72	7.644
P	—1	—9	1.665

The further treatment of Eq. (17) consists of the calculation of the momentum integrals. For this purpose let us introduce the following invariant functions of the momentum p (private communication from H. Mitter).

$$\int \frac{d^4q d^4r}{N_1}\hat{S} = -\frac{\pi^4}{\varkappa^2}\hat{p}M\left(-\frac{p^2}{\varkappa^2}\right),$$
$$\int \frac{d^4q d^4r}{N_1}\hat{S}(QR) = -\frac{\pi^4}{\varkappa^2}\hat{p}L\left(-\frac{p^2}{\varkappa^2}\right),$$
$$\int \frac{d^4q d^4r}{N_1}(QS) = -\frac{\pi^4}{\varkappa}N\left(-\frac{p^2}{\varkappa^2}\right) \quad (25)$$

With these and the relation $G(p) = \hat{p}/p^2$, we rewrite Eq. (17) in the form

$$\psi(p) = \frac{1}{3}\left(\frac{\varkappa l}{4\pi}\right)^4\left[aM\left(-\frac{p^2}{\varkappa^2}\right) + bL\left(-\frac{p^2}{\varkappa^2}\right)\right.$$
$$\left. + 2ai\varkappa\frac{\hat{p}}{p^2}N\left(-\frac{p^2}{\varkappa^2}\right)\right]\psi(p). \quad (26)$$

Squaring this equation, we bring it to the scalar form

$$\frac{1}{3}\left(\frac{\varkappa l}{4\pi}\right)^4\left[aM\left(-\frac{p^2}{\varkappa^2}\right) + bL\left(-\frac{p^2}{\varkappa^2}\right)\right.$$
$$\left.\pm 2a\sqrt{\frac{\varkappa^2}{-p^2}}N\left(-\frac{p^2}{\varkappa^2}\right)\right] = 1. \quad (27)$$

Since p is the momentum of the proton,

$$p^2 = -m_p^2, \quad (28)$$

and from Eq. (27) we find

$$\left(\frac{4\pi}{\varkappa l}\right)^4 = \frac{1}{3}\left[aM\left(\frac{m_p^2}{\varkappa^2}\right) + bL\left(\frac{m_p^2}{\varkappa^2}\right)\right.$$
$$\left.\pm 2a\frac{\varkappa}{m_p}N\left(\frac{m_p^2}{\varkappa^2}\right)\right]. \quad (29)$$

From this it is clear that the value of the constant $(4\pi/\kappa l)^4$ depends essentially on the choice of the quantity κ appearing in the propagation function. Values of the constants for $\kappa = m_p$ are shown in the table.

In conclusion I express my sincere gratitude to Prof. W. Heisenberg and Dr. H. Mitter for their interest in this work and for supplying numerical data omitted from their paper.[3]

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