CYLINDRICAL AND PLANE MAGNETOHYDRODYNAMIC WAVES

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Cylindrical waves produced in a conducting medium by a magnetic field are considered. The two cases when the field is directed along the z axis and along the angle φ are analyzed. Special attention is paid to "sound" waves as well as to those possessing velocities close to that of light.

A study of plane and particularly cylindrical magnetohydrodynamic waves is of considerable interest both from the physical and from the analytical points of view.

In the present paper we confine ourselves to the case of infinite conductivity and isentropic motion, with the condition that the magnetic field is perpendicular to the velocity. We shall carry out the study first in the general relativistic form with strong fields and large energy densities, and shall then proceed to examine "classical" relatively weak waves.

1. FUNDAMENTAL EQUATIONS

The fundamental equations of isentropic cylindrical waves can be written in the relativistic case

$$\frac{1}{\theta^2}\left(\frac{\partial a}{\partial t}+a\frac{\partial a}{\partial r}\right)+c^2\left(\frac{\partial \ln w^*}{\partial r}+\frac{a}{c^2}\frac{\partial \ln w^*}{\partial t}\right)=0; \tag{1}$$

$$-\left(\frac{\partial \ln V}{\partial t}+a\frac{\partial \ln V}{\partial r}\right)+\frac{1}{\theta^2}\left(\frac{\partial a}{\partial r}+\frac{a}{c^2}\frac{\partial a}{\partial t}\right)+\frac{Na}{r}=0. \tag{2}$$

Here a is the velocity, w^* the total heat content, V the specific volume, $\theta^2 = 1 - a^2/c^2$, $N = 1$ for cylindrical waves, and $N = 0$ for plane waves. The equations can be derived by using the condition $\partial T_{ik}/\partial x_k = 0$, where T_{ik} is the total energy-momentum tensor of the field and of the medium.[1]

If the magnetic-field vector is perpendicular to the plane of motion of the medium, we have

$$HV = b = \text{const}. \tag{3}$$

If the magnetic-field vector lies in the plane of motion (but is, naturally, perpendicular to the velocity vector), then

$$\frac{HV}{r} = b = \text{const}. \tag{4}$$

Relations (3) and (4) can be combined into one,

$$HV = br^m, \tag{5}$$

where $m = 0$ when $N = 0$ and $m = 0$ or 1 when $N = 1$. From the relation $dw^* = Vd(p + p') = V\,dp^*$, where

$$dp' = \frac{Hr^{-m}}{4\pi}d(Hr^m) = \frac{1}{4\pi V}d(H^2V) \tag{6}$$

(p' is the supplementary field pressure), we obtain

$$w^* = pV + \rho Vc^2 + \frac{H^2V}{4\pi}. \tag{7}$$

Inserting now the value of H from (5) we get

$$dw^* = V\,dp - \frac{b^2}{4\pi V^2}r^{2m}\,dV + \frac{2mb^2r^{2m-1}}{4\pi V}\,dr$$

$$= V\,dp - \frac{H^2\,dV}{4\pi} + \frac{2mH^2V\,dr}{4\pi r}. \tag{8}$$

Now Eq. (1) can be rewritten

$$\frac{1}{\theta^2}\left(\frac{\partial a}{\partial t}+a\frac{\partial a}{\partial r}\right)$$

$$-\omega^{*2}\left(\frac{\partial \ln V}{\partial r}+\frac{a}{c^2}\frac{\partial \ln V}{\partial t}\right)+\frac{2mc^2VH^2}{4\pi r w^*}=0. \tag{9}$$

The characteristics of the system of equations (9) and (2) can be written

$$\frac{dr}{dt}=\frac{a\pm\omega^*}{1\pm a\omega^*/c^2}. \tag{10}$$

The following relations hold along the lines

$$\frac{da}{\theta^2}=\pm\omega^*\,d\ln V\mp\omega^*\frac{Na}{r}\,dt-\frac{2mc^2VH^2}{4\pi r w^*}\,dt$$

$$=\pm\omega^*\,d\ln V\mp\omega^*\frac{Na}{r}\,dt-\frac{2mc^2b^2r}{4\pi V w^*}\,dt, \tag{11}$$

where

$$\frac{\omega^{*2}}{c^2}=\frac{V}{w^*}\left(-V\frac{dp}{dV}+\frac{H^2}{4\pi}\right); \tag{12}$$

ω^* is the magneto-gasdynamic velocity of sound. Inasmuch as

$$\frac{V}{w^*}=\frac{1}{p+\rho c^2+H^2/4\pi}, \qquad -V\frac{dp}{dV}=\frac{\omega^2}{c^2}(p+\rho c^2),$$

where $\omega = (dp/d\rho)^{1/2}$ is the ordinary velocity of sound, we get

$$\frac{\omega^{*2}}{c^2} = \frac{(p+\rho c^2)\,\omega^2/c^2 + H^2/4\pi}{p+\rho c^2 + H^2/4\pi}, \tag{13}$$

Let us now calculate the value of ω^{*2}/c^2 for an ideal gas, assuming that for the isentropic process $p = AV^{-k}$, where A is a constant. Here, starting with the relation $-dV/V = c^2 d\rho/(p+\rho c^2)$, we obtain

$$\rho V = \alpha + \frac{pV}{(k-1)c^2}, \qquad \alpha = \rho_a V_a\left(1 - \frac{p_a}{(k-1)\rho_a c^2}\right),$$

where ρ_a and V_a are the initial values of ρ and V at $p = p_a$. It is obvious that

$$\frac{\omega^{*2}}{c^2} = \frac{kAV^{2-k} + r^{2m}b^2/4\pi}{\alpha Vc^2 + kAV^{2-k}/(k-1) + r^{2m}b^2/4\pi}.$$

For ordinary gas $\alpha = 1$; for an ultrarelativistic or a photon gas, when $p = (k-1)\rho c^2$, we get $\alpha = 0$.

Let us proceed now to examine several problems.

2. STATIONARY FLOWS

In the case of stationary flows the system (1) and (2) admits of the following integrals

$$w^*/\theta = w_0 = \text{const}, \tag{14}$$

$$\frac{ar^N}{V\theta} = \frac{\overline{m}}{A} = m = \text{const}, \tag{15}$$

where w_0^* is the heat contents at rest, A = 1 or 2π respectively for N = 0 or 1, and $\overline{m}$ is the total mass flow per second.

Relations (14) and (15) can be rewritten

$$\frac{a^2}{c^2} = \left(1 + \frac{c^2 r^{2m}}{m^2V^2}\right)^{-1} = 1 - \frac{w^{*2}}{w_0^{*2}}$$
$$= 1 - \frac{1}{w_0^{*2}}(pV + \rho Vc^2 + H^2V/4\pi)^2. \tag{16}$$

From (16) we obtain an expression for V

$$\left(\alpha c^2 + \frac{kA}{k-1}V^{1-k} + \frac{b^2r^{2m}}{4\pi V}\right)^2 = w_0^{*2}\left(1 + \frac{m^2V^2}{c^2r^{2m}}\right)^{-1}.$$

The case when N = 0 (and consequently m = 0) is of no interest; in this case V = const.

Let us consider two interesting types of motion, when N = 1 and m = 0 and 1. It is easy to explain first that the motion in these cases can be defined only within a certain region. In fact, since

$$r^{2m} = (m/c)^2\, w^{*2}V^2/(w_0^{*2} - w^{*2}),$$

we have at $dr^{2m} = 0$

$$-d\ln w^*/d\ln V = 1 - w^{*2}/w_0^{*2} = a^2/c^2.$$

Furthermore

$$dw^* = V\,dp - \frac{H^2\,dV}{4\pi}, \qquad -\frac{d\ln w^*}{d\ln V} = \frac{\omega^{*2}}{c^2},$$

so that

$$\frac{\omega^{*2}}{c^2} = \frac{\omega_{cr}^{*2}}{c^2} = \frac{a_{cr}^2}{c^2} = 1 - \frac{w_{cr}^{*2}}{w_0^{*2}},$$

i.e., the condition of critical flow is obtained.

If m = 0, we can readily determine $V = V_{cr}$; $r = r_{cr} = r_{min}$, and the motion is defined in the region $r_{min} \le r < \infty$. When $\alpha \ne 0$ and $r \to \infty$ we have $V \to \infty$; $a = a_{max} < c$, with

$$r_{min} \approx \frac{mV}{c}\,\frac{\alpha c^2}{(w_0^{*2} - \alpha^2c^4)^{1/2}}.$$

If $\alpha = 0$, then

$$r \approx \frac{m}{cw_0^*}\left(\frac{kA}{k-1}\right)^{1/2} V^{(2-k)/2}$$

and

$$a \to a_{max} = c.$$

If m = 1, we obtain by eliminating r^2

$$\left[\frac{k}{k-1}AV^{1-k} + \alpha c^2 + \frac{b^2m^2V}{4\pi c^2} + \frac{1}{-1 + \left\{\left[kAV^{1-k}\frac{2-k}{k-1} + \alpha c^2\right]/w_0^*\right\}^{-2/3}}\right]^3$$
$$= w_0^{*2}\left(\frac{2-k}{k-1}kAV^{1-k} + \alpha c^2\right),$$

which determines $V = V_{cr}$.

If $\alpha \ne 0$, then, as seen from an analysis of this equation, we have two values of $V = V_{cr}$, corresponding to two extremal values of r, $r = r_{min}$ and $r = r_{max}$. Thus, the motion will be defined in the region $r_{min} \le r \le r_{max}$. When $r \ge r_{max}$, the stationary mode of flow is no longer possible, and the current will pulsate there.

If $\alpha = 0$ (ultrarelativistic gas), then $r_{max} \to \infty$, and thus the stationary flow will be defined in the region $r_{cr} \leqslant r < \infty$.

3. NONSTATIONARY WAVES

We shall study strong "sound" waves in an ultrarelativistic gas, when the pulsation occurs at velocities of flow close to the velocity of light so that $1 - a/c \ll 1$. Now Eqs. (2) and (9) become

$$\frac{1}{\theta^2}\left[\frac{\partial(a/c)}{c\partial t} + \frac{\partial(a/c)}{\partial r}\right] = \frac{\omega^{*2}}{c^2}\left(\frac{\partial\ln V}{c\partial t} + \frac{\partial\ln V}{\partial r}\right)$$
$$-\frac{2\,mH^2V}{4\pi rw^*} = -\frac{N}{r} + \frac{\partial\ln V}{c\partial t} + \frac{\partial\ln V}{\partial r}. \tag{17}$$

Hence, since $H^2V = b^2r^{2m}/V$, we have

$$\left(1 - \frac{\omega^{*2}}{c^2}\right)\left(\frac{\partial\ln V}{c\,\partial t} + \frac{\partial\ln V}{\partial r}\right) = \frac{N}{r} - \frac{2\,mb^2r^{2m}}{4\pi Vrw^*}. \tag{18}$$

Here

$$1-\frac{\omega^{*2}}{c^2}=\frac{kAV^{1-k}(2-k)/(k-1)+\alpha c^2}{kAV^{1-k}/(k-1)+\alpha c^2+b^2r^{2m}/4\pi V},$$

$$w^*=\frac{kAV^{1-k}}{k-1}+\alpha c^2+\frac{b^2r^{2m}}{4\pi V}.$$

The solution of Eq. (18) involves no great difficulties in the case of an ultrarelativistic gas, $\alpha = 0$. In the case when $N = 0$, we have $V = F(r-ct)$, but a more detailed analysis is of no interest.

Let us consider the case when $N = 1$:

$$\frac{\partial \ln V}{c\partial t}+\frac{\partial \ln V}{\partial r}=\frac{1}{(2-k)r}\left[1+\frac{(k-1)b^2r^{2m}}{4\pi kAV^{2-k}}(1-2m)\right],$$

$$V^{2-k}=r\,[F(r-ct)-(k-1)b^2r^{2m-1}/4\pi kA]; \qquad (19)$$

with this

$$\frac{1}{\theta^2}\left[\frac{\partial(a/c)}{c\partial t}+\frac{\partial(a/c)}{\partial r}\right]$$

$$=-\frac{N}{r}+\frac{1}{(2-k)r}\left[\frac{4\pi kArF(r-ct)-2m(k-1)b^2r^{2m}}{4\pi kArF(r-ct)-(k-1)b^2r^{2m}}\right].$$

The solution of this equation is of the form

$$1-\frac{a}{c}=r^{-2(k-1)/(2-k)}f(r-ct)$$

$$\times\left[1-\frac{(k-1)b^2r^{2m-1}}{4\pi kAF(r-ct)}\right]^{-2/(2-k)}. \qquad (20)$$

It is obvious that

$$1-\frac{a}{c}=r^{-2(k-1)/(2-k)}\frac{f_1(r-ct)}{V^2}r^{2/(2-k)}=f_1(r-ct)\frac{r^2}{V^2}.$$

It follows from this that when $a \approx c$ we also have

$$\frac{ar}{V\theta}=f_1(r-ct),$$

which is an analogue of the continuity equation for the stationary case, when $f_1(r-ct) = \text{const}$. In this case we deal with a source of variable flow.

An investigation of relation (19) and (20) shows that when $m = 0$ the motion is defined over the entire space $0 < r < \infty$.

When $m = 1$, waves of this type can exist only at $0 \le r \le r_{max}$, when $1 - a/c \to 1$ and $V \to 0$. When $r > r_{max}$, motions can exist only when $a/c \ll 1$.

We now show that if there is no magnetic field, Eqs. (17) assume the form

$$\frac{1}{\theta^2}\left[\frac{\partial(a/c)}{c\partial t}+\frac{\partial(a/c)}{\partial r}\right]$$

$$=\frac{N}{r}\frac{\omega^2/c^2}{1-\omega^2/c^2}=\frac{\omega^2}{c^2}\left(\frac{\partial \ln V}{c\partial t}+\frac{\partial \ln V}{\partial r}\right). \qquad (21)$$

In the case $\alpha = 0$ we have $\omega^2/c^2 = k-1$, and therefore Eq. (21) can be rewritten

$$\frac{1}{\theta^2}\left[\frac{\partial(a/c)}{c\partial t}+\frac{\partial(a/c)}{\partial r}\right]=\frac{k-1}{2-k}\frac{N}{r},$$

$$\frac{\partial \ln V}{c\partial t}+\frac{\partial \ln V}{\partial r}=\frac{N}{(2-k)r}.$$

The solution of these equations is

$$1-\frac{a}{c}=f(r-ct)\,r^{-2N(k-1)/(2-k)}, \qquad V=\varphi(r-ct)\,r^{N/(2-k)}.$$

Obviously the following relations hold

$$\frac{ra}{V\theta}=f_1(r-ct), \qquad \frac{w}{\theta}=f_2(r-ct).$$

The second relation is an analogue of the Bernoulli equation. With this

$$p=AV^{-k}=\Phi(r-ct)\,r^{-kN/(2-k)}.$$

If $N = 2$ and $k = 4/3$,

$$p=\Phi(r-ct)\,r^{-4}, \qquad (22)$$

hence

$$4\pi r^2p=F=\frac{\Phi(r-ct)}{r^2}, \qquad (23)$$

where F is the force acting on the surface $4\pi r^2$. This force is inversely proportional to the square of the distance r from the center of the body that radiates the ultrarelativistic (or photon) gas.

Let us proceed now to investigate sound waves. The principal equations for sound waves, when $a \ll c$, become

$$\frac{\partial a}{\partial t}-\omega^{*2}\frac{\partial \ln V}{\partial r}+\frac{2mVH^2c^2}{4\pi rw^*}=0, \qquad (24)$$

$$-\frac{\partial \ln V}{\partial t}+\frac{\partial a}{\partial r}+\frac{Na}{r}=0. \qquad (25)$$

if $m = 0$, the problem of investigating sound wave is simply solved. Here $\omega^* = \omega_0^* = \text{const}$, and the system (24) and (25) can be written

$$\frac{\partial a}{\partial t}=\frac{\omega_0^{*2}}{V_0}\frac{\partial V}{\partial t}, \qquad \frac{\partial V}{V_0\partial t}=\frac{\partial a}{\partial r}+\frac{Na}{r},$$

with $V = V_0 + \Delta V$. Hence, introducing the velocity potential defined by the relations

$$a=\frac{\partial\varphi}{\partial r}, \qquad V=\frac{V_0}{\omega_0^{*2}}\frac{\partial\varphi}{\partial t},$$

we arrive at the classical wave equation

$$\omega_0^{*-2}\frac{\partial^2\varphi}{\partial t^2}=\frac{\partial^2\varphi}{\partial r^2}+\frac{N}{r}\frac{\partial\varphi}{\partial r}$$

(where $N = 0$ or 1), the solution of which is well known.

If we have $m = 1$ and when $N = 1$, the problem is much more complicated. Since $HV = br$, then $HV \to \infty$ when $r \to \infty$. We may have here $p \approx \rho c^2$ and $H^2 = \rho c^2$. It is therefore necessary to integrate the system

$$\frac{\partial a}{\partial t}+\frac{b^2c^2r}{2\pi Vw^*}=\omega^{*2}\frac{\partial \ln V}{\partial r}, \quad (26)$$

$$\frac{\partial \ln V}{\partial t}=\frac{\partial a}{\partial r}+\frac{a}{r}. \quad (27)$$

In considering sound waves, we should put

$$V=V_0(r)+\Delta V, \qquad H=H_0(r)+\Delta H.$$

Since $H_0V_0 = HV = br$, then

$$\Delta H=-\Delta V\frac{H_0}{V_0}=-\Delta V\frac{br}{V_0^2}, \qquad H=\frac{br}{V_0}\left(1-\frac{\Delta V}{V_0}\right).$$

With this, the system of equations beomes

$$\frac{\partial \xi}{\partial r}=\frac{r}{V_0}\frac{\partial \Delta V}{\partial t},$$

$$\frac{\partial \xi}{\partial t}=-\frac{b^2c^2r^2}{2\pi V_0 w_0^*}+\frac{r\omega_0^{*2}}{V_0}\frac{\partial \Delta V}{\partial r}, \qquad \text{where } \xi=ar.$$

It is obvious that, by introducing

$$\eta=\Delta V-\frac{b^2c^2}{2\pi}\int\frac{r\,dr}{w_0^*\omega_0^{*2}}=\Delta V-\frac{b^2}{4\pi}\int\frac{V_0\,dr^2}{AkV_0^{2-k}+b^2r^2/4\pi}, \quad (28)$$

we arrive at a simpler system of equations

$$\frac{\partial \xi}{\partial r}=\frac{r}{V_0}\frac{\partial \eta}{\partial t}, \qquad \frac{\partial \xi}{\partial t}=\frac{r}{V_0}\omega_0^{*2}\frac{\partial \eta}{\partial r}.$$

Hence, eliminating η, we obtain the equation

$$\frac{\partial^2 \xi}{\partial r^2}+\frac{d\ln(V_0/r)}{dr}\frac{\partial \xi}{\partial r}=\frac{c^2}{\omega_0^{*2}}\frac{\partial^2 \xi}{c^2\,\partial t^2},$$

the solution of which is also possible in explicit form for the set of relations $V_0 = V_0(r)$.

Relation (28) can be meaningful only under the condition that both terms in the right half are of the same order of smallness. This can occur either if b is small, i.e., the field is weak, or if we assume that

$$\frac{\partial \Delta V}{\partial r}\sim\frac{b^2}{2\pi}\frac{V_0 r}{kAV_0^{2-k}+b^2r^2/4\pi},$$

i.e., by assuming that the derivatives of the functions ΔV and ξ can be arbitrary when b is arbitrary, a fact that leads to high frequency waves of low amplitude.

Depending on the law $V_0 = V_0(r)$, the velocity of sound ω_0^* may increase with the distance, decrease with it, or remain constant as a particular case.

Assuming $\alpha \neq 0$ and specifying $V_0 = \beta r^\nu$ at $k > 1$, we find that if $\nu < 2$, we get $\omega_0^* \to c$ as $r \to \infty$.

If $\nu > 2$, then $\omega_0^* \to 0$ as $r \to \infty$; when $r = 0$ we get in both cases $\omega_0^* = \sqrt{k-1}\,c$. If $\alpha = 0$, we have $\omega_0^* = \text{const}$ when $\nu = 2/(2-k)$.

When $\nu < 2/(2-k)$ and $r \to \infty$ we get $\omega_0^* \to c$; when $r = 0$ we get $\omega_0^* = \sqrt{k-1}\,c$.

When $\nu > 2/(2-k)$ and $r \to \infty$, we have $\omega_0^* = \sqrt{k-1}\,c$; when $r = 0$ we get $\omega_0^* \to c$.

We can, in particular, assume the field to be constant everywhere; then $V_0 \sim r$, i.e., $\nu = 1$.

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[1] Baum, Kaplan, and Stanyukovich, Введение в космическую газодинамику (Introduction to Cosmic Gas Dynamics), Part 3, Ch. 2, Fizmatizdat, 1958.

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