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ACTION AS A SPACE COORDINATE. X.

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The difficulties encountered in five-optics in formulation of the spinor equations are surmounted in the present paper. It is shown that the requirement of invariance with respect to physically permissible transformations leads to correct spinor equations.

The requirement of physical admissibility separates subgroups of general transformations of the four-dimensional space-time and gauge transformations from the general transformation group of five-dimensional space. Restriction of the group of permissible transformations does in no means signify that the five-dimensional conception is not valid in principle. The most important feature of the five-dimensional theory is the periodic dependence (with period h) of the fields on the action coordinate. This fact and its consequences cannot be reduced to a simple unification of the four-coordinate point-transformation group with the gauge-transformation group. Another essentially five-dimensional effect is the existence of the scalar χ-field whose appearance in the theory of the field of a charged material particle yields formulas that differ from those of present-day gravitation theory.

1. INTRODUCTION

FIVE-dimensional optics,[1] the subject of this series of communications (1949—1953), begins with the observed deep symmetry of the laws of nature in three spatial coordinates, time, and action. This symmetry permits the relativistic problem (classical and quantum) of the motion of a charged material point in a specified gravitational and electromagnetic field to be formulated as a problem in five-dimensional optics (geometrical and wave), concerning the propagation of wave fields in the Riemann configuration space of coordinates, time, and action. The metric tensor of this space is

$$G_{\mu\nu} = \begin{pmatrix} g_{ik} + (1+\chi)\, g_i g_k & (1+\chi)\, g_k \\ (1+\chi)\, g_i & (1+\chi) \end{pmatrix}, \tag{1.1}$$

where g_{ik} are gravitational potentials, $g_i = (e/mc^2)A_i$ are electromagnetic potentials, and χ is a new scalar field that appears in five-dimensional optics.

The five-dimensional space of five-optics is topologically closed in the fifth action coordinate, and the period of the fifth coordinate is equal to Planck's constant h. This means that all the components of the metric tensor $G_{\mu\nu}$ are periodic functions of the fifth coordinate, with a period equal to h.

In the classical approximation $h \to 0$ the periodicity condition imposed on the field $G_{\mu\nu}$, degenerates into a condition that all $G_{\mu\nu}$ be independent of the fifth action coordinate, i.e., into the so-called cylindricity condition of the earlier five-dimensional theories (Kalutza-Klein-Fock).

It has been shown in the papers of this series that in the case of tensor particles (particles with integral spin) an account of the external gravitational and electromagnetic fields is made possible by a general covariant formulation of the equations of motion for a free particle. A connection was established between five-dimensional tensor analysis and the so-called D-formalism (replacement of the operators ∂_k by the operators $\partial_k - i\,(e/\hbar c)\,A_k$).

In the previous papers "general covariance of five-dimensional equations" meant covariance under general transformations of all five coordinates (x^i, $x^5 = S$)

$$x^{i'} = x^i + f^i(x^1, x^2, x^3, x^4, S),$$
$$S' = S + f(x^1, x^2, x^3, x^4, S) \tag{1.2}$$

with that substantial limitation, that only those functions f^i, f are admitted, which are periodic with respect to the coordinate S (with period h),

$$f^i(\ldots, S+h) = f^i(\ldots, S),$$
$$f(\ldots, S+h) = f(\ldots, S), \tag{1.3}$$

since only such transformations are admitted by the topological structure of five-dimensional space.

In the case of spinor particles (particles with half-integer spin), five-optics encountered a considerable difficulty, which was emphasized in the papers of this series and remained unsurmountable. Specifically, the general-covariant formulation of the equation for the free electron has led to an incorrect expression for the current-density vector

$$\bar{\psi}\gamma_k\psi + \frac{i}{8}\partial_l[\bar{\psi}(\gamma_k\gamma_l - \gamma_l\gamma_k)\psi], \qquad (1.4)$$

in which the second term differs from that in the correct expression. Further, in the presence of an external electromagnetic field, the following incorrect expression was obtained

$$\{\gamma_k(\partial_k - ieA_k) + m - (ie/8\ m)\gamma_k\gamma_l F_{kl}\}\psi = 0. \qquad (1.5)$$

This differs from the correct one by a term proportional to F_{kl}.

We shall show in the present communication where the error lies in the general-covariant formulation of the field equations, and shall derive correct equations for spinor particles.

2. PHYSICALLY-ADMISSIBLE TRANSFORMATIONS OF FIVE-DIMENSIONAL SPACE

Let us investigate whether all the transformations of group (1.2) are physically admissible. We start with the premise that if the metric potentials $G_{\mu\nu}$ do not depend on the fifth action coordinate, and consequently transitions of particles from one charged state into another with emission of charged quanta are forbidden, then this forbiddenness should be retained in any other physically admissible coordinate system.

This requirement is incompatible with the requirement of covariance with respect to the general transformations (1.2), and leads to a narrowing of this group to two subgroups: groups of point transformations of space-time, and groups of gauge transformations

$$x^{i'} = x^i + f^i(x^1, x^2, x^3, x^4), \quad S' = S + f(x^1, x^2, x^3, x^4). \quad (2.1)$$

The narrowing of the group of admissible transformation does not mean giving up five-dimensionality. The most important feature of the five-dimensional theory is the periodic dependence of the fields on the action coordinate, with period h. This fact and its consequences cannot be reduced to a simple unification of a group of point transformations of four coordinates with a group of gauge transformations.

An important five-dimensional effect is also the scalar χ-field, the appearance of which in the problem of the field of a charged material point leads to formulas that differ from the formulas of modern theory of gravitation.

Thus, the geometry of five-dimensional space of five-optics is Riemannian only in a limited sense. A two-dimensional model of such a space is a cylinder with a constant radius $h/2\pi$. The admissible transformations represent all possible extensions along the axis of the cylinder and "rotation of wheels". These lead to changes in the coordinate lines on the surface. Here, however, cross sections perpendicular to the axis cylinder transform into themselves, while the radius of the cylinder remains unchanged. Thus, Planck's constant h has an invariant geometrical meaning only for the subgroup of physically admissible transformations.

Many authors have noted that spinor fields can be introduced into Riemann space only with the aid of local systems of orthogonal references, which admit local rotations. A systematic reference-system analysis, which permits unique formulation of the equations for tensor and spinor fields, was proposed by the author.[2] The principal premises and formulas of the reference-system analysis are given in the appendix.

Let us now consider the physical limitations that are superimposed on the choice of local coordinate systems, i.e., on the rotation of four-dimensional references.

In the four-dimensional case the choice of one of the vectors of the reference is limited to the physical condition that the fourth vector of the reference must be time-like, thus restricting the permissible rotations of the reference. A transition from one reference to another one, connected with the initial orthogonal transformation, denotes physically the transition from one Einsteinian lift into another one, moving uniformly and in a straight line relative to the first one. The group of local rotations is none other than the local Lorentz group.

In special relativity theory the equations of the field are invariant under two groups of transformations: a) translation group, $x^{i'} = x^i + a^i$, with which the laws of energy and momentum conservation are related, b) Lorentz group $x^{i'} = \Sigma_k l_{ik} x^k$, with which are connected the laws of conservation of angular momentum and the symmetry of the energy-momentum tensor.

In general relativity theory, the roles of these two groups are assumed by the following: a) group of general transformations of coordinates, $x^{i'} = x^i + f^i(x^1, x^2, x^3, x^4)$ and b) local Lorentz group $\Omega^{i'}(k) = L(k,l)\,\Omega^i(l)$. Were the world to contain only tensor fields and no spinor ones, it would not be necessary to introduce the Lorentz local group, and only transformations of group a) need be considered.

The existence of spinor fields, the components of which transform in accordance with the representations of the local Lorentz groups, indicates the fundamental significance of this group also in general relativity theory.

In the case of five-dimensional space, one of the directions—the "direction" of action—is physically separate. While the space-time coordinates can be measured by physical instruments, the action coordinate cannot be measured by an instrument. The restriction to the group of admissible transformations (2.1) shows that the "direction" of action is conserved under all admissible point transformations. We can therefore choose for the fifth vector $\Omega^{\sigma}(5)$ a vector tangent to the coordinate line x^5, and admit only orthogonal transformations that leave the vector $\Omega^{\sigma}(5)$ invariant. Obviously, the group of these transformations is indeed the Lorentz group of local transformations, i.e., the transition from one Einstein lift to another.

We have formulated above a principle, by which the metric or the components of the reference are independent of the fifth coordinate in physically admissible coordinate system regardless of the choice of the system. It follows therefore that the coefficients of orthogonal transformations L (n, m) cannot depend on the fifth coordinate x^5 if this transformation is physically admissible.

In our earlier papers, the physical limitations on the group of point transformations and the group of reference relations were not considered. It was assumed that the equation should be covariant with respect to extensive groups of all point transformations (1.2) and five-dimensional rotations of the reference. This has led to an incorrect equation for spinor particles. We shall show that the restriction to groups of permissible transformations makes it possible to obtain a correct equation for the electron.

Let us construct, in accordance with the above, a reference field.

Let x^1, x^2, x^3, x^4 be space-time coordinates, which determine the localization of an event in space-time, and x^5 the action coordinate. Then the components of the metric five-tensor $G_{\mu\nu}$ are periodic functions of the coordinate x^5. To separate in the five-dimensional space the four-dimensional subspace (space + time) and the orthogonal linear action subspace, which is orthogonal to the other subspace, we begin to orthogonalize the metric tensor

$$G_{\mu\nu}=\begin{pmatrix} g_{ik}+(1+\chi)g_ig_k & (1+\chi)g_k \\ (1+\chi)g_i & 1+\chi \end{pmatrix}. \tag{2.2}$$

For this purpose we chose a unit vector $\Omega^{\sigma}(5)$ tangent to the action lines

$$\begin{aligned}&\Omega^{\sigma}(5)=(0,0,0,0,1/\sqrt{1+\chi}),\\ &\Omega_{\sigma}(5)=(\sqrt{1+\chi}\,g_1,\sqrt{1+\chi}\,g_2,\\ &\quad\sqrt{1+\chi}\,g_3,\sqrt{1+\chi}\,g_4,\sqrt{1+\chi}).\end{aligned} \tag{2.3}$$

We choose four unit vectors $\Omega_{\sigma}(n)$, (n = 1, 2, 3, 4), orthogonal to each other and orthogonal to the vector $\Omega^{\sigma}(5)$. In virtue of $\Omega^{\sigma}(5)\,\Omega_{\sigma}(n) = \delta(5, n)$, all the components of $\Omega_5(n)$ should vanish. Denoting

$$\Omega_{\sigma}(n)=\{\omega_i(n),0\}, \tag{2.4}$$

we have

$$\Omega_{\sigma}(\alpha)=\begin{pmatrix} \omega_1(1) & \omega_1(2) & \omega_1(3) & \omega_1(4) & \sqrt{1+\chi}\,g_1 \\ \omega_2(1) & \omega_2(2) & \omega_2(3) & \omega_2(4) & \sqrt{1+\chi}\,g_2 \\ \omega_3(1) & \omega^3(2) & \omega_3(3) & \omega_3(4) & \sqrt{1+\chi}\,g_3 \\ \omega_4(1) & \omega_4(2) & \omega_4(3) & \omega_4(4) & \sqrt{1+\chi}\,g_4 \\ 0 & 0 & 0 & 0 & \sqrt{1+\chi} \end{pmatrix}. \tag{2.5}$$

Calculation of the covariant components of the reference field yields

$$\Omega^{\sigma}(\alpha)=\begin{pmatrix} \omega^1(1) & \omega^1(2) & \omega^1(3) & \omega^1(4) & 0 \\ \omega^2(1) & \omega^2(2) & \omega^2(3) & \omega^2(4) & 0 \\ \omega^3(1) & \omega^2(2) & \omega^3(3) & \omega^3(4) & 0 \\ \omega^4(1) & \omega^4(2) & \omega^4(3) & \omega^4(4) & 0 \\ -\omega^i(1)\,g_i & -\omega^i(2)\,g_i & -\omega^i(3)\,g_i & -\omega^i(4)\,g_i & 1/\sqrt{1+\chi} \end{pmatrix} \tag{2.6}$$

Reference tensor analysis (see Appendix) introduces, along with ordinary derivatives ∂_{σ}, the reference derivatives $D(\alpha) = \Omega^{\sigma}(\alpha)\,\partial_{\sigma}$. We note that reference derivatives $D(\alpha)$, for the foregoing choice of the five-dimensional reference, have the form

$$D(k)=\omega^i(k)(\partial_i-g_i\partial_5),\qquad D(5)=(1+\chi)^{-1/2}\partial_5 \tag{2.7}$$

If we neglect the χ-field, confine ourselves to a harmonic dependence of the functions on the coordinate x^5, and replace the operators ∂_5 by im, then D(k) goes into the operator of generalized momentum of a particle in an electromagnetic field.

We shall henceforth consider only reference-component quantities since spinors have only reference components.

If we restrict ourselves to physically admissible transformations, the quantities that are invariants of point transformations and tensors of local rotations are

$$T(\alpha),\qquad D(5)\,T(\alpha),\qquad \nabla(\beta)\,T(\alpha), \tag{2.8}$$

where $T(\alpha)$ are the invariant components of the tensors, $D(\alpha)$ is the reference derivative, and $\nabla(\alpha)$ is the invariant derivative. Definitions of these quantities and their properties are given in the Appendix.

3. FIVE-DIMENSIONAL FIELD EQUATIONS

According to five-optics, allowance is made in particle wave equations for the external electromagnetic field by a general-covariant formulation (with the limitations indicated in Sec. 2) of the equations for the free particles in the Riemannian five-dimensional space. We agree from now on to distinguish between "five-dimensional" quantities or operators, which will be denoted by capital letters, and "four-dimensional" quantities or operators, denoted by lower-case letters (see Appendix for symbols).

Five-dimensional quantities:

$$G_{\mu\nu},\ \Omega_\mu(\alpha),\ G^{\mu}_{\nu\sigma},\ \Delta(\alpha,\beta,\gamma),\ \Lambda = \det|\Omega_\sigma(\alpha)| = \lambda\sqrt{1+\chi}.$$

Four-dimensional quantities:

$$g_{ik}, \omega_i(k),\ \Gamma^l_{ik},\ \delta(i,k,l),\ \lambda = \det|\omega_i(k)|.$$

Five-dimensional operators:

$$D(\alpha),\quad \nabla(\alpha),\quad B(\alpha).$$

Four-dimensional operators:

$$d(k),\quad \delta(k),\quad b(k).$$

Let us consider the five-dimensional Lagrange integral

$$J = \frac{1}{2i}\int d^5x\Lambda\,\{\widetilde{W}\mu(\alpha)(D(\alpha)W) - (D(\alpha)\widetilde{W})\mu(\alpha)W\}, \qquad (3.1)$$

where, according to (2.7)

$$D(k)W = (d(k)W - g(k)\partial_5 W),$$
$$D(k)\widetilde{W} = (d(k)\widetilde{W} - g(k)\partial_5\widetilde{W}),$$
$$D(5)W = (1+\chi)^{-1/2}\partial_5 W,\ D(5)\widetilde{W} = (1+\chi)^{-1/2}\partial_5\widetilde{W}.$$

The integral (3.1) can be written in the form

$$J = \frac{1}{2i}\int d^5x\cdot\lambda\sqrt{1+\chi}\,\{\widetilde{W}(d(k) - g(k)\partial_5)\mu(k)W$$
$$- [(d(k) - g(k)\partial_5)\widetilde{W}]\mu(k)W$$
$$+ \frac{1}{\sqrt{1+\chi}}(\widetilde{W}\mu(5)\partial_5 W - (\partial_5\widetilde{W})\mu(5)W)\}. \qquad (3.2)$$

The integral (3.2) is invariant under general admissible point coordinate transformations (2.1), but still does not admit of a free four-dimensional rotatability of the reference field in space-time. This free rotatability, however, appears immediately if we replace the reference derivatives in (3.2) by invariant derivatives:

$$d(k)W \to \delta(k)W = d(k)W - b(k)W,$$
$$d(k)\widetilde{W} \to \delta(k)\widetilde{W} = d(k)W + \widetilde{W}b(k). \qquad (3.3)$$

We obtain a new Lagrange integral $\widetilde{J}$, which differs from the integral J by the additional term

$$\widetilde{J} - J = -\frac{1}{2i}\int d^5x\cdot\lambda\sqrt{1+\chi}\,[\widetilde{W}(\mu(k)b(k) + b(k)\mu(k))W]. \qquad (3.4)$$

The Lagrangian $\widetilde{J}$ satisfies all the requirements of five-optic covariance, i.e., it remains invariant under transformations of the (local) Lorentz group and under transformations of groups (2.1).

If the metric fields are independent of the coordinate x^5, then the transitions from one charged state into another are forbidden, and the spinors W, $\widetilde{W}$ depend harmonically on the coordinate x^5.

$$W = U(x^1, x^2, x^3, x^4)\exp(imx^5),$$
$$\widetilde{W} = \widetilde{U}(x^1, x^2, x^3, x^4)\exp(-imx^5). \qquad (3.5)$$

Inserting (3.5) into (3.4) and integrating over x^5 we get

$$\widetilde{J} = \frac{1}{2i}\int d^4x\cdot\lambda\sqrt{1+\chi}\,\{\omega^l(k)\widetilde{U}\mu(k)(\partial_l U - img_l U)$$
$$+ \frac{2im}{\sqrt{1+\chi}}\widetilde{U}\mu(5)U - \omega^l(k)(\partial_l\widetilde{U} + img_l\widetilde{U})\mu(k)U$$
$$- \widetilde{U}(\mu(k)b(k) + b(k)\mu(k))U\}. \qquad (3.6)$$

We now introduce the notation

$$U = \psi,\quad i\widetilde{U}\mu(5) = \bar{\psi},\quad i\mu(k)\mu(5) = \gamma(k),$$
$$\sqrt{1+\chi}\,\omega^i(k) = \widetilde{\omega}^i(k)$$

and rewrite (3.6) in the form

$$\widetilde{J} = \frac{1}{2i}\int d^4x\cdot\lambda\,\{\widetilde{\omega}^i(k)\,[\bar{\psi}\gamma(k)(\partial_i\psi - img_i\psi)$$
$$- (\partial_i\bar{\psi} + img_i\bar{\psi})\gamma(k)\psi]$$
$$+ 2m\bar{\psi}\psi - \bar{\psi}(\gamma(k)b(k) + b(k)\gamma(k))\psi\}. \qquad (3.7)$$

We see that the external electromagnetic field is computed by formulating equations for the free electron, covariant with respect to groups of admissible transformations. The scalar χ-field, which is specific for five-optics, is calculated by replacing the true gravitational references $\omega^i(k)$ by the effective gravitational references $\widetilde{\omega}^i(k)$.

It was shown in the earlier communications that in five-optics the energy-momentum four-tensor and the current vector are combined into a common energy-momentum-charge five-tensor:

$$\theta^\sigma(\alpha) = \Lambda^{-1}\delta\widetilde{J}/\delta\Omega_\sigma(\alpha), \qquad (3.8)$$

where $\widetilde{J}$ is an invariant Lagrange integral

$$\widetilde{J} = \int d^5x\cdot\Lambda\mathscr{L}. \qquad (3.9)$$

It was also shown that the invariance of the Lagrangian $\mathscr{L}$ with respect to the group of local five-dimensional rotations leads to the symmetry of the five-tensor $\theta(\alpha,\beta) = \theta(\beta,\alpha)$.

We restrict ourselves to requiring invariance with respect to physically admissible rotations, i.e., the four-dimensional Lorentz group. Therefore the five-tensor of the energy-momentum-charge, generally speaking, will not be symmetrical. Only the energy-momentum four-tensor will be symmetrical, $\theta(m, n) = \theta(n, m)$ $(m, n = 1, 2, 3, 4)$, but in general $\theta(5, n) \neq \theta(n, 5)$. Thus in accordance with the general theory of relativity, we obtain a symmetrical energy-momentum tensor. In our previous papers we obtained an incorrect expression for the electron current vector, since the Lagrange integral was assumed invariant with respect to the five-dimensional rotation group. We obtain a correct expression by calculating the functional derivative of $\widetilde{J}$ with respect to $\Omega_i(5) = g_i\sqrt{1+\chi}$. From (3.6) we get

$$\Lambda\theta^i(5) = \delta\widetilde{J}/\delta\Omega_i(5)$$
$$= -\lambda m\widetilde{U}\mu(k)\,\omega^i(k)\,U = \lambda m\,(\bar{\psi}\gamma(k)\,\psi). \tag{3.10}$$

Taking (2.5) into account, we obtain

$$\theta_5^i = \Omega_5(\alpha)\,\theta^i(\alpha) = \Omega_5(5)\,\theta^i(5) = m\widetilde{\omega}^i(k)\,(\bar{\psi}\gamma(k)\,\psi), \tag{3.11}$$

i.e., a correct expression for the current vector.

We see that by giving up the extensive group of transformations (1.2) and restricting ourselves to the narrower transformation group (2.1), together with foregoing the free rotatability of the five-reference, we obtain correct equations for the spinor field in the presence of an external field. The absence of invariance relative to five-dimensional (local) Lorentz group and the general group of point transformations of all five coordinates is a manifestation of the absence of a principle of equivalence for the electromagnetic field. The absoluteness of the "direction" of the lines of action and "the relativity" of the directions of all the remaining coordinates lines indeed shows the impossibility of producing a coordinate system in which the electromagnetic field (in the small) would be "transformed."

Proceeding to tensor fields, we note that a particular case is possible when the requirement of covariance of the field equations with respect to the narrow group of transformations (2.1) involves automatically the covariance with respect to the total group (1.2). This case occurs for the meson field considered in the earlier communications.

For example, the equations for scalar and vector meson fields in five-dimensional covariant notation have the form

$$\partial_\lambda \sqrt{|G|}\,W^\lambda = \sqrt{|G|}\,Q, \quad \partial_\mu W_\lambda - \partial_\lambda W_\mu = 0, \tag{3.12}$$

$$\partial_\mu \sqrt{|G|}\,W^{\lambda\mu} = \sqrt{|G|}\,Q^\lambda,$$
$$\partial_\nu W_{\lambda\mu} + \partial_\lambda W_{\mu\nu} + \partial_\mu W_{\nu\lambda} = 0. \tag{3.13}$$

In these cases it makes no difference whether we use "five-dimensional" or "four-dimensional" covariant differentiation, inasmuch as the Christoffel symbols cancel each other out. The situation is analogous for equations for pseudo-vector and pseudoscalar meson fields.

Consequently, everything said in the preceding communications with respect to meson fields remains fully in force, the energy-momentum-charge five-tensor remains symmetrical, and the equations for the fields are correct.

APPENDIX

At each point of space we introduce a field of covariant reference $\Omega_i(n)$. We denote by $\Omega^i(n)$ the reciprocal reference, satisfying the condition

$$\Omega_i(n)\,\Omega^i(m) = \delta(n, m). \tag{A1}$$

If we express $\Omega^i(n)$ in terms of the contravariant components of the vector $\Omega_i(n)$, we introduce into the space a Riemannian matrix, whose metric tensor g_{ik} is determined by

$$g_{ik} = \Omega_i(n)\,\Omega_k(n) \tag{A2}$$

(we agree to drop the summation sign over the reference indices). The metric is invariant with respect to local orthogonal transformation of references:

$$\Omega^{i'}(n) = L(n, m)\,\Omega^i(m). \tag{A3}$$

Along with the covariant and contravariant tensor components we introduce invariant components, in accordance with the formula

$$T(k) = \Omega_i(k)T^i = \Omega^i(k)\,T_i. \tag{A4}$$

The quantities $T(n)$ are invariants of the group of point transformations and tensors of the local Lorentz group.

Along with the ordinary derivative ∂_k we introduce the reference derivative

$$D(n) = \Omega^k(n)\,\partial_k. \tag{A5}$$

Quantities of the type $D(n)\,T(m)$ are invariants of the group of point transformations, but are not the tensors of the local Lorentz group if the coefficients $L(n,m)$ depend on the coordinates. It is possible to introduce an invariant vector derivative defined by

$$\nabla(n)\,T(m) = \Omega^i(n)\,\Omega^k(m)\,\nabla_i T_k, \tag{A6}$$

where ∇_i is a symbol of covariant vector differentiation of a vector. This quantity is an invariant of point transformations and a tensor of second rank of the rotation group. Calculation yields

$$\nabla(n)T(m) = D(n)T(m) - \Delta(n, m, l)T(l), \quad \text{(A7)}$$

where $\Delta(n, m, l) = -\Delta(n, l, m)$ are called Ricci symbols and are expressed in terms of components of the reference $\Omega^i(n)$ in the following manner:

$$\Delta(n, m, l) = {}^1/_2\{(\Omega^s(m)\Omega^r(l) - \Omega^s(l)\Omega^r(m))\partial_s\Omega_r(n)$$
$$+ (\Omega^s(l)\Omega^r(n) - \Omega^s(n)\Omega^r(l))\partial_s\Omega_r(m)$$
$$- (\Omega^s(n)\Omega^r(m) - \Omega^s(m)\Omega^r(n))\partial_s\Omega_r(l)\}. \quad \text{(A8)}$$

In reference analysis, the $\Delta(n, m, l)$ play a role analogous to the Christoffel symbols in tensor analysis.

The spinors ψ and $\bar{\psi}$ are defined as quantities having the following transformation properties:

a) They are invariant under general point coordinate transformations

$$\psi'(x') = \psi(x), \quad \bar{\psi}'(x') = \bar{\psi}(x). \quad \text{(A9)}$$

b) Under free rotations of the reference $\Omega^{i\prime}(n) = L(m, n)\,\Omega^i(m)$ they transform in accordance with

$$\psi' = S\psi, \quad \bar{\psi}' = \bar{\psi}S^{-1}, \quad \text{(A10)}$$

where the matrices S are connected with the matrix $\|L(m, n)\|$ by the ordinary relation

$$S^{-1}\gamma(n)S = L(n, m)\gamma(m), \quad \text{(A11)}$$

where $\gamma(k)$ are four matrices that satisfy the conditions

$$\gamma(n)\gamma(m) + \gamma(m)\gamma(n) = 2\delta(n, m).$$

From the laws of spinor transformations it follows that the quantities $\bar{\psi}\gamma(k)\psi$ form a four-vector with reference components. The components of the spinor are analogues of the reference components of the tensors with that substantial difference, that for spinors there exist no contravariant or covariant components, and that it is therefore impossible to introduce spinors in Riemann spaces, retaining the introduction of the metric with the aid of the tensor field g_{ik}.

The reference derivative of a spinor is determined by the formula

$$D(k)\psi = \Omega^i(k)\partial_i\psi, \quad \text{(A12)}$$

which remains invariant under point transformation of the coordinates.

The invariant derivative of a spinor is introduced by the formula

$$\nabla(k)\psi = D(k)\psi - B(k)\psi,$$
$$\nabla(k)\psi = D(k)\bar{\psi} + \bar{\psi}B(k), \quad \text{(A13)}$$

and the matrices B (k) are defined such as to obtain the correct expression for the invariant derivative of the vector.

We have

$$\nabla(k)(\bar{\psi}\gamma(n)\psi) = D(k)(\bar{\psi}\gamma(n)\psi)$$
$$- \Delta(k, n, m)(\bar{\psi}\gamma(m)\psi), \quad \text{(A14)}$$

from which it follows that

$$B(k) = {}^1/_4\,\Delta(k, n, m)\gamma(n)\gamma(m). \quad \text{(A15)}$$

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[1] Yu. B. Rumer. Исследования по 5-оптике, (Investigations in 5-Optics), GTTI, 1956. This monograph contains all the previous articles of this series, which have been revised and supplemented.

[2] Yu. B. Rumer, J. Exptl. Theoret. Phys. (U.S.S.R.) **25**, 271 (1953).

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