THE INFLUENCE OF ANISOTROPY ON THE THERMAL CONDUCTIVITY OF SUPERCONDUCTORS

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We have evaluated the electronic part of the thermal conductivity of a superconductor taking anisotropy into account. We show that the temperature dependence of the thermal conductivity may be different along different crystallographic axes for uniaxial crystals.

THE electronic part of the thermal conductivity of superconductors was evaluated by a number of authors[2,3] on the basis of the theory of Bardeen, Cooper, and Schrieffer.[1] In these papers an isotropic model of a superconductor was considered. In the following we shall give a similar evaluation taking anisotropy into account. It will be shown that in some cases the temperature dependence of the coefficient of thermal conductivity may change with direction.

Bogolyubov, Tolmachev, and Shirkov[4] obtained the energy spectrum of a superconductor taking anisotropy into account by the same method as was used for the isotropic case. One can in that case start either from Fröhlich's Hamiltonian, or from the model Hamiltonian of Bardeen's. In the given case the model Hamiltonian can be written in the form

$$H = \sum (E_{\mathbf{k}} - \mu) a_{\mathbf{k}}^{+} a_{\mathbf{k}} + \tfrac{1}{2} \sum g(\mathbf{k}_1', \mathbf{k}_2'; \mathbf{k}_1, \mathbf{k}_2)\, a^{+}_{\mathbf{k}_1' 1/2} a^{+}_{\mathbf{k}_2' -1/2} a_{\mathbf{k}_1 1/2} a_{\mathbf{k}_2 -1/2}, \tag{1}$$

where μ is the chemical potential. The function $g(\mathbf{k}_1', \mathbf{k}_2'; \mathbf{k}_1, \mathbf{k}_2)$ satisfies the conditions

$$g(\mathbf{k}_1', \mathbf{k}_2'; \mathbf{k}_1, \mathbf{k}_2) = g(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_1', \mathbf{k}_2') = g(-\mathbf{k}_1, -\mathbf{k}_2; -\mathbf{k}_1', -\mathbf{k}_2'). \tag{2}$$

because of its invariance with respect to a permutation of the particles and to spatial inversion. Performing, furthermore, a transformation from the particle operators $a_{\mathbf{k}}$ to the excitation operators $\alpha_{\mathbf{k}}$ according to

$$a_{\mathbf{k} 1/2} = u_{\mathbf{k}} \alpha_{\mathbf{k}0} + v_{\mathbf{k}} \alpha^{+}_{\mathbf{k}1}, \quad a_{\mathbf{k} -1/2} = u_{\mathbf{k}} \alpha_{\mathbf{k}1} - v_{\mathbf{k}} \alpha^{+}_{\mathbf{k}0}, \quad u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1 \tag{3}$$

and considering in the Hamiltonian only the pair interactions for particles with equal, but opposite momenta, we find the excitation spectrum in the form

$$\varepsilon_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2(\mathbf{e}_{\mathbf{k}})}, \qquad \xi_{\mathbf{k}} = E_{\mathbf{k}} - \mu \tag{4}$$

($\mathbf{e}_{\mathbf{k}}$ is a unit vector in the $\mathbf{k}$ direction). The anisotropic gap $\Delta(\mathbf{e}_{\mathbf{k}})$ is determined by the condition*

$$\Delta(\mathbf{e}_{\mathbf{k}}) = \sum_{\mathbf{k}'} g(\mathbf{k}, -\mathbf{k}; \mathbf{k}', -\mathbf{k}')\, \Delta(\mathbf{e}_{\mathbf{k}'}) / 2\varepsilon_{\mathbf{k}'}. \tag{5}$$

The transformation parameters $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are in this case equal to

$$u_{\mathbf{k}}^2 = 1/2\{1 + \xi_{\mathbf{k}}/\varepsilon_{\mathbf{k}}\}, \qquad v_{\mathbf{k}}^2 = 1/2\{1 - \xi_{\mathbf{k}}/\varepsilon_{\mathbf{k}}\}. \tag{6}$$

We shall assume that the electronic thermal conductivity is caused by the scattering of the excitations by impurities. The Hamiltonian for the interaction of an electron with an impurity is equal to

$$H_{int} = \sum V_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}}^{+} a_{\mathbf{k}'}. \tag{7}$$

Performing the u, v transformation (3) we get the Hamiltonian for the interaction of the excitation with an impurity

$$H_{int} = \sum V_{\mathbf{k}\mathbf{k}'} (u_{\mathbf{k}} u_{\mathbf{k}'} - v_{\mathbf{k}} v_{\mathbf{k}'})\, \alpha_{\mathbf{k}}^{+} \alpha_{\mathbf{k}'}. \tag{8}$$

The transport equation for the excitations

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}} \frac{\partial \varepsilon}{\partial \mathbf{k}} - \frac{\partial n}{\partial \mathbf{k}} \frac{\partial \varepsilon}{\partial \mathbf{r}} = I(n) \tag{9}$$

[I(n) is the collision integral] assumes the following form

$$\frac{n}{T^2} \xi_{\mathbf{k}} \left(\frac{\partial E}{\partial \mathbf{k}} \nabla T \right) = I(n). \tag{10}$$

under stationary conditions and in the presence of a temperature gradient.

We shall perform the further calculations for the case of sufficiently low temperatures when the inequality

$$T \ll \Delta_{min}. \tag{11}$$

*We have assumed that the function g depends only on the angles characterizing the directions $\mathbf{k}$ and $\mathbf{k}'$.

is satisfied. The magnitude of the gap $\Delta(\mathbf{e_k})$ depends on the direction. It follows that the gap will take on minimum values $\Delta_{\min}$ for some extremal directions. It is also clear that there will be for each extremal direction a corresponding equivalent extremal direction in the opposite direction. For the case of temperatures near T_c when the values of Δ are small, an analysis of the transport equation in the anisotropic case becomes not very instructive. At low temperatures, however, when the value of Δ practically ceases to depend on the temperature, one can solve the problem completely and one can take as the distribution function for the excitations the classical Boltzmann function.

Since the number of excitations moving in a given direction will basically be determined by the exponent $\exp(-\Delta/T)$, it is natural that the largest contribution to the energy current will be given by excitations moving near to the extremal directions. It is thus sufficient to perform the analysis of the transport equation only in the neighborhood of these directions. The problem consists thus in finding the distribution function for the excitations moving near the extremal directions.

For elastic scattering by an impurity only the energy is conserved in the anisotropic case. As to the momentum of the excitation, its absolute magnitude is clearly not conserved. The collision integral $I(n)$ is in the case under consideration equal to

$$I(n) = -\frac{2\pi}{\hbar}\int |V_{\mathbf{kk'}}|^2 (u_\mathbf{k} u_{\mathbf{k'}} - v_\mathbf{k} v_{\mathbf{k'}})^2 \times \delta(\varepsilon_\mathbf{k} - \varepsilon_{\mathbf{k'}})(n_\mathbf{k} - n_{\mathbf{k'}})\, d\tau_{\mathbf{k'}}. \tag{12}$$

We have used here (8) and the fact that the distribution function is classical. We further use Eqs. (6) and integrate the δ-function in (12) over $d\xi_{\mathbf{k'}}$. We get as a result

$$I(n) = -\frac{\pi}{2\hbar}\int |V_{\mathbf{kk'}}|^2 \frac{(\xi_\mathbf{k} + \xi_{\mathbf{k'}})^2}{\varepsilon_\mathbf{k} |\xi_{\mathbf{k'}}|}\left(\frac{\partial^2\tau}{\partial\xi_{\mathbf{k'}}\partial o'}\right) do' (n_\mathbf{k} - n_{\mathbf{k'}}) \tag{13}$$

(do' is an element of solid angle).

We write the required non-equilibrium function $n_\mathbf{k}$ in the form

$$n_\mathbf{k} = n_\mathbf{k}^0 (1 + \chi_\mathbf{k}), \tag{14}$$

where $n_\mathbf{k}^0$ is the equilibrium distribution function. Substituting (13) into (10) we get then the following integral equation to determine the function $\chi_\mathbf{k}$:

$$-\frac{\varepsilon_\mathbf{k}}{T^2}\left(\frac{\partial E}{\partial \mathbf{k}} \nabla T\right) = \frac{\pi}{2\hbar}\int |V_{\mathbf{kk'}}|^2 \frac{(\xi_\mathbf{k} + \xi_{\mathbf{k'}})^2}{\xi_\mathbf{k} |\xi_{\mathbf{k'}}|}\left(\frac{\partial^2\tau}{\partial\xi_{\mathbf{k'}}\partial o'}\right)(\chi_\mathbf{k} - \chi_{\mathbf{k'}})\, do'. \tag{15}$$

This equation can easily be solved for the extremal directions. We consider the case when there are only two (opposite) extremal directions.

Let the gradient ∇T be directed along the extremal direction and let us denote by ϑ the angle between ∇T and the vector $\mathbf{k}$. We introduce, moreover, the following notation (for the sake of simplicity we have assumed Δ to be independent of the azimuthal angle φ):

$$\eta_\mathbf{k}^2 = \xi_\mathbf{k}^2 + \Delta\Delta''\vartheta^2, \qquad y = \xi_\mathbf{k} / \eta_\mathbf{k} \qquad (\Delta'' = \partial^2\Delta / \partial\vartheta^2). \tag{16}$$

Taking into account that the temperature is low we can everywhere in (15) neglect the quantity $\xi_\mathbf{k}$ compared to Δ. As a result the integral Eq. (15) assumes the following form:

$$\frac{1}{\eta_\mathbf{k}^2} Ay = V_0^2 \int_{-1}^{1} (y + y')^2 (\chi_y - \chi_{y'})\, dy' + V_\pi^2 \int_{-1}^{1} (y + y')^2 (\chi_y + \chi_{y'})\, dy', \tag{17}$$

where

$$A = -\frac{\hbar}{\pi^2}\left(\frac{\partial E}{\partial k}\right)_F^2 \frac{\Delta^2\Delta''}{T^2}\left(\frac{\partial^2\tau}{\partial k \partial o}\right)_F^{-1} |\nabla T|, \tag{18}$$

V_0 and V_π are the matrix elements for scattering over an angle 0 and an angle π, respectively, for excitations moving initially in the extremal direction. The first integral arises from integrating in (15) over the region in the neighborhood of the extremal direction, and the second integral from the region near the opposite direction. In deriving (17) we have used the fact that $\eta_\mathbf{k}^2 = \eta_{\mathbf{k'}}^2$, because of the conservative law for energy. The quantity Δ'' occurring in the equations is the second derivative of the gap Δ with respect to the angle ϑ near its extrema (it is obvious that $\Delta' = 0$). Derivatives of E and τ with respect to k are taken in the extremal direction on the Fermi surface. Finally we have used the obvious consequence of Eq. (15), namely, that χ_y' for the direction opposite to the extremal direction differs from χ_y only in sign.

From Eq. (17) one sees easily that χ_y is an odd function of y, i.e.

$$\chi_y = -\chi_{-y}. \tag{19}$$

To find χ_y from (17) we get, if we take this fact into account, the following equation

$$\chi_y = A\psi_y / \eta_\mathbf{k}^2,$$

$$y = 2(V_0^2 + V_\pi^2)(y^2 + \tfrac{1}{3})\psi_y + 2y(V_\pi^2 - V_0^2)\int_{-1}^{1} \psi_{y'} y' dy'. \tag{20}$$

The solution of Eq. (20) is elementary

$$\phi_y = y/2\left(y^2 + \tfrac{1}{3}\right)[(V_\pi^2 + V_0^2) + 2(1 - \pi/3\sqrt{3})(V_\pi^2 - V_0^2)]. \qquad (21)$$

There only remains for us to evaluate the energy current

$$\mathbf{q} = \int \varepsilon_{\mathbf{k}} \frac{\partial \varepsilon}{\partial \mathbf{k}} n^0 \chi_{\mathbf{k}} d\tau_{\mathbf{k}}. \qquad (22)$$

We only need perform the integration in (22) over the regions near the extremal directions. The distribution function near the extremal directions can be written in the form

$$n^0 = \exp\{-\Delta/T - \eta_{\mathbf{k}}^2/2\Delta T\}. \qquad (23)$$

Substituting (21) into (22) we get after some simple transformations

$$q = \frac{2\pi A e^{-\Delta/T}}{\Delta\Delta''}\left(\frac{\partial^2\tau}{\partial k \partial o}\right)_F \frac{1}{(V_\pi^2 + V_0^2) + 2(1 - \pi/3\sqrt{3})(V_\pi^2 - V_0^2)} \times \int_0^\infty \eta_{\mathbf{k}} d\eta_{\mathbf{k}} \exp\left\{-\frac{\eta_{\mathbf{k}}^2}{2\Delta T}\right\} \int_{-1}^{1} \frac{y^2 dy}{y^2 + 1/3}. \qquad (24)$$

From this we finally get for the coefficient of thermal conductivity along the extremal direction

$$\varkappa_\parallel \approx \frac{8\hbar\Delta^2}{5\pi T}\left(\frac{\partial E}{\partial k}\right)_F^2 \frac{1}{1.8V_\pi^2 + 0.2V_0^2} e^{-\Delta/T}. \qquad (25)$$

To evaluate the coefficient of thermal conductivity along directions perpendicular to the extremal one, it is necessary to solve anew the integral Eq. (15). We now take the temperature gradient along an axis perpendicular to the extremal direction (x axis). We get then easily the following equation for the function χ, if we go over to the variables (16),

$$\cos\varphi \frac{A}{\eta_{\mathbf{k}}\sqrt{\Delta\Delta''}} y\sqrt{1 - y^2} = (V_0^2 + V_\pi^2)\int_{-1}^{1} (y + y')^2 (\chi_y - \chi_{y'})\, dy'. \qquad (26)$$

The solution of this equation is also obtained elementarily and we get

$$\chi_y = \phi_y A \cos\varphi / \eta_{\mathbf{k}} \sqrt{\Delta\Delta''}(V_0^2 + V_\pi^2), \qquad (27)$$

$$\phi_y = \frac{y}{2(y^2 + 1/3)}\left(\sqrt{1 - y^2} + \frac{\pi}{6}\,\frac{1}{1 - 2(1 - \pi/3\sqrt{3})}\right). \qquad (28)$$

The heat current in this case is equal to

$$q_\perp = \int \varepsilon_{\mathbf{k}} \frac{\partial \varepsilon}{\partial k} \vartheta \cos\varphi\, n^0 \chi_{\mathbf{k}} d\tau_{\mathbf{k}}. \qquad (29)$$

Substituting here χ from (27) and going over to the variables $\eta_{\mathbf{k}}$ and y we get

$$q_\perp = \frac{2\pi A e^{-\Delta/T}}{(\Delta\Delta'')^2 (V_0^2 + V_\pi^2)} \times \int_0^\infty \eta_{\mathbf{k}}^3 d\eta_{\mathbf{k}} \exp\left\{-\frac{\eta_{\mathbf{k}}^2}{2\Delta T}\right\} \int_{-1}^{1} \phi_y \sqrt{1 - y^2}\, y dy, \qquad (30)$$

or finally

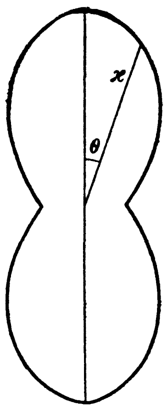
$$\varkappa_\perp \approx \frac{10\hbar\Delta^2}{3\pi\Delta''}\left(\frac{\partial E}{\partial k}\right)_F^2 \frac{1}{V_0^2 + V_\pi^2} e^{-\Delta/T}. \qquad (31)$$

The ratio $\kappa_\perp/\kappa_\parallel$ is equal to

$$\varkappa_\perp/\varkappa_\parallel \approx (2T/\Delta'')(1.8V_\pi^2 + 0.2V_0^2)/(V_\pi^2 + V_0^2). \qquad (32)$$

The thermal conductivity in a direction perpendicular to the extremal one is thus less than its extremal value roughly by a factor T/Δ. The main (exponential) part of the temperature dependence of κ is here, of course, the same in all directions. A qualitative picture of the dependence of the coefficient of thermal conductivity on the angle θ, reckoned from the extremal direction, at low temperatures will be in the form of a rosette, shown in the figure and determined by the following obvious relation

$$\varkappa = \varkappa_\parallel \cos^2\theta + \varkappa_\perp \sin^2\theta. \qquad (33)$$



Dependence of the thermal conductivity on the angle θ (for the case $\varkappa_\perp/\varkappa_\parallel = 1/5$).

The situation considered here, where there is only one pair of extremal directions, is clearly rare. It could be realized in the case of a uniaxial crystal with the extremal direction along the main axis of symmetry. More typical will be the case where there are several extremal directions which are not lying in one plane. One can see that the temperature dependence of the thermal conductivity will in that case be unique in all directions in the approximation considered here.

The preliminary results of N. V. Zavaritskiĭ on the thermal conductivity of Ga in the superconducting state[5] indicate the presence of a different temperature dependence along the directions of the crystallographic axes. If these results are confirmed this will mean that a situation takes place in Ga where there are two extremal directions for Δ.

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