# THE RADIATION REACTION IN THE MOTION OF A CHARGE IN A MEDIUM

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The radiation reaction force is computed for a charge that moves in a medium which, for generality, is taken to be anisotropic and gyrotropic. The radiation force in the medium can be important in cases in which the particle moves in a magneto-active plasma, in channels and slits in dielectrics, or in waveguides. At velocities greater than the phase velocity of light in the medium, the radiation force that affects the oscillation because of the anomalous Doppler effect has a different sign than that due to dissipation associated with the normal Doppler effect. The total radiation force which affects the amplitude of the oscillations of the particle in an isotropic medium corresponds to dissipation for motion at velocities greater than the velocity of light. However this dissipation force may be appreciably smaller than the dissipation associated with motion at velocities smaller than the velocity of light. In an isotropic medium the oscillations can be strengthened instead of attenuated. The reduction in the radiative dissipation force may be related to the peculiarities of the anomalous Doppler effect as found in the quantum-mechanical analysis and the instability of particle beams which move at velocities greater than the velocity of light.

SINCE the presence of a medium has a pronounced effect on the nature of the electromagnetic waves produced by a moving particle it is clear that the radiation reaction forces in the medium are affected by the medium. As an example we may note that an oscillator of frequency $\omega$ in an isotropic plasma with a refractive index $n^2 = 1 - 4\pi e^2 N / m\omega^2$ will not radiate when $4\pi e^2 N/m > \omega^2$ if $n^2 < 0$; in a magneto-active plasma in the nonrelativistic approximation no radiation is produced by an electron which rotates in a magnetic field $H_0$ at a frequency $\omega_H = eH_0/mc$ (cf. reference 1). In both of these cases the radiation force is obviously zero; in vacuum this force is $f_0 = (2e^2/3c^3)\ddot{v}$. On the other hand, in uniform motion in a medium, if $v > c/n(\omega)$, Cerenkov radiation occurs; in the isotropic case the radiation force associated with this radiation is[2]

$$ f_{Cer} = -\frac{e^2 V}{c^2 v} \int_{c/n \leqslant v} (1 - c^2/v^2 n^2(\omega)) \omega d\omega. \qquad (1) $$

In light of the above alone, it is of interest to compute the radiation reaction forces for arbitrary motion of a charge in an arbitrary medium. To the best of our knowledge, this problem has not yet been investigated.

The apparent reason for this is that the radiation force for motion in a medium is usually much smaller than the braking forces associated with ionization losses. Thus, the Cerenkov radiation losses, which may be considered radiation losses, amount to only a small fraction of the total losses even in a transparent but dense medium. In the case of nonuniform motion of a charge the situation is generally the same.

There are, however, cases of practical interest in which it is important to take account of the radiation forces for motion in a medium. This situation arises in the motion of particles in channels or slits in a medium, motion in a vacuum close to a medium, and motion in waveguides. If the channel is narrow the radiation produced is essentially the same as the radiation in a continuous medium.[3,4] Thus, motion in channels or slits in a medium can in most cases be treated in the same way as motion through a medium (the channel is essentially a mechanism for avoiding the ionization losses). The radiation forces can also be important for motion of a charge in a rarefied plasma (in particular, in a magneto-active plasma).

The problem of radiation forces in a medium acquires a special interest at velocities greater than the velocity of light. For these cases the radiation force which affects the amplitude of the particle oscillations can be very small and even zero [this is due to the nature of the anomalous Doppler effect (cf. references 5 and 6)]. We

shall pay special attention to this point since it is intimately related to the problem of the stability of beams which travel faster than the velocity of light.

Before dealing with these problems, we compute the radiation reaction force in a medium for the general case.

1. To compute the radiation reaction force we first assume that the charge is not a point charge but is characterized by a density distribution $\rho = eg(\mathbf{r}-\mathbf{R})$, where $\mathbf{R}(t)$ is the radius vector to the center of gravity of the charge and $\int g(\mathbf{r}-\mathbf{R})\,d\mathbf{r} = 1$. Under these conditions the field equations and the equations of motion assume the form

$$\operatorname{curl}\mathbf{H} = \frac{4\pi}{c}ev g(\mathbf{r}-\mathbf{R}) + \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}, \quad \operatorname{div}\mathbf{D} = 4\pi eg(\mathbf{r}-\mathbf{R}),$$

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \quad \operatorname{div}\mathbf{H} = 0; \qquad (2)$$

$$\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}\right) = e\left\{\mathbf{E}^{(0)} + \frac{1}{c}\mathbf{v}\times\mathbf{H}^{(0)}\right\}$$

$$+ e\int\left\{\mathbf{E}(\mathbf{r}) + \frac{1}{c}\mathbf{v}\times\mathbf{H}(\mathbf{r})\right\}g(\mathbf{r}-\mathbf{R})\,d\mathbf{r}, \qquad (3)$$

where the charge is assumed to move as a whole with velocity $\mathbf{v} = \dot{\mathbf{R}}$; $\mathbf{H}^{(0)}$ and $\mathbf{E}^{(0)}$ are the external fields, $\mathbf{E}$ and $\mathbf{H}$ are the fields produced by the charge itself and for simplicity the medium is assumed to have a magnetic permeability of unity ($\mu = 1$).

Having in mind applications to a magneto-active plasma we shall assume that the medium is anisotropic and gyrotropic, writing

$$\tilde{D}_\alpha^{(\omega)} = \varepsilon_{\alpha\beta}(\omega)\tilde{E}_\beta(\omega), \quad \alpha, \beta = 1, 2, 3, \qquad (4)$$

where the double subscripts indicate summation and the symbol $\omega$ indicates that the Fourier components are taken and the real fields are $\mathbf{D} = \tilde{\mathbf{D}} + \tilde{\mathbf{D}}^*$ and so on. The tensor $\epsilon_{\alpha\beta}(\omega)$ is assumed to be Hermitian ($\epsilon_{\alpha\beta} = \epsilon_{\beta\alpha}^*$), i.e., absorption is neglected; in principle it is possible to remove this restriction.

For an arbitrary medium the only effective method for solving the problem is expansion of the fields into normal plane waves.[7-9] Thus, we write

$$\mathbf{A} = \tilde{\mathbf{A}} + \tilde{\mathbf{A}}^*, \quad \tilde{\mathbf{A}} = \sqrt{4\pi c}\sum_{\lambda,\,j=1,2}\frac{q_{\lambda j}(t)\,\mathbf{a}_{\lambda j}}{n_{\lambda j}}e^{i\mathbf{k}_\lambda\mathbf{r}},$$

$$\mathbf{E} = \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\varphi, \quad \mathbf{H} = \operatorname{curl}\mathbf{A}, \qquad (5)$$

where $n_{\lambda j}$ is the refractive index and $\mathbf{a}_{\lambda j} = \mathbf{a}_{\lambda j1} + i\mathbf{a}_{\lambda j2}$ $(j = 1, 2)$ is the complex polarization vector which corresponds to the normal (extraordinary or ordinary) wave; the summation over $\mathbf{k}_\lambda$ in (5) is taken over the upper hemisphere in $\mathbf{k}$-space.

In order to simplify the relations it is convenient to impose the following condition on $\mathbf{A}$

$$\varepsilon_{\alpha\beta}\partial\tilde{A}_\alpha/\partial x_\beta + \text{c.c.} = 0.$$

The equations for $\mathbf{A}$ and $\varphi$ which follow from (2) and (6) are as follows:

$$\Delta\tilde{\mathbf{A}} - \operatorname{grad}\operatorname{div}\tilde{\mathbf{A}} - \frac{1}{c^2}\varepsilon_{\alpha\beta}\frac{\partial^2\tilde{A}_\beta}{\partial t^2}\mathbf{e}_\alpha - \frac{1}{c}\varepsilon_{\alpha\beta}\frac{\partial^2\tilde{\varphi}}{\partial t\partial x_\beta}\mathbf{e}_\alpha + \text{c.c.}$$

$$= -\frac{4\pi}{c}ev g(\mathbf{r}-\mathbf{R}),$$

$$\varepsilon_{\alpha\beta}\frac{\partial^2\tilde{\varphi}}{\partial x_\alpha\partial x_\beta} + \text{c.c.} = -4\pi eg(\mathbf{r}-\mathbf{R}), \qquad (7)$$

where $\mathbf{e}_\alpha$ is a unit vector along the $\alpha$ axis.

In the normalization of the potentials which has been chosen the part of the field $\mathbf{E}$ which is equal to $-\nabla\varphi$ makes no contribution to the radiation reaction and will not be considered below.

Substituting Eq. (5) in the expansion in (7), after multiplication by $\sqrt{4\pi c}\,(\mathbf{a}_{\lambda j}^*/n_{\lambda j})e^{-i\mathbf{k}_\lambda\mathbf{r}}$ and integration over the volume we have

$$\ddot{q}_{\lambda j} + \omega_{\lambda j}^2 q_{\lambda j}$$

$$= \sqrt{4\pi}e\int g(\mathbf{r}'-\mathbf{R})\frac{(\mathbf{va}_{\lambda j}^*)}{n_{\lambda j}}\exp(-i\mathbf{k}_\lambda\mathbf{r}')\,d\mathbf{r}' = f(t), \qquad (8)$$

where $\omega_{\lambda j}^2 = k_\lambda^2 c^2/n_{\lambda j}^2$ (this calculation is considered in greater detail in references 7 and 8). It should be noted that in using the quantity $n_{\lambda j}$ in Eq. (5) and below, because this quantity is a function of $\omega$ we are not being consistent since the time dependence of $q_{\lambda j}(t)$ is still not known. Actually, however, in the calculation of the radiated energy for the radiation reaction force the quantity $q_{\lambda j}$ refers to the radiation field and the entire calculation can be carried out assuming at the outset that $n_{\lambda j}$ is constant and by taking account of the dependence of $n_{\lambda j}$ on $\omega$ in the final result before integration over frequency (cf. reference 7).

It follows from Eq. (8) that

$$q_{\lambda j}(t) = q_{\lambda j}(0) + \frac{1}{\omega_{\lambda j}}\int_0^t f(t')\sin\omega_{\lambda j}(t-t')\,dt',$$

$$\tilde{\mathbf{A}} = 4\pi ce\sum_{\lambda j}\frac{\mathbf{a}_{\lambda j}}{n_{\lambda j}\omega_{\lambda j}}e^{i\mathbf{k}_\lambda\mathbf{r}}$$

$$\times\int_{\mathbf{r}'}\int_0^t g(\mathbf{r}'-\mathbf{R}')(\dot{\mathbf{R}}'\mathbf{a}_{\lambda j}^*)e^{-i\mathbf{k}_\lambda\mathbf{r}'}\sin\omega_{\lambda j}(t-t')\,dt'd\mathbf{r}' + \tilde{\mathbf{A}}(0), \qquad (9)$$

where $\mathbf{R}' = \mathbf{R}(t')$ and $\dot{\mathbf{R}}' = \dot{\mathbf{R}}(t') = \mathbf{v}'(t')$. We neglect the free solution $\mathbf{A}(0)$ below. Computing $\mathbf{E}$ and $\mathbf{H}$ from Eq. (9) and substituting in Eq. (3) we have

$$\frac{d}{dt}\frac{mv}{\sqrt{1-v^2/c^2}} = F^{(0)} - \frac{e^2}{2\pi^2}\sum_{j=1,2}\int g(r-R)g(r'-R')e^{ik(r-r')}$$

$$\times\left\{\frac{a_j(v'a_j^*)}{n_j^2}\cos[\omega_j(t-t')] - i\, v\times(k\times a_j)\right.$$

$$\left.\times\frac{v'a_j^*}{n_j^2\omega_j}\sin[\omega_j(t-t')]\right\}dt'dk dr'dr + \text{c.c.}, \qquad (10)$$

where

$$F^{(0)} = e\{E^{(0)} + c^{-1} v\times H^{(0)}\},$$

and as usual we carry out the integration over $k$ (density of states $dk_\lambda/8\pi^3$). The second term in the curly brackets in Eq. (10) is to be associated with the magnetic radiation reaction force; in vacuum in the nonrelativistic approximation this term is $v/c$ times smaller than the first and is neglected. In the presence of a medium the magnetic field $H \sim nE$ [for a plane wave $\sim e^{i\omega t - ikr}$ we have $k\times E = \omega H/c$, $k^2 = (\omega n/c)^2$] and the magnetic reaction force differs from the electric reaction force by a factor of approximately $vn/c$. For motion at velocities greater than that of light, in which we shall be especially interested, $vn/c > 1$; thus, the magnetic force cannot be neglected even if $v/c \ll 1$. Obviously the magnetic force does no work.

Below we shall compute only that part of the reaction force which does not depend on the dimensions of the particle. Hence, till we drop the term containing the electromagnetic mass it is convenient to take the form factor $g(r-R)$ as a $\delta$-function, $\delta(r-R)$, cutting off the integration over $k$ at $k_{max} = \omega_{max}/c \sim 2\pi/r_0$, where $r_0$ is the radius of the particle. We write Eq. (10) in the form

$$\frac{d}{dt}\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right) = F^{(0)} - \frac{e^2}{2\pi^2}\sum_{j=1,2}\int_0^t\int_0^{k_{max}}\left\{\frac{a_j(v'a_j^*)}{n_j^2}\cos\omega_j(t-t')\right.$$

$$\left. - i\, v\times(k\times a_j)\frac{va_j^*}{n_j^2\omega_j}\sin\omega_j(t-t')\right\}e^{ik(R-R')}dt'dk$$

$$+ \text{c.c.} = F^{(0)} + f_r. \qquad (11)$$

The method of computing the radiation reaction forces which is being used here is convenient either in an isotropic medium or in vacuum (cf. references 10 and 11). To verify this statement and to check the results we consider the non-relativistic motion of a particle in a vacuum. In this case, as the $a_j$ we take the real unit vectors which are perpendicular to $k$ and to each other; one of these, which lies in the plane defined by the vectors $k$ and $v'$, will be denoted simply by $a$. The second term in the integral in Eq. (11) is neglected since

it is of order $v/c$. Introducing polar coordinates in $k$-space ($k^2 = \omega^2/c^2$) we have:

$$m\frac{dv}{dt} = F^{(0)} - \frac{e^2}{\pi^2 c^3}\int_0^t\int_0^{\omega_{max}}\int_\Omega a(v(t')a)\cos\omega(t-t')$$

$$\times \cos(kR(t) - R(t'))\omega^2 d\omega d\Omega dt'. \qquad (12)$$

Writing

$$\cos\omega(t-t') \equiv -\frac{1}{\omega}\frac{d}{dt'}[\sin\omega(t-t')]$$

and integrating by parts over $t'$ we transform the integral in Eq. (12) to the form

$$\int\frac{d}{dt'}\{a(v'a)\cos k(R-R')\}\sin\omega(t-t)\omega d\omega d\Omega dt',$$

where we have dropped the term that depends on the initial conditions and exists only when $t \lesssim 1/\omega_{max}$ (this is considered in greater detail in references 10 and 11). Carrying out another integration by parts and integrating with respect to $d\Omega = \sin\theta\, d\theta\, d\varphi$, we have

$$m\frac{dv}{dt} = F^{(0)} - \frac{4e^2}{3\pi}\frac{\omega_{max}}{c^3}\frac{dv}{dt} + \frac{2e^2}{\pi c^3}\int_0^t\int_0^{\omega_{max}}\frac{d^2a(va')\cos k(R-R')}{dt'^2}$$

$$\times\cos\omega(t-t')dt'd\omega\sin\theta d\theta. \qquad (12a)$$

The second term on the right side of Eq. (12a) is proportional to the electromagnetic mass $4e^2\omega_{max}/3\pi c^3 \sim e^2/r_0 c^2$ and must be discarded since by $m$ we understand the experimental value of the mass (this procedure, which has been used for many years, is identical with mass renormalization, which is sometimes assumed to be of recent origin). The third term on the right side of Eq. (12a), if one neglects quantities which approach zero as $\omega_{max} \to \infty$ (i.e., $r_0 \to 0$), is the well-known expression $(2e^2/3c^3)\dddot{R}$, since in the nonrelativistic case $\cos k(R-R')$ varies much slower with $t'$ than $\cos\omega(t-t')$ while

$$\lim_{\omega_{max}\to\infty}\int_0^t\int_0^{\omega_{max}}\varphi(t')\cos\omega(t-t')dt'd\omega = \frac{1}{2}\pi\varphi(t).$$

As a second example we consider uniform motion ($R = vt$) in an isotropic medium with velocity $v > c/n$. The polarization vector $a$ may be conveniently chosen as in the preceding case and the magnetic force vanishes identically as follows immediately from symmetry considerations. As a result we obtain the reaction force

$$f_r = f_{Cer} = -\frac{2e^2}{\pi c^3}\int_0^t\int_0^{\omega_{max}}\int_0^{\pi/2} a(va)\cos\omega(t-t')$$

$$\times\cos kv(t-t')n\omega^2\sin\theta dt'd\omega d\theta.$$

Integration over $t'$ leads to the appearance of two terms, one of which contains the factor

$$\frac{\sin \omega \,[1 - (nv'/c)\cos\theta]\,t}{1 - (nv'/c)\cos\theta} \to \pi\delta\left(1 - \frac{nv}{c}\cos\theta\right) \quad \text{for } t \to \infty.$$

Since $\cos\theta = c/nv$, we have Eq. (1), since the term which does not contain the $\delta$-function does not make any real contribution in the expression for the force.

In the general case of accelerated motion in an arbitrary medium the reaction force $\mathbf{f_r}$ is determined by Eq. (11). Since $\mathbf{f_r}$ is small, in the calculation the quantities $\mathbf{R}$ and $\mathbf{v} = \dot{\mathbf{R}}$ can be the functions which correspond to the motion of the particle with the radiation reaction neglected.

**2.** We consider motion of an oscillator at velocities greater than that of light in an isotropic medium, a case which is of interest in connection with microwave radiation.[12]

Choosing the velocity of the reciprocatory motion $\mathbf{v_0}$ along the $z$ axis, we start with an oscillator which oscillates parallel to $\mathbf{v_0}$. In this case the vector quantities in the problem are as follows:

$$\mathbf{R} = \{0,\ 0,\ v_0 t + R_0 \sin\Omega t\}, \quad \mathbf{v} = \{0,\ 0,\ v_0 + v_a \cos\Omega t\},$$

$$v_a = R_0\Omega, \quad \mathbf{a_1} = \{1,\ 0,\ 0\}, \quad \mathbf{a_2} = \{0,\ \cos\theta,\ -\sin\theta\},$$

$$\mathbf{k} = \{0,\ k\sin\theta,\ k\cos\theta\}. \tag{13}$$

Then, from Eq. (11), for the force $\mathbf{f_{rz}}$ we have

$$f_{rz} = -\frac{e^2}{2\pi^2}\int_0^t\int_0^{k_{max}} \frac{\sin^2\theta}{n^2}(v_0 + v_a\cos\Omega t')\cos\omega(t - t')$$

$$\times \exp\{ik\cos\theta\,[v_0(t - t')$$

$$+ R_0(\sin\Omega t - \sin\Omega t')]\}\,dt'dk, \tag{14}$$

where we have neglected the complex conjugate expression which is implied wherever complex quantities are used. At the outset we shall not assume that the oscillation amplitude is small; in Eq. (14) we use the well-known expansion

$$\exp(ik\cos\theta R_0\sin\Omega t) = \sum_{s=-\infty}^{+\infty} J_s(kR_0\cos\theta)\,e^{is\Omega t}. \tag{15}$$

As a result we obtain

$$f_{rz} = -\frac{e^2}{2\pi^2}\sum_{s,\,s'=-\infty}^{\infty}\int_0^t\int_0^{k_{max}}\frac{\sin^2\theta}{n^2}\cos\omega(t - t')\,J_s G_{s'}$$

$$\times \exp\{i\,[kv_0\cos\theta\cdot(t - t') + \Omega(st - s't')]\}\,dkdt',$$

$$G_{s'}(x) = J_{s'}(x)(v_0 + v_a s'/x). \tag{16}$$

Integrating over $t'$ and keeping only the term which corresponds to the $\delta(y)$-function [$\lim(\sin\alpha y/\pi y)$ for $\alpha \to \infty$] we have

$$f_{rz} = -\frac{e^2}{4\pi}\sum_{ss'}\int \frac{\sin^2\theta}{n^2}J_s(kR_0\cos\theta)\,G_{s'}(kR_0\cos\theta)$$

$$\times e^{i\Omega(s-s')t}\delta(y)\,dk,$$

$$f_{rx} = f_{ry} = 0, \quad y = \omega - s'\Omega - kv_0\cos\theta, \quad k = \omega n/c. \tag{17}$$

The work of the radiation reaction force on the particle is

$$A = \int_0^T vf_{rz}\,dt = v_0\int_0^T f_{rz}\,dt + v_a\int_0^T \cos\Omega t\,f_{rz}\,dt = A_0 + A_a, \tag{18}$$

where $T = 2\pi l/\Omega$, $l = 1, 2, 3, 4, \ldots$, i.e., $T$ is equal to a whole number of periods.

For high values of $T$, only those terms for which $s - s' = 0$ and $s - s' = \pm 1$ make a contribution to the integrals in Eq. (18). Thus, since

$$J_{s'-1}(x) + J_{s'+1}(x) = (2s'/x)J_{s'}(x),$$

we have

$$A = -\frac{e^2 T}{c^3}\sum_{s=-\infty}^{\infty}\int \sin^2\theta\, G_s^2(kR_0\cos\theta)\,n(\omega)\sin\theta\omega^2\delta(y)\,d\theta d\omega. \tag{19}$$

The term with $s = 0$ yields the Cerenkov radiation and will not be considered below (the usual formula for the intensity of the Cerenkov radiation[2] is obtained from Eq. (19) when $s = 0$ if the quantity $kR_0\cos\theta$, which in this case is $\omega R_0/v_0$, is much smaller than unity). For an oscillator characterized by small amplitudes, where

$$kR_0 = (\omega/c)n(\omega)R_0 \ll 1, \tag{20}$$

we need consider only those terms for which $s = \pm 1$.

It is apparent that when (20) is satisfied

$$G_{\pm 1}^2(kR_0\cos\theta) = \tfrac{1}{4}(v_0 \pm v_a/kR_0\cos\theta)^2(kR_0\cos\theta)^2$$

$$= v_a^2/4(1 - \beta_0 n\cos\theta)^2,$$

since the vanishing of the argument of the $\delta$-function yields the Doppler condition

$$\omega = \Omega/|1 - \beta_0 n(\omega)\cos\theta|, \quad \beta_0 = v_0/c. \tag{21}$$

In this case the anomalous Doppler effect, ($\beta n\cos\theta > 1$), corresponds to $s = -1$, while the normal effect ($\beta n\cos\theta < 1$) corresponds to $s = +1$. Integrating over $\theta$ we have ($A = A^+ + A^-$)

$$A_{s=1} \equiv A^+$$

$$= -\frac{e^2 R_0^2 T}{4c^3\beta_0}\int_{\beta_0 n(\omega)\cos\theta < 1}\omega^3\left\{1 - \frac{1}{\beta_0^2 n^2(\omega)}\left(1 - \frac{\Omega}{\omega}\right)^2\right\}d\omega,$$

$$A_{s=-1} \equiv A^-$$

$$= -\frac{e^2 R_0^2 T}{4c^3\beta_0}\int_{\beta_0 n(\omega)\cos\theta > 1}\omega^3\left\{1 - \frac{1}{\beta_0^2 n^2(\omega)}\left(1 + \frac{\Omega}{\omega}\right)^2\right\}d\omega. \tag{22}$$

where $\cos\theta$ is expressed in terms of $\omega$ by means of Eq. (21). It is apparent that this result can also be obtained directly from Eq. (14) if the condition in (20) is used at the beginning.

From Eq. (22), or, directly from Eq. (19), when (20) applies, with $n = \text{const}$ we have

$$A^+ = -\frac{e^2\Omega^4 R_0^2 nT}{4c^3} \int_{\theta>\theta_0} \frac{\sin^3\theta\, d\theta}{(1-\beta_0 n\cos\theta)^5},$$

$$A^- = -\frac{e^2\Omega^4 R_0^2 nT}{4c^3} \int_{\theta<\theta_0} \frac{\sin^3\theta\, d\theta}{(\beta_0 n\cos\theta - 1)^5}, \qquad (23)$$

where $\theta_0$ is the Cerenkov angle ($\cos\theta_0 = c/n v_0$).

In a number of cases it is convenient to approximate the function $n(\omega)$ by a step function (motion in channels, radiation from bunches, cf. reference 6): $n(\omega) = n$ when $\omega < \omega_c$ and $n(\omega) = 1$ when $\omega > \omega_c$. Under such conditions, as before, the relations in (23) are used but for the radiation frequency $\omega < \omega_c$ in $A^+$ the limits of integration are $\cos^{-1}[(1-\Omega/\omega_c)/\beta_0 n]$ and $\pi$ while in $A^-$ the limits of integration are $0$ and $\cos^{-1}[(1+\Omega/\omega c)/\beta_0 n]$.

The quantity $W^\pm = -A^\pm$ is the energy radiated inside the Cerenkov cone ($A^-$) and outside the cone ($A^+$). In this case Eq. (23) coincides with that given by Frank.[13]

We now divide the work $A^\pm$ into the parts $A_0^\pm$ and $A_a^\pm$, where $A_a^\pm$ corresponds to the work associated with the attenuation or amplification of the particle oscillations [cf. Eq. (18)]. It is apparent that $A_0^\pm$ is always negative whereas

$$A_a^+ = -\frac{e^2\Omega R_0^2 T}{4c^3\beta_0} \int_{\beta_0 n\cos\theta<1} \omega^2\left\{1-\frac{1}{\beta_0^2 n^2}\left(1-\frac{\Omega}{\omega}\right)^2\right\} d\omega, \qquad (24a)$$

$$A_a^- = \frac{e^2\Omega R_0^2 T}{4c^3\beta_0} \int_{\beta_0 n\cos\theta>1} \omega^2\left\{1-\frac{1}{\beta_0^2 n^2}\left(1+\frac{\Omega}{\omega}\right)^2\right\} d\omega \qquad (24b)$$

(we may recall that $A_a = A_a^+ + A_a^-$ and the total work $A = A_0 + A_a$, where $A_0 = A_0^+ + A_0^-$). For $n = \text{const}$ or a "step"

$$A_a^\pm = \mp (e^2\Omega^4 R_0^2 nT / 4c^3) \int (1-\beta_0 n\cos\theta)^{-4}\sin^3\theta\, d\theta. \qquad (25)$$

Thus, the radiation which appears in the region outside the Cerenkov cone leads to a decay of oscillations ($A_a^+ < 0$) whereas the radiation inside the cone corresponds to the anomalous Doppler effect and enhancement of the oscillations ($A_a^- > 0$). This result is in complete agreement with that obtained from quantum mechanical considerations.[5,6]

One is easily convinced that the limits of integration over $\omega$ in Eq. (24a) are further apart than in Eq. (24b); for example, with $n = \text{const}$, the limits in Eq. (24a) are $\infty$, $\Omega/(1+\beta_0 n)$ while in Eq. (24b)

these limits are $\infty$, $\Omega/(\beta_0 n - 1)$. The integrand in Eq. (24a) is also larger than that in Eq. (24b). For this reason it is always true that $|A_a^+| \geq A_a^-$ where $|A_a^+| = A_a^-$, if in the actual region of integration $\beta_0 n(\omega) \to \infty$. Thus, at greater-than-light velocities the damping of the oscillations of the oscillator is weakened and can even almost vanish although amplification can never occur.

We now consider an oscillator that vibrates perpendicularly to the velocity $\mathbf{v}_0$; in this case

$$\mathbf{R} = \{R_0\sin\Omega t, 0, v_0 t\}, \quad \mathbf{v} = \{v_a\cos\Omega t, 0, v_0\},$$

$$\mathbf{a}_1 = \{\sin\varphi, -\cos\varphi, 0\},$$

$$\mathbf{a}_2 = \{\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta\};$$

$$\mathbf{k} = \{k\sin\theta\cos\varphi, k\sin\theta\sin\varphi, k\cos\theta\}, \qquad (26)$$

where the polar axis (z) has again been chosen in the direction of the velocity $\mathbf{v}_0$. Proceeding as above we obtain

$$f_{rz} = -\frac{e^2}{4\pi} \sum_{s,\,s'=-\infty}^{\infty} \int J_s(\zeta)\, G_{s'}'(\zeta)$$

$$\times \left\{\sin\theta - \frac{v_a n}{c}[\cos\Omega t(\cos\theta\sin\varphi + \cos\varphi)]\right\}$$

$$\times e^{i\Omega(s-s')t}\delta(y)\frac{d\mathbf{k}}{n^2};$$

$$G_{s'}'(\zeta) = \left[v_0\sin\theta - \frac{v_a(\sin\varphi + \cos\theta\cos\varphi)s'}{kR_0\sin\theta\cos\varphi}\right] J_s(\zeta);$$

$$\zeta = kR_0\sin\theta\cos\varphi; \quad y = \omega - s'\Omega - kv_0\cos\theta. \qquad (27)$$

If the condition in (20) is satisfied

$$A_z^+ = \int_0^T v_0 f_{rz}\, dt$$

$$= -\frac{e^2 R_0^2 T}{8c^3\beta_0} \int \omega^3\left(1-\frac{\Omega}{\omega}\right)\left[\frac{\Omega^2}{\omega^2} + \frac{1}{\beta_0^2 n^2(\omega)}\left(\varkappa + \frac{\Omega}{\omega}\right)^2\right] d\omega,$$

$$A_z^- = -\frac{e^2 R_0^2 T}{8c^3\beta_0} \int \omega^3\left(1+\frac{\Omega}{\omega}\right)\left[\frac{\Omega^2}{\omega^2} + \frac{1}{\beta_0^2 n^2(\omega)}\left(\varkappa - \frac{\Omega}{\omega}\right)^2\right] d\omega, \qquad (28)$$

where

$$\varkappa = \beta_0^2 n^2(\omega) - 1.$$

Furthermore,

$$f_{rx} = -\frac{e^2}{4\pi}\sum_{ss'}\int J_s G_{s'}'\left[(\sin\varphi + \cos\theta\cos\varphi)\right.$$

$$\left. - \frac{v_0 n}{c}(\cos\varphi + \sin\varphi\cos\theta)\right]e^{i\Omega(s-s')t}\frac{d\mathbf{k}}{n^2}; \qquad (29)$$

$$A_x \equiv A_a = \int_0^T v_a f_{rx}\, dt = A_a^+ + A_a^-,$$

$$A_a^+ = -\frac{e^2 R_0^2\Omega T}{8c^3\beta_0} \int \omega^2\left[\frac{\Omega^2}{\omega^2} + \frac{1}{\beta_0^2 n^2(\omega)}\left(\varkappa + \frac{\Omega}{\omega}\right)^2\right] d\omega,$$

$$A_a^- = \frac{e^2 R_0^2\Omega T}{8c^3\beta_0} \int \omega^2\left[\frac{\Omega^2}{\omega^2} + \frac{1}{\beta_0^2 n^2(\omega)}\left(\varkappa - \frac{\Omega}{\omega}\right)^2\right] d\omega. \qquad (30)$$

In this case it is also true that $|A_a^+| \geq A_a^-$. For helical motion, for example along a magnetic field, when $\Omega = \omega_H = (eH_0/mc)\sqrt{1 - v^2/c^2}$, it is necessary to set $R = \{R_0 \cos \Omega t, R_0 \sin \Omega t, v_0 t\}$; it is easy to compute the radiation force $f_r$:

$$f_{rz} = -\frac{e^2}{c^3} \sum_{s=-\infty}^{\infty} \int n\omega^2 \delta(y) \Phi_s \sin\theta d\theta d\omega,$$

$$\Phi_s = J_s^2(kR_0 \sin\theta)\left[-\sin\theta + \frac{\Omega s}{\omega \sin\theta}\right]\left[\frac{sv_a \cos\theta}{kR_0 \sin\theta} - v_0 \sin\theta\right]$$

$$+ \frac{v_a^2 n}{c}\cos\theta J_s'^2(kR_0 \sin\theta);$$

$$f_{rx} = -\frac{e^2 \sin\Omega t}{c^3}\sum_{s=-\infty}^{\infty}\int n\omega^2\delta(y)\Phi_s^a \sin\theta d\theta d\omega,$$

$$\Phi_s^a = (n\beta_0 \cos\theta - 1)v_a J_s'^2(kR_0 \sin\theta)$$

$$+ \frac{(n\beta_0 - \cos\theta)s}{kR_0 \sin\theta}\left(\frac{sv_a \cos\theta}{kR_0 \sin\theta} - v_0 \sin\theta\right)J_s^2(kR_0 \sin\theta) \quad (31)$$

where $f_{ry}$ differs from $f_{rx}$ by the substitution of $-\sin\Omega t$ for $\cos\Omega t$. The work performed by this force and its components $A_a^\pm$ is equal to the expressions in (28) and (30) multiplied by 2 if the condition in (20) is satisfied.

If the charge moves in a very narrow channel or slit in the case of oscillations along $v_0$ [cf. Eq. (13)] there is no change in the radiation intensity (the work $A^\pm$) and the reaction force. For an oscillator which oscillates along the x axis [cf. Eq. (26)] the radiation intensity remains unchanged for motion in a narrow slit in the plane $y = 0$. Finally, in the case of helical motion which takes place in a narrow cylindrical channel of circular cross section the radiation intensity is multiplied by the factor $4\epsilon^2/(\epsilon+1)^2$. The remarks concerning channels follow from the results obtained in reference 4.

3. In the case of an anisotropic medium we may consider motion along the axis of a uniaxial nongyrotropic crystal, with the electron assumed to be oscillating in the same direction. Under these conditions

$$R = \{0, 0, v_0 + R_0 \sin\Omega t\}, \quad k = \{0, k\sin\theta, k\cos\theta\};$$

$$a_1 = \{0, \cos\theta + K_1 \sin\theta, -\sin\theta + K_1 \cos\theta\};$$

$$a_2 = \{1, 0, 0\}, \quad K_1 = (n_1^2 - \epsilon_\perp)\cos\theta / \epsilon_\perp \sin\theta,$$

$$n_1^{-2} = \sin^2\theta / \epsilon_\parallel + \cos^2\theta / \epsilon_\perp, \quad (32)$$

where $n_1$ is the index of refraction for the extraordinary wave which, in the present case, is the only one radiated: $K_1$ is the ratio of the electric field components parallel and perpendicular to the vector $k$ in the extraordinary wave; this electric vector is parallel to the polarization vector $a_1$, the length of which satisfies the condition $n_1^{-2}\{\epsilon_\parallel a_z^2 + \epsilon_\perp a_y^2\} = 1$ (cf. reference 7).

Substituting Eq. (32) in Eq. (11), we obtain results similar to those obtained for the isotropic medium, where we have in Eq. (21) $n_{1,2}(\theta, \omega)$ in place of $n(\omega)$. Here we present only the expressions that correspond to Eqs. (22) and (24):

$$A_{s=\pm 1} \equiv A^\pm$$

$$= -\frac{e^2 R_0^2 T}{4c^3 \beta_0}\int \frac{\omega^3 \epsilon_\perp^2(\omega)\sin^2\theta d\omega}{(\epsilon_\perp \sin^2\theta + \epsilon_\parallel \cos^2\theta)^2 |1 - (ctg\,\theta / n_1)\,\partial n_1 / \partial\theta|},$$
$$(33)$$

$$A_a^\pm = \mp\frac{e^2\Omega R_0^2 T}{4c^3 \beta_0}\int\frac{\omega^3 \epsilon_\perp^2(\omega)\sin^2\theta\, d\omega}{(\epsilon_\perp \sin^2\theta + \epsilon_\parallel \cos^2\theta)^2 |1 - (ctg\,\theta / n_1)\,\partial n_1 / \partial\theta|}$$

$$= \mp\frac{e^2\Omega^4 R_0^2 T}{4c^3}\int\frac{n_1^5 \sin^3\theta d\theta}{\epsilon_\parallel^2 |1 - \beta_0 n_1 \cos\theta|^3 |1 - \beta_0 \cos\theta\partial(\omega n_1)/\partial\omega|},$$
$$(34)$$

where $A = A^+ + A^- = A_0 + A_a = A_0^+ + A_0^- + A_a^+ + A_a^-$ [cf. Eqs. (18) and (22)] while $\cos\theta$, $\sin\theta$ and $n_1(\omega, \theta)$ must be expressed in terms of $\omega$ through the Doppler expression (21).

Thus, for the normal effect in Eqs. (33) and (34)

$$s = 1, \quad \cos\theta = (1 - \Omega/\omega)/\beta_0 n_1[\omega, \theta(\omega)]$$

and the integration is taken over the region for which $\beta_0 n_1 \cos\theta < 1$; for the anomalous effect

$$s = -1, \quad \cos\theta = (1 + \Omega/\omega)/\beta_0 n_1$$

and the integration is taken over the region for which $\beta_0 n_1 \cos\theta > 1$. We may note that the factor

$$\epsilon_\perp^2 \sin^2\theta / (\epsilon_\perp \sin^2\theta + \epsilon_\parallel \cos^2\theta)^2 = n_1^4 \sin^2\theta / \epsilon_\parallel^2,$$

which appears in Eqs. (33) and (34) is also equal to $a_{1z}^2$, when $\epsilon_\perp = \epsilon_\parallel$ equals $\sin^2\theta$ and this corresponds to the transition from Eqs. (22) to (24).

In an anisotropic medium it is possible for amplification as well as attenuation to occur (in this case $|A_a^+| > A_a^-$). As an example we consider the following idealized case: $\epsilon_\perp$ and $\epsilon_\parallel$ are independent of frequency and $\epsilon_\perp > 0$ and $\epsilon_\parallel < 0$. In this case $n_1^2(\theta_\infty) = \infty$ at the angle $\theta_\infty$ determined from the condition $\epsilon_\perp \sin^2\theta_\infty + \epsilon_\parallel \cos^2\theta_\infty = 0$. Furthermore $n_1^2$ is a minimum and equal to $\epsilon_\perp$ when $\theta = 0$ while $\pi/2 > \theta > \theta_\infty$ for $n_1^2 < 0$ and propagation is impossible. If now $\beta_0 \epsilon_\perp > 1$, it is always possible to choose $\epsilon_\parallel$ in such a way that the Cerenkov angle $\theta_0 = \cos^{-1}(1/\beta_0 n_1)$ is larger than $\theta_\infty$. Under these conditions there is no Cerenkov radiation and in the forward direction (for $\theta < \pi/2$) only anomalous Doppler waves are radiated. In the backward direction $(\pi - \theta < \theta_\infty)$ normal Doppler waves are radiated but here $(1 - \beta_0 n\cos\theta) = (1 + \beta_0 n |\cos\theta|)$ and, as follows from Eq. (34), the work $|A_a^+|$ is always smaller than the work $A_a^-$.

It should be noted that at large values of n both in anisotropic and in isotropic media it may turn

out that Eq. (20) is not satisfied; a number of the formulas given here are then inapplicable. This situation is especially important for anisotropic media in cases in which the signs of $\epsilon_{||}$ and $\epsilon_{\perp}$ are different and $n_2$ becomes infinite over a wide range of frequencies.

The case of helical motion of a charge in a magneto-active plasma* has been considered in detail in another paper.[14] Here we limit ourselves to the remark that if there is an anomalous Doppler region we know that there will be a weakening of the radiation dampening which leads to a reduction of the particle velocity perpendicular to the external magnetic field. At certain values of the parameters the damping becomes negative, i.e., oscillations are excited.

4. As noted above, the difference in the signs of the forces which affect the oscillatory motion of a particle for the normal and anomalous Doppler effects are in complete agreement with the results which follow from simple quantum mechanical considerations.[5,6] In particular, in the anomalous Doppler effect the system makes a transition from a lower level to a higher level; this corresponds to excitation rather than dissipation. Hence, from the quantum-mechanical point of view the system (for example a moving oscillator) must have some probability of making transitions to higher and higher oscillatory levels so long as these transitions are consistent with the condition $\beta_0 n(\omega) \cos \theta > 1$. At the same time, in the classical approximation, in an isotropic medium the oscillations are always attenuated, i.e., the system makes transitions to lower levels although the rate of "fall" may be slowed down considerably at velocities greater than that of light. There is actually no inconsistency. Consider a wave packet, composed of wave functions with approximately the same energies (energies of oscillatory motion). At velocities below the velocity of light there are only transitions to lower levels and, aside from the reduction in the mean energy of the packet, the width of the energy spectrum is increased. At velocities greater than the velocity of light it is also possible to have transitions to upper levels and the variation of the energy spectrum of the packet is different; in particular, in the total ensemble of states there will be a state with energy greater than the initial energy. In this sense we have excitation of transverse oscillations. However, in the classical

approximation, i.e., if one neglects the spread of the packet, only the change in the mean energy of the packet is important. This change, as has been shown, always corresponds to a reduction in mean energy in an isotropic medium. It can be shown that at velocities greater than the velocity of light the system becomes unstable to some extent. In the first place we have the excitation of oscillations in the quantum approximation, as already noted; secondly, in an anisotropic medium it is possible to have excitation of oscillations even in the classical approximation; in the third place (and in practice this is the most important) there is an instability and excitation of oscillations when we consider a particle bunch moving at velocities greater than light rather than a single particle.[15] This excitation is associated with the fact that the beam exhibits negative absorption (reabsorption) in the region of the anomalous Doppler frequencies (in this region, in absorbing a photon the system makes a transition in the downward direction whereas a transition in the upward direction corresponds to induced emission). It is completely obvious that this instability of "superlight" bunches, which occurs in an isotropic medium even in the classical approximation, is closely associated with the problem of radiation reaction which has been considered above in connection with a single particle.

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*This case is automatically realized for motion of an electron in a plasma in a magnetic field.

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