CHARGED PARTICLE ENERGY LOSSES DUE TO EXCITATION OF PLASMA OSCILLATIONS

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Beam electrons and plasma oscillations are regarded as two subsystems. A kinetic equation describing the interaction between the beam and plasma is obtained on the assumption that the beam does not change the properties of the plasma and that the plasma state is specified by its equilibrium parameters. The expression for the decelerating force calculated on the basis of this equation includes losses due to electron-electron collisions as well as those due to the excitation of plasma oscillations. A more general case is considered in which neither of the subsystems is in thermal equilibrium. The solution of a set of nonlinear equations for the beam electron distribution function and the electric potential is considered for this particular case. The results are used to account for the rapid energy transfer from beam electrons to plasma electrons, which was first observed by Langmuir.

In calculating the energy losses of electrons moving through a plasma it is customary to consider separately the losses resulting from short-range interactions (electron-electron collisions) and those resulting from the excitation of plasma oscillations.

The calculation for electron collisions results in the following expression for the decelerating force:

$$F_1 = (e\omega_L/v_0)^2 \ln (r_d/a), \tag{1}$$

where e is the electron charge, $\omega_L = \sqrt{4\pi e^2 n/m}$ is the Langmuir frequency, v_0 is the velocity of electrons entering the plasma, $a = e^2/mv_0^2$ and r_d is the Debye shielding distance. Equation (1) can be obtained from Landau's kinetic equation[1] or from the corresponding Fokker-Planck equation, where (1) represents the systematic frictional force exerted by plasma electrons on a beam electron.[1,9]

The decelerating force resulting from the distant part of the interaction is usually calculated in the approximation of the given particle motion. The following expression is obtained:[2-8]*

$$F_2 = (e\omega_L/v_0)^2 \ln (v_0/v_T). \tag{2}$$

It follows from (1) and (2) that the energy losses resulting from near and distant interactions are of the same order of magnitude.

It was established very early by Langmuir that an electron beam in a plasma is scattered much more rapidly than (1) and (2) indicates. Langmuir suggested that this extremely rapid scattering is associated with the excitation of plasma oscillations; this was later confirmed experimentally (see reference 10, for example).

The problem has been investigated theoretically in papers by Vlasov,[2] Bohm and Gross[3] and others. In these papers it is assumed that the velocities of electrons entering the plasma are modulated as they traverse the double space-charge sheath and that the electrons then form bunches of different densities, as in a klystron oscillator. Regions of maximum beam density are also regions of strong scattering. However, as will be seen below, there is considerable analogy with the operation of a traveling-wave tube.[11] We shall now briefly indicate the results obtained in the two parts of the present work.

Following Bohm and Pines,[12] the Hamiltonian for beam and plasma electrons is

$$H = \frac{1}{2m}\sum_{1\leqslant i\leqslant N}\left(\mathbf{P}_i - \frac{e}{c}\mathbf{A}(\mathbf{q}_i)\right)^2 + \frac{1}{8\pi}\int E^2 d\mathbf{q}, \tag{3}$$

where $\mathbf{A}$ is the vector potential of the longitudinal electric field ($\mathbf{E} = -(1/c)\,\partial \mathbf{A}/\partial t$; $\operatorname{curl} \mathbf{A} = 0$), and N is the number of electrons in the system. With $\mathbf{A}$ as the field coordinate we have the momentum $\Pi = -\mathbf{E}/4\pi c$. The Fourier series for $\mathbf{A}$ and Π are

*The results obtained by various workers differ in the logarithmic term.

$$\mathbf{A}=\sqrt{\frac{4\pi c^2}{V}}\sum_{\mathbf{k},j}\mathbf{a}_\mathbf{k}Q_\mathbf{k}^{(j)}\begin{smallmatrix}\sin\mathbf{kq}\\ \cos\mathbf{kq}\end{smallmatrix},\quad \Pi=\frac{1}{\sqrt{4\pi c^2V}}\sum_{\mathbf{k},j}\mathbf{a}_\mathbf{k}P_\mathbf{k}^{(j)}\begin{smallmatrix}\sin\mathbf{kq}\\ \cos\mathbf{kq}\end{smallmatrix}. \quad (4)$$

Here $\mathbf{a_k}$ is a unit vector. In (4) and hereinafter the upper function pertains to $j = 1$ and the lower function to $j = 2$.

The state variables of the system will be the electron coordinates and momenta, $\mathbf{q}_i$ and $\mathbf{P}_i$, and the coordinates and momenta, $Q_\mathbf{k}^{(j)}$ and $P_\mathbf{k}^{(j)}$, of plasma oscillators with wave numbers $k < k_d$. Here $k_d \approx 1/r_d$. Substituting (4) into (3) and separating terms with $k < k_d$ from those with $k > k_d$, we obtain the following Hamiltonian in linear approximation:

$$H=\sum_i\frac{P_i^2}{2m}-\frac{e}{m}\sqrt{\frac{4\pi}{V}}\sum_{i,k<k_d}(\mathbf{P}_i\mathbf{a}_\mathbf{k})(Q_\mathbf{k}^{(1)}\sin\mathbf{kq}_i+Q_\mathbf{k}^{(2)}\cos\mathbf{k}\,\mathbf{q}_i)$$
$$+\frac{1}{2}\sum_{k<k_d,j}(P_\mathbf{k}^{(j)2}+\omega_L^2Q_\mathbf{k}^{(j)2})+\frac{1}{2}\sum_{i,j}U(|\mathbf{q}_i-\mathbf{q}_j|). \quad (5)$$

Here the first term represents the kinetic energy of the electrons, the third term is the energy of the plasma oscillations with frequency ω_L and $k < k_d$, the second term is the interaction energy of plasma oscillations and electrons, and the last term represents the screened (near) part of the electron interaction energy.

We introduce the distribution function of electrons and plasma oscillations $f(\mathbf{q}_i, \mathbf{P}_i, Q_\mathbf{k}^{(j)}, P_\mathbf{k}^{(j)}, t)$, which specifies the probabilities of different states of the system. By means of (5) we obtain the following expression for f:

$$\frac{\partial f}{\partial t}+\sum_i\left\{\frac{\mathbf{P}_i}{m}-\frac{e}{m}\sqrt{\frac{4\pi}{V}}\sum_{\mathbf{k},j}\mathbf{a}_\mathbf{k}Q_\mathbf{k}^{(j)}\begin{smallmatrix}\sin\mathbf{kq}_i\\ \cos\mathbf{kq}_i\end{smallmatrix}\right\}\frac{\partial f}{\partial\mathbf{q}_i}$$
$$+\sum_{\mathbf{k},j}\left\{P_\mathbf{k}^{(j)}\frac{\partial f}{\partial Q_\mathbf{k}^{(j)}}-\omega_L^2Q_\mathbf{k}^{(j)}\frac{\partial f}{\partial P_\mathbf{k}^{(j)}}\right\}$$
$$-\sum_i\frac{\partial}{\partial\mathbf{q}_i}\sum_jU(|\mathbf{q}_i-\mathbf{q}_j|)\frac{\partial f}{\partial\mathbf{P}_i}$$
$$\pm\frac{e}{m}\sqrt{\frac{4\pi}{V}}\sum_{i,\mathbf{k},j}\left\{(\mathbf{P}_i\mathbf{a}_\mathbf{k})\mathbf{k}Q_\mathbf{k}^{(j)}\begin{smallmatrix}\sin\mathbf{kq}_i\\ \sin\mathbf{kq}_i\end{smallmatrix}\frac{\partial f}{\partial\mathbf{P}_i}\right.$$
$$\left.+(\mathbf{P}_i\mathbf{a}_\mathbf{k})\begin{smallmatrix}\sin\mathbf{kq}_i\\ \cos\mathbf{kq}_i\end{smallmatrix}\frac{\partial f}{\partial P_\mathbf{k}^{(j)}}\right\}=0. \quad (6)$$

In the first part of the present paper we obtain from (6) approximate kinetic equations for the electron distribution function $f_1(\mathbf{q}, \mathbf{P}, t)$ and for $F_1(Q_\mathbf{k}^{(i)}, P_\mathbf{k}^{(i)}, t)$, which is the coordinate and momentum distribution of plasma oscillations with the wave vector $\mathbf{k}$.

The kinetic equation obtained for f_1 differs from the familiar equation of Landau by taking the excitation of plasma waves into account besides electron-electron collisions. In linear approximation this becomes the Fokker-Planck equation in momentum space, with the systematic frictional term consisting of two parts corresponding to (1) and (2) for the decelerating force.

The systematic frictional term in the kinetic equation for F_1 corresponds to the damping coefficient of plasma oscillations obtained by Landau.[13]

As already noted, under certain conditions the transfer of energy from nonequilibrium electrons to plasma electrons occurs at distances considerably smaller than the relaxation lengths obtained by means of (1) and (2). The existence of this Langmuir effect indicates that the kinetic equation used in the first part of the present paper does not determine electron deceleration in all cases. In deriving this kinetic equation for the electrons we assume equilibrium states of the plasma electrons surrounding a given beam electron and of the plasma oscillations. However, with a sufficiently high concentration of nonequilibrium electrons (such as beam electrons entering the plasma) these conditions are not satisfied and a set of simultaneous nonlinear equations for the beam and plasma must be solved to determine the beam deceleration. This will be done in the second part of the present paper.

1. DERIVATION OF KINETIC EQUATIONS FOR f_1 AND F_1

When (6) is integrated over all variables except the coordinate and momentum of a single particle, and then over all variables except the coordinates and momentum of a single oscillator with wave number k, we obtain the first two of a chain of equations for the distribution functions:

$$\frac{\partial f_1}{\partial t}+\frac{\mathbf{P}}{m}\frac{\partial f_1}{\partial\mathbf{q}}-\frac{e}{m}\sqrt{\frac{4\pi}{V}}\int\sum_{\mathbf{k},j}\mathbf{a}_\mathbf{k}Q_\mathbf{k}^{(j)}\begin{smallmatrix}\sin\mathbf{kq}\\ \cos\mathbf{kq}\end{smallmatrix}\frac{\partial\Phi_2}{\partial\mathbf{q}}dQ_\mathbf{k}^{(j)}\,dP_\mathbf{k}^{(j)}$$
$$\pm\frac{e}{m}\sqrt{\frac{4\pi}{V}}\sum_{\mathbf{k},j}\int(\mathbf{Pa}_\mathbf{k})\,\mathbf{k}Q_\mathbf{k}^{(j)}\begin{smallmatrix}\cos\mathbf{kq}\\ \sin\mathbf{kq}\end{smallmatrix}\frac{\partial\Phi_2}{\partial\mathbf{P}}dQ_\mathbf{k}^{(j)}\,dP_\mathbf{k}^{(j)}$$
$$-n\frac{\partial}{\partial\mathbf{q}}\int U(|\mathbf{q}-\mathbf{q}'|)\frac{\partial f_2}{\partial\mathbf{P}}d\mathbf{q}'\,d\mathbf{P}'=0, \quad (7)$$

$$\frac{\partial F_1}{\partial t}+\sum_j P_\mathbf{k}^{(j)}\frac{\partial F_1}{\partial Q_\mathbf{k}^{(j)}}-\sum_j\omega_L^2Q_\mathbf{k}^{(j)}\frac{\partial F_1}{\partial P_\mathbf{k}^{(j)}}$$
$$+n\frac{e}{m}\sqrt{\frac{4\pi}{V}}\int\sum_j(\mathbf{P}\,\mathbf{a}_\mathbf{k})\begin{smallmatrix}\sin\mathbf{kq}\\ \cos\mathbf{kq}\end{smallmatrix}\frac{\partial\Phi_2}{\partial P_\mathbf{k}^{(j)}}d\mathbf{q}\,d\mathbf{P}=0. \quad (8)$$

Here $\Phi_2(\mathbf{q}, \mathbf{P}, Q_\mathbf{k}^{(j)}, P_\mathbf{k}^{(j)}, t)$ is the second mixed distribution function, $f_2(\mathbf{q}, \mathbf{P}, \mathbf{q}', \mathbf{P}', t)$ is the second electron distribution function and $n = N/V$ is the average number of electrons per unit volume.

Equations (7) and (8) relate the first and second distribution functions. In a similar manner we can obtain equations for the three functions Φ_2, f_2, F_2 (second distribution functions).

The equations for the second distribution functions contain third distribution functions etc. In order to obtain an approximate closed set of equation we follow Bogolyubov and Gurov[14] in introducing the following approximate distribution functions:

$$f_3 = f_1 f_1 f_1, \quad \Phi_3 = f_1 f_1 F_1 \text{ and so on} \tag{9}$$

$$f_2 = f_1 f_1 + G(\mathbf{q}, \mathbf{P}, \mathbf{q}', \mathbf{P}', t) \tag{10}$$

$$\Phi_2 = f_1 F_1 + g(\mathbf{q}, \mathbf{P}, Q_{\mathbf{k}}^{(j)}, P_{\mathbf{k}}^{(j)}, t), \tag{11}$$

where G and g are correlation functions which are proportional to a small parameter (the ratio of the interaction energy to the kinetic energy).

We shall first consider the case $f_1(\mathbf{q}, \mathbf{P}, t) = f_1(\mathbf{P}, t)$, i.e., a uniform first distribution function of the electrons. In this approximation, using the approximations introduced above for the distribution functions, from (7), (8) and the corresponding equations for the second distribution functions we obtain the following equations for f_1, F_1, g, and G:

$$\frac{\partial f_1}{\partial t} \pm \frac{e}{m}\sqrt{\frac{4\pi}{V}}\sum_{\mathbf{k},j}\int(\mathbf{P}\mathbf{a}_{\mathbf{k}})\,\mathbf{k}Q_{\mathbf{k}}^{(j)}\,{}^{\cos\mathbf{kq}}_{\sin\mathbf{kq}}\,F_1 dQ_{\mathbf{k}}^{(j)}dP_{\mathbf{k}}^{(j)}\frac{\partial f_1}{\partial \mathbf{P}} = \frac{e}{m}\sqrt{\frac{4\pi}{V}}$$
$$\times\sum_{\mathbf{k},j}\int\left\{\mathbf{a}_{\mathbf{k}}Q_{\mathbf{k}}^{(j)}\,{}^{\sin\mathbf{kq}}_{\cos\mathbf{kq}}\frac{\partial g}{\partial \mathbf{q}} \mp \mathbf{k}(\mathbf{P}\mathbf{a}_{\mathbf{k}})\,Q_{\mathbf{k}}^{(j)}\,{}^{\cos\mathbf{kq}}_{\sin\mathbf{kq}}\frac{\partial g}{\partial \mathbf{P}}\right\}dQ_{\mathbf{k}}^{(j)}dP_{\mathbf{k}}^{(j)}$$
$$+ n\frac{\partial}{\partial \mathbf{q}}\int U(|\mathbf{q}-\mathbf{q}'|)\frac{\partial G}{\partial \mathbf{P}}d\mathbf{q}'\,d\mathbf{P}'; \tag{12}$$

$$\frac{\partial F_1}{\partial t} + \sum_j\left(P_{\mathbf{k}}^{(j)}\frac{\partial F_1}{\partial Q_{\mathbf{k}}^{(j)}} - \omega_L^2\,Q_{\mathbf{k}}^{(j)}\frac{\partial F_1}{\partial P_{\mathbf{k}}^{(j)}}\right)$$
$$+ n\frac{e}{m}\sqrt{\frac{4\pi}{V}}\int(\mathbf{P}\mathbf{a}_{\mathbf{k}})^{\sin\mathbf{kq}}_{\cos\mathbf{kq}}f_1 d\mathbf{q}\,d\mathbf{P}\,\frac{\partial F_1}{\partial P_{\mathbf{k}}^{(j)}}$$
$$= -n\frac{e}{m}\sqrt{\frac{4\pi}{V}}\sum_j\int(\mathbf{P}\,\mathbf{a}_{\mathbf{k}})^{\sin\mathbf{kq}}_{\cos\mathbf{kq}}\frac{\partial}{\partial P_{\mathbf{k}}^{(j)}}g\,d\mathbf{q}\,d\mathbf{P}, \tag{13}$$

$$\frac{\partial G}{\partial t} + \frac{\mathbf{P}}{m}\frac{\partial G}{\partial \mathbf{q}} + \frac{\mathbf{P}'}{m}\frac{\partial G}{\partial \mathbf{q}'}$$
$$= \frac{\partial}{\partial \mathbf{q}}U(|\mathbf{q}-\mathbf{q}'|)\left\{\frac{\partial f_1}{\partial \mathbf{P}}f_1 - f_1\frac{\partial f_1}{\partial \mathbf{P}'}\right\} = 0; \tag{14}$$

$$\frac{\partial g}{\partial t} + \frac{\mathbf{P}}{m}\frac{\partial g}{\partial \mathbf{q}} + \sum_j\left\{P_{\mathbf{k}}^{(j)}\frac{\partial g}{\partial Q_{\mathbf{k}}^{(j)}} - \omega_L^2\,Q_{\mathbf{k}}^{(j)}\frac{\partial g}{\partial P_{\mathbf{k}}^{(j)}}\right\}$$
$$= \mp\frac{e}{m}\sqrt{\frac{4\pi}{V}}\sum_j(\mathbf{P}\mathbf{a}_{\mathbf{k}})\,\mathbf{k}Q_{\mathbf{k}}^{(j)}\,{}^{\cos\mathbf{kq}}_{\sin\mathbf{kq}}\frac{\partial f_1}{\partial \mathbf{P}}F_1$$
$$- \frac{e}{m}\sqrt{\frac{4\pi}{V}}\sum_j(\mathbf{P}\,\mathbf{a}_{\mathbf{k}})^{\sin\mathbf{kq}}_{\cos\mathbf{kq}}\frac{\partial F_1}{\partial P_{\mathbf{k}}^{(j)}}f_1. \tag{15}$$

If the initial distributions for G and g are known, we obtain equations for f_1 and F_1 by solving (14) and (15) and eliminating G and g from (12) and (13). Usually only the initial values of f_1 and F_1 are known, in which case, as in reference 15, we can obtain approximate kinetic equations for f_1 and F_1, which are valid only in such large time intervals that the initial values of G and g are no longer significant.

The solution of (14) in this approximation can be represented by

$$G = \frac{\partial}{\partial \mathbf{q}}\int_0^\infty U\left(\left|\mathbf{q}-\mathbf{q}' - \frac{(\mathbf{P}-\mathbf{P}')}{m}\tau\right|\right)d\tau\left\{\frac{\partial f_1}{\partial \mathbf{P}}f_1 - f_1\frac{\partial f_1}{\partial \mathbf{P}'}\right\}. \tag{16}$$

By using (16) to eliminate G from the last term of (12), we obtain an expression corresponding to the right-hand side of Eq. (10.21) in Bogolyubov's book.[15] This is the approximate kinetic equation for a system of particles with Coulomb interaction, which was derived by Landau. The only difference is that in our case the expansion of the interaction potential energy contains only terms with wave numbers $k > k_d$. Therefore by linearizing this term and assuming that f_1 differs very little from a Maxwellian distribution, we obtain two terms describing diffusion and systematic friction. The corresponding coefficients are finite for small k since the potential energy $U(|\mathbf{q}-\mathbf{q}'|)$ contains only terms with $k > k_d$ in the expansion according to wave numbers. For large k the region of integration is limited by the condition $k \sim mv_0^2/e^2$. The resulting decelerating force agrees with (1).

In order to obtain a closed equation for f_1 we must eliminate the correlation function g from (12). To obtain an equation for f_1 that is accurate up to quadratic terms in the ratio of potential energy to kinetic energy, we may substitute for F_1 in (15) the equilibrium distribution for oscillations:

$$F_1^{(0)} = A\exp\{-P_{\mathbf{k}}^{(j)2}/2\varkappa T - \omega_L^2\,Q_{\mathbf{k}}^{(j)2}/2\varkappa T\}. \tag{17}$$

The solution for g then becomes

$$g = \mp\frac{e}{m}\sqrt{\frac{4\pi}{V}}\int_0^\infty\sum_j(\mathbf{P}\mathbf{a}_{\mathbf{k}})\,\mathbf{k}\left(Q_{\mathbf{k}}^{(j)}\cos\omega_L\tau - \frac{P_{\mathbf{k}}^{(j)}}{\omega_L}\sin\omega_L\tau\right)$$
$$\times{}^{\cos}_{\sin}\left[\mathbf{k}\left(\mathbf{q}-\frac{\mathbf{P}}{m}\tau\right)\right]d\tau F_1^{(0)}\frac{\partial f_1}{\partial \mathbf{P}} + \frac{e}{m}\sqrt{\frac{4\pi}{V}}\frac{1}{\varkappa T}\int_0^\infty\sum_j(\mathbf{P}\,\mathbf{a}_{\mathbf{k}})$$
$$\times{}^{\sin}_{\cos}\left[\mathbf{k}\left(\mathbf{q}-\frac{P}{m}\tau\right)\right]\{P_{\mathbf{k}}^{(j)}\cos\omega_L\tau$$
$$+ Q_{\mathbf{k}}^{(j)}\omega_L\sin\omega_L\tau\}\,d\tau F_1^{(0)}f_1. \tag{18}$$

Substituting (18) into (12) and integrating over $Q_{\mathbf{k}}^{(j)}$ and $P_{\mathbf{k}}^{(j)}$, we obtain the following kinetic equation for f_1:

$$\frac{\partial f_1}{\partial t} = n\frac{\partial}{\partial \mathbf{q}}\int U(|\mathbf{q}-\mathbf{q}'|)\frac{\partial G}{\partial \mathbf{P}}d\mathbf{q}'d\mathbf{P}'$$
$$+ \frac{\partial}{\partial P_\alpha}D_{\alpha\beta}\frac{\partial f_1}{\partial P_\beta} + \frac{\partial}{\partial \mathbf{P}}(\mathbf{A}f_1). \tag{19}$$

Here G is given by (16); for the diffusion coefficient $D_{\alpha\beta}$ and for the coefficient **A** of systematic friction due to the excitation of random plasma oscillations we obtain

$$D_{\alpha\beta}=\frac{e^2\varkappa T}{2\pi}\int\delta(\omega_L-\mathbf{kP}/m)\,a_{k_\alpha}a_{k_\beta}\,d\mathbf{k}, \tag{20}$$

$$\mathbf{A}=\frac{e^2}{2\pi m}\int\mathbf{a_k}(\mathbf{Pa_k})\,\delta(\omega_L-\mathbf{kP}/m)\,d\mathbf{k}. \tag{21}$$

The decelerating force F_2 is obtained from (21)

Transforming to spherical coordinates in (21) with the z axis along P, we obtain

$$F_2=e^2\omega_L\int_{-1}^{1}\int_{0}^{k_d}k\,dk\,\delta\left(\omega_L-\frac{kP}{m}y\right)y\,dy,\qquad y=\cos\theta. \tag{22}$$

It follows from (22) that the decelerating force F_2 differs from zero only when $P/m \geq \omega_L/k$, which indicates that particles with the momentum P can excite waves only with the wave number $k > \omega_L m/P$. We therefore have

$$F_2=\frac{e^2\omega_L^2}{v^2}\int_{\omega_L/v}^{k_d}\frac{dk}{k}=\frac{e^2\omega_L^2}{v^2}\ln\frac{v}{v_T}. \tag{23}$$

Here $v = P/m$ and $v_T = r_d/\omega_L$ is the thermal velocity. The total force of systematic friction is given by the sum of F_1 and F_2 and does not depend on the choice of k_d. (Compare with the corresponding results obtained by Vlasov in reference 2.)

By using (7) and the corresponding equation for the second distribution functions we can obtain a kinetic equation for f_1 in the inhomogeneous case. This complicated equation will not be presented here, but in the second part of the present paper we shall use a specific example to show that under certain conditions when the inhomogeneity of the distribution function is taken into account the decelerating force acting on beam electrons can be considerably greater than would follow (1) and (23).

We shall now consider the diffusion coefficient. The only nonvanishing terms in (20) are those with $\alpha = \beta$. With the z axis along P, the integral gives

$$D_{33}=\frac{e^2\varkappa T}{v^3}\omega_L^2\ln\frac{v}{v_T},$$

$$D_{11}=D_{22}=\frac{me^2\omega_L^2}{4v}\quad\text{for } v\gg v_T. \tag{24}$$

In reference 9 Temko has calculated the diffusion coefficients associated with the screened part of the interaction.

We note that in the stationary case the kinetic equation (19) is satisfied by a Maxwellian distribution.

We shall now consider the kinetic equation for the coordinate and momentum distribution function of plasma oscillations, which is obtained under the aforementioned assumptions by eliminating the correlation function g from (13) and (15). We insert the equilibrium value of f_1 (the Maxwellian distribution) in the right-hand side of (15). The calculation gives the following equation for F_1:

$$\frac{\partial F_1}{\partial t}+\sum_j\left\{P_{\mathbf{k}}^{(j)}\frac{\partial F_1}{\partial Q_{\mathbf{k}}^{(j)}}-\left[\omega_{\mathbf{k}}^2Q_{\mathbf{k}}^{(j)}+\varkappa T\frac{\omega_{\mathbf{k}}^2}{\omega_L^2}\frac{\partial}{\partial Q_{\mathbf{k}}^{(j)}}\right]\frac{\partial F_1}{\partial P_{\mathbf{k}}^{(j)}}\right\}$$
$$=2\gamma\varkappa T\sum_j\frac{\partial^2F_1}{\partial P_{\mathbf{k}}^{(j)2}}+2\gamma\sum_j\frac{\partial}{\partial P_{\mathbf{k}}^{(j)}}(P_{\mathbf{k}}^{(j)}F_1). \tag{25}$$

Here

$$\omega_k^2=\omega_L^2+\frac{3\varkappa T}{m}k^2,\qquad \gamma=\sqrt{\frac{\pi}{8}}\frac{\omega_L}{r_d^3k^3}\exp(-1/2r_d^2k^2). \tag{26}$$

From (25) we obtain an equation for the amplitudes $Q_{\mathbf{k}}^{(j)}$, averaged by means of the distribution function F_1, of plasma oscillations with different numbers:

$$\ddot{\overline{Q}}_{\mathbf{k}}^{(j)}+2\gamma\dot{\overline{Q}}_{\mathbf{k}}^{j}+\omega_{\mathbf{k}}^2\overline{Q}_{\mathbf{k}}^{(j)}=0,\qquad \overline{Q}_{\mathbf{k}}=\int Q_{\mathbf{k}}F_1dQ_{\mathbf{k}}\,dP_{\mathbf{k}}. \tag{27}$$

The damping coefficient γ of plasma oscillations agrees with that calculated by Landau.[13] We also obtain various statistical parameters of plasma oscillations from (25).

2. NONLINEAR THEORY OF PLASMA OSCILLATIONS EXCITED BY AN ELECTRON BEAM

The kinetic equations for f_1 and F_1 were derived on the assumption that at initial time the plasma oscillations (in the case of f_1) or the electrons (in the case of F_1) are in thermal equilibrium. There are many problems in which this is not the case and neither of the subsystems (electrons and plasma oscillations) is in thermal equilibrium. We shall now consider one of these problems.

An electron beam enters the plasma in the x direction at the point $x = 0$ with a velocity that exceeds the thermal velocity of plasma electrons. We shall show that the decelerating force acting on the beam electrons due to the excitation of plasma oscillations is considerably greater than that given by (1) and (2).

We at the very start separate beam electrons and plasma electrons in the Hamiltonian (5). Then the two equations (7) and (8) for the first distribution functions are supplemented by another equation for the beam electron distribution function.

When determining the decelerating forces in the homogeneous case considered above it was important to take into account the correlation between plasma electron variables and the variables pertaining to plasma oscillations, since in this homogeneous case the self-consistent term vanishes.

In the inhomogeneous case, with organized oscillations of the entire system excited at the expense of electron beam energy, in approximating the second distribution function of electrons and plasma oscillations we may set

$$\Phi_2(q, \mathbf{P}, Q_{\mathbf{k}}^{(j)}, P_{\mathbf{k}}^{(j)}, t) = f_1 F_1, \quad (28)$$

since in the inhomogeneous case this multiplicative term is the principal term.

Taking (28) for the second distribution function, we arrive at a system of self-consistent equations for the distribution functions of electrons and plasma oscillations. We do not present these equations because, farther along, in order to simplify a comparison of our results with those of other writers, we shall use a system of self-consistent equations for the electron distribution function and scalar electric potential which were first investigated by Vlasov and thereafter by many other writers:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{\partial \varphi}{\partial x}\frac{\partial f}{\partial v} = 0, \quad (29)$$

$$\frac{\partial^2 \varphi(x, t)}{\partial x^2} = 4\pi e\left\{\int_{-\infty}^{\infty} f(x, v, t)\, dv - n_+\right\}. \quad (30)$$

We assume that the charge of the electrons is neutralized by the positive ion background.

In solving our problem of electron deceleration through the excitation of plasma waves we must obtain a wave solution of (29) and (30) which satisfies the given boundary conditions at $x = 0$. If φ is assumed to be a known function, by solving (29) with respect to φ and eliminating f from (30) we obtain an equation for the electric potential.

From the solution of the linearized equations (29) and (30) it follows[3,11,16-18] that longitudinal plasma waves are generated growing in the x direction, the phase velocity of which is smaller than the average velocity v of the beam electrons. The rate of growth of the plasma waves depends on the velocity and concentration of the beam electrons, and with a sufficiently small concentration the growth may be as small as desired.

For a sufficiently slow growth of plasma waves the solution for the potential in nonlinear approximation can be obtained in the form

$$\varphi(x, t) = \varphi_0(x)\sin(\omega t - kx + \Psi(x)), \quad (31)$$

where $\varphi_0(x)$ and $\Psi(x)$ are the slowly varying amplitude and phase, respectively.

For a steady wave, i.e., the amplitude and phase are independent of x, the potential is a function of only $x - v_{ph}t$. Then the solution of (29) with respect to φ becomes

$$f(x, v, t) = \Phi\left(\pm\left[(v - v_{ph})^2 - 2\frac{e}{m}\varphi(x - v_{ph}t)\right]^{1/2} + v_{ph}\right), \quad (32)$$

where Φ is an arbitrary function; the + sign is taken for $v > v_{ph}$ and the − sign for $v < v_{ph}$. Eliminating f from (30) by means of (32), we obtain the following equation for φ:

$$\frac{\partial^2 \varphi}{\partial x^2} = 4\pi e\left\{\int \Phi(v)\left[1 + \frac{2e\varphi(x - v_{ph}t)}{m(v - v_{ph})^2}\right]^{-1/2} dv - n_+\right\}. \quad (33)$$

With a Maxwellian distribution used for Φ, (33) agrees with the equation given by Bohm and Gross.[3]

Akhiezer and Lyubarskiĭ[19] have solved an equation similar to (33)* for zero temperature of plasma and beam electrons. In another paper Akhiezer, Lyubarskiĭ, and Faĭnberg[20] have solved the more general equation for nonzero plasma temperature. In this case Φ may be represented by

$$\Phi = \Phi_0(mv^2/2) + n_1\delta(v - \bar{v}). \quad (34)$$

Here n_1 is the beam electron concentration, $\bar{v}$ is the electron velocity and Φ_0 is an arbitrary function of the energy.

These solutions cannot be used directly in the problem of electron beam deceleration, since with small thermal losses the transfer of energy from the beam to the wave occurs in the region of wave buildup. We must therefore consider the process whereby the wave is established.

It also remains an open question whether the solution for a growing wave will approach a solution satisfying (33). Yet from this particular solution we can infer that different conditions govern the application of the linear approximation to plasma and beam electrons. Indeed, when Φ in the right-hand side of (33) is replaced by (34) or a more general expression allowing for the thermal spread of beam electrons, it is easily seen that when $v_{ph} \gg \sqrt{\kappa T/m}$ the linear approximation is valid for plasma electrons if $e\varphi \ll mv_{ph}^2/2$ and for beam electrons if $e\varphi \ll m(v_{ph} - \bar{v})^2/2$. When $m(v_{ph} - \bar{v})^2 \ll mv_{ph}^2$ nonlinear effects will be manifested for beam electrons at considerably lower potentials than for plasma electrons. The use of the nonlinear equation only for plasma electrons can, of course, be justified only if the solution yields a steady value of the amplitude such that $e\varphi_{st} \ll mv_{ph}^2/2$.

Thus for a low concentration of beam electrons (but sufficiently large to maintain plasma oscillations at the expense of beam energy), we can replace (29) and (30) by a set of equations for the beam electron distribution f_1 alone, with φ given by the wave equation for the plasma wave. The phase velocity and damping coefficient of the plasma wave will then be taken from the linear theory of plasma oscillations.[2,13]

*The equations differ because of different constants of integration.

We thus arrive at the following set of equations:

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} + \frac{e}{m}\frac{\partial \varphi}{\partial x}\frac{\partial f_1}{\partial v} = 0, \tag{35}$$

$$\frac{\partial^2 \varphi}{\partial x^2} + 2\frac{\gamma}{v_\Phi}\frac{\partial \varphi}{\partial x} - \frac{1}{v_\Phi^2}\frac{\partial^2 \varphi}{\partial t^2} = 4\pi e\left\{\int f_1 dv - n_{+1}\right\}, \tag{36}$$

$$\varphi = \varphi_0(x)\sin(\omega_k t - kx + \Psi(x)). \tag{37}$$

Here ω_k has its previous meaning given in (26); γ is the damping coefficient, which includes both that obtained by Landau [see (26)] and possible damping resulting from collisions. The functions $\varphi_0(x)$ and $\Psi(x)$ are, as previously, the slowly varying amplitude and phase.

With (37) for the potential, we solve (35) for a given beam electron distribution function at $x = 0$. We denote the known function f_1 at $x = 0$ by $f_1^{(0)}(v^{(0)})$, with $\int f_1^{(0)} dv^{(0)} = n_{+1}$.* The superscript 0 pertains to the point $x = 0$.

Since, for a given φ, (35), is a first-order linear differential equation, its solution is determined by the solution of the characteristic equation. In view of the slow amplitude and phase variations this equation can be written as

$$\frac{d^2x}{dt} \approx -\frac{ek}{m}\varphi_0(x)\cos(\omega_k t - kx + \Psi(x)). \tag{38}$$

When the functions $v^{(0)}(t, x, v)$ and $t^{(0)}(t, x, v)$ are obtained from (38), the solution of (35) can be written as

$$f_1(x, v, t) = f_1^{(0)}(v^{(0)}(x, v, t)).$$

Using this solution, we obtain the following expressions for the density and current:

$$\rho = -e\int f_1^{(0)}(v^{(0)}(x, v, t))\,dv;$$

$$j = -e\int v f_1^{(0)}(v^{(0)}(x, v, t))\,dv.$$

Substituting this expression for ρ into the right-hand side of (36), we obtain a nonlinear equation for the potential. For low electron beam intensity the right-hand side of this equation is small; we can therefore apply a familiar method in the theory of nonlinear oscillations to obtain simpler equations for the wave amplitude and phase. This necessitates finding the Fourier components of ρ, assuming the beam amplitude and phase to be constant during integration.

Let

$$\rho = \rho^{(1)}\cos(\omega_k t - kx) + \rho^{(2)}\sin(\omega_k t - kx);$$

then

$$\rho^{(i)} = -\frac{e}{\pi}\int_0^{2\pi}\int {\cos \atop \sin}[\omega_k t - kx] f_1^{(0)}(v^{(0)}(x, v, t))\,dv\,d(kx),$$
$$i = 1,2. \tag{39}$$

Equation (39) can be simplified by using the Liouville theorem $dx\,dv = dx^{(0)}dv^{(0)}$, or since $dx = v\,dt$, $dx^{(0)} = v^{(0)}dt^{(0)}$, we have $dx\,dv = v^{(0)}dt^{(0)}dv^{(0)}$. After the substitution of variables $t, v \to t^{(0)}, v^{(0)}$ in (39) we obtain the following expression for the Fourier components of the density:

$$\rho^{(i)} = -\frac{e}{\pi v_{ph}}\int_0^{2\pi}\int {\cos \atop \sin}[\omega_k t(t^{(0)}, v^{(0)}, x) - kx] f_1^{(0)}(v^{(0)})\,v^{(0)}dv^{(0)}d(\omega_k t^{(0)}). \tag{40}$$

We can obtain $\rho^{(i)}$ from (40) if $t(t^{(0)}, v^{(0)}, x)$ is known. When we return to the equation of motion (38) and assume that the variable t in the right-hand side of this equation is a known function of the coordinates $v^{(0)}$ and $t^{(0)}$, the energy integral of (38) can be represented as

$$v = v^{(0)}\left\{1 - \frac{2ek}{mv^{(0)2}}\int_0^x \varphi_0(x')\cos[\omega_k t^{(0)} + \omega_k(t - t^{(0)}) - kx' + \Psi(x')]\,dx'\right\}^{1/2},$$

and for the function $t(t^{(0)}, v^{(0)}, x)$ we obtain the integral equation

$$t - t^{(0)} = \frac{1}{v^{(0)}}\int_0^x\left\{1 - \frac{2ek}{mv^{(0)2}}\int_0^{x'}\varphi_0(x'')\cos[\omega_k t^{(0)} + \omega_k(t - t^{(0)}) - kx'' + \Psi(x'')]\,dx''\right\}^{-1/2}dx'. \tag{41}$$

Because of the complexity of (41) we can solve our problem without the use of numerical methods only in certain special cases. We shall now consider some of these cases.

Let μ be a small parameter. We consider the case where the steady amplitude of the oscillations is such that $e\varphi_0/mv^{(0)2} \sim \mu^2$ and the maximum of the excitation pertains to the waves for which $(v^{(0)} - v_{ph})v^{(0)} \sim \mu$. Under these conditions and with slowly varying amplitude and phase $\varphi_0(x)$ and $\Psi(x)$, (41) can be simplified by expanding the square root in a series of which only the first two terms are retained. (41) then becomes

$$t - t^{(0)} = \frac{x}{v^{(0)}} + \frac{ek}{mv^{(0)3}}\int_0^x (x - x')\varphi_0(x')\cos[\omega_k t^{(0)} + \omega_k(t - t^{(0)}) - kx' + \Psi(x')]\,dx'. \tag{42}$$

Equation (42) contains two parameters of length. One of these parameters, $A = k(v^{(0)} - v_{ph})/v^{(0)}$ is determined from the excess of the electron stream velocity over the phase velocity of the rapidly growing plasma wave at $x = 0$; the second

*If the electron beam is modulated with respect to density or velocity the boundary form of the distribution will, of course, differ.

parameter α characterizes the rate of change of wave amplitude and phase. Let us now consider the case when $\alpha/A \sim \mu$.

We denote the ratio $e\varphi_0/m(v^{(0)}-v_{ph})^2$ by X and obtain an approximate solution of (42) as a power series in X, assuming $X \lesssim 1$. Retaining terms up to X^3 inclusively, we obtain the following expression for $t-t^{(0)}$:

$$\omega(t-t^{(0)}) = \frac{\omega x}{v^{(0)}} - \left(X - \tfrac{5}{16}X^3\right)\cos[\omega t^{(0)} - Ax + \Psi] - \tfrac{1}{8}X^2 \sin 2[\omega t^{(0)} - Ax + \Psi]. \tag{43}$$

Substituting this into the integrand of (40), we integrate over $t^{(0)}$, separate the terms in X to X^3 and use the formula

$$\int \frac{f_1(v-\bar{v})}{(v-v_{ph})^n}dv = ⨍ \frac{f_1(v-\bar{v})}{(v-v_{ph})^n}dv + i\pi \frac{1}{(n-1)!}\left(\frac{\partial^n f_1}{\partial v^n}\right)_{v=v_{ph}},$$

in which $⨍$ denotes that the principal value of the integral is taken. In view of (39) we now obtain the following expression for the beam density:

$$\rho = -\frac{e^2}{m}⨍\left[1 - \frac{3}{8}\left(\frac{e\varphi_0}{m(v-v_{ph})^2}\right)^2\left(\frac{v\Phi}{v}\right)\right]^2 \frac{f_1(v-\bar{v})}{(v-v_{ph})^2}dv\cdot\varphi + \frac{\pi e^2}{mk}\left[\left(\frac{\partial f_1}{\partial v}\right)_{v=v_{ph}} - \frac{3}{8\cdot 4!}\left(\frac{e\varphi_0}{m}\right)^2\left(\frac{\partial^5 f_1}{\partial v^5}\right)_{v=v_{ph}}\right]\frac{\partial\varphi}{\partial x}. \tag{44}$$

Substituting this into the right-hand side of (36), we obtain a nonlinear equation for the potential when the solution of this equation is sought in the form (37).

When the ratio of beam and plasma electron concentrations is such that the parameter characterizing slowness of wave amplitude and phase variations is of the same order of magnitude as the parameter characterizing smallness of the right-hand side of the equation for φ, we equate terms of the same order of smallness and obtain the following equations for the wave amplitude and phase:

$$d\varphi_0/dx = \alpha\varphi_0 - \beta\varphi_0^3, \tag{45}$$

$$\frac{d\Psi}{dx} = -\frac{2\pi e^2}{mk}⨍\left[1 - \frac{3}{8}\left(\frac{e\varphi_0}{m(v-v_{ph})^2}\right)^2\left(\frac{v_{ph}}{v}\right)^2\right]\frac{f_1(v-\bar{v})}{(v-v_{ph})^2}dv. \tag{46}$$

The following notation has been used in (45):

$$\alpha = \frac{2\pi^2 e^2}{mk}\left(\frac{\partial f_1}{\partial v}\right)_{v=v_{ph}} - \frac{\gamma}{v_{ph}}, \quad \beta = \frac{3}{8\cdot 4!}\frac{2\pi^2 e^2}{mk}\left(\frac{e}{m}\right)^2\left(\frac{\partial^5 f_1}{\partial v^5}\right)_{v=v_{ph}}.$$

For self excitation of oscillations the coefficient α must be positive. When the damping coefficient γ is given essentially by the damping coefficient of plasma oscillations, i.e., when collisions play a small part, the condition for self excitation becomes

$$(\partial f^{(0)}/\partial v)_{v=v_{ph}} > 0,$$

where $f^{(0)}$ is the distribution function of all electrons (of both the plasma and beam) at $x = 0$. This self-excitation condition corresponds to that given by other authors.[3,16-18]

To a sufficient degree of accuracy, the distribution $f_1^{(0)}$ can now be specified as

$$f_1^{(0)} = n_1(m/2\pi\kappa T_1)^{1/2}\exp[-m(v-\bar{v})^2/2\kappa T_1].$$

Here n_1, T_1 and $\bar{v}$ are the concentration, temperature, and velocity of beam electrons. With this boundary distribution function the autoexcitation condition is satisfied for the phase velocity region $\Delta_{ph}v \sim \sqrt{\kappa T_1/m}$. The coefficient α is maximal for a wave with the phase velocity $\bar{v} - v_{ph} = \sqrt{T_1\kappa/m}$. Since $\bar{v}$, $v_{ph} \gg \sqrt{\kappa T_1/m}$ and $v_{ph} = \omega_k/k \approx \omega_L/k$, we obtain $k \approx \omega_L/\bar{v}$ as the wave number of the most rapidly growing wave. For this wave we have

$$\alpha = k\left[\sqrt{\frac{\pi}{22}}\frac{n_1}{n}\frac{\bar{v}^2 m}{\kappa T_1} - \frac{\gamma}{\omega_L}\right], \quad \beta = \frac{3}{32}\sqrt{\frac{\pi}{22}}\frac{n_1}{n}\frac{\bar{v}^2 m e^2}{(\kappa T_1)^3}.$$

It follows from the expression for α that for given values of the parameters $\bar{v}$, n, T_1 and γ a lower limit always exists for the concentration of beam electrons which can accompany autoexcitation of oscillations. At lower concentrations an equilibrium velocity distribution for electron motion through the plasma is established only as a result of the relaxation processes described in the first part of the present paper.

The solution of (45) for the amplitude is given by

$$\varphi_0(x) = \varphi^{(0)}e^{\alpha x}\left[1 + \frac{\alpha}{\beta}\ ^{(0)2}(e^{2\alpha x} - 1)\right]^{-1/2}. \tag{47}$$

Here $\varphi^{(0)}$ is the amplitude at $x = 0$. Let φ_{st} be the steady-state value of the amplitude. For small x $(x \ll 1/\alpha)$ the solution of (47) increases exponentially with x. For large x the amplitude approaches

$$\varphi_{st} = \sqrt{\alpha/\beta}. \tag{48}$$

When γ is so small that the second term in the expression for α can be neglected we have $e\varphi_{st} \approx 3\kappa T_1$.

It is evident from the above equations that the steady-state wave amplitude approaches zero for $T_1 \to 0$ i.e., for a single-velocity beam. It follows from the expression for α that in this case the self-excitation condition for plasma waves is not fulfilled. This does not mean, of course, that plasma waves are not excited when a single-velocity beam passes through a plasma. The solution being considered here is obtained when $\alpha \ll A$ and $X \lesssim 1$, under which conditions plasma waves actually do not arise in the hydrodynamic approximation. The solution for $\alpha \sim A$ must be considered to describe wave excitation in this case.

We shall now consider (46) for the variation of phase; this equation determines the variation of plasma wave number with increasing x ($\Delta k = -d\Psi/dx$). It follows from (46) that Δk is given by two terms, one of which depends on the amplitude. Both terms are of the order of $\mu^2 k$ and are thus small compared with Δk, defined as the initial difference between the velocity of beam electrons and the wave velocity, which is of the order of μ.

We now estimate the distance in which the energy of beam electrons is transformed into the energy of plasma oscillations. For this purpose we require the ratio of the flux of plasma wave electrical energy in the region where a steady wave has already been established, to the electronic energy flux at $x = 0$. This ratio is represented by

$$S_{\text{wave}}/S_{\text{el}} = (n/n_1)(e\varphi_0/m\bar{v}^2)^2. \qquad (49)$$

The order of magnitude of the ratio is estimated as follows. Since $e\varphi_0/m\bar{v}^2 \sim \mu^2$ and $\alpha/k \sim \mu^2$, it follows from the expression for α that the ratio of beam and plasma electron concentrations is $n_1/n \sim \mu^4$. From (49) we find that for these values of the parameters the ratio of the energy fluxes is of the order of unity. Thus the beam energy is transformed into plasma wave excitation in the distance l, which is equal in magnitude to the distance within which a wave is established. Denoting the plasma wavelength by λ, we have

$$\lambda \ll l \lesssim 1/\alpha.$$

With currents of 20 – 25 ma, $\kappa T \approx 1$ ev, $m\bar{v}^2/2 \approx 20$ ev, $n_1 = 3 \times 10^8\ \text{sec}^{-1}$ and $n \approx 3 \times 10^{10}\ \text{sec}^{-1}$, l is of the order of a centimeter. When these same numerical data are used, the relaxation length calculated from (1) and (2) is about 10^5 cm.

Looney and Brown[21] have observed standing waves in a plasma traversed by an electron beam. Standing waves arise when a reflecting electrode is present. In order to determine the conditions for the generation of standing plasma waves by an electron beam the solution for the electric field can be obtained in the form

$$\varphi_s = \varphi_0(t)\sin(\omega t + \Psi(t))\sin(s\pi x/L), \qquad s = 1, 2, \ldots,$$

where $\varphi(t)$ and $\Psi(t)$ are the wave amplitude and phase, which vary slowly with time, and L is the length of the plasma in the direction of beam motion. The calculation shows that for $n_1/n \ll 1$ the conditions for autoexcitation are best satisfied for frequencies and wavelengths given by

$$\omega_L \approx s\pi\bar{v}/L, \qquad \lambda_s \approx 2L/s.$$

It follows that a transition from the fundamental oscillatory mode to higher modes occurs only with increased plasma electron concentration or reduced average velocity of beam electrons.

In the experiments of Looney and Brown the beam electron concentration was the basic factor, i.e., $n_1/n \gg 1$. An analysis of this case will regard the beam as the initial wave system. Under these conditions the square of the oscillatory frequency for a given mean velocity will be proportional to the beam electron concentration or to the current.

Spatial periodicity was detected differently in the well-known work of Merrill and Webb,[10] which we shall not discuss here. We note only that growing plasma waves can be detected by measuring the root-mean-square potential difference between two probes. When one of the probes is moved along the beam axis this quantity will be a periodic function of the probe separation with increasing amplitude. The spatial period of this function is the length of the most rapidly growing plasma wave,

$$\lambda = (2\pi\bar{v}/\omega_L)(1 - \sqrt{\kappa T_1/m}/\bar{v}).$$

The experiments of Merrill and Webb also indicated the existence of growing plasma waves; the observed spatial periodicity was in good agreement with the value of λ derived from this formula.

We know that plasma waves can arise at the expense of the energy of relative electron-ion motion. The passage of strong current pulses can in this way cause an appreciable rise of the plasma temperature.[22] Plasma waves can be established by pulses of $\sim 10^{-3}$ sec duration. The amount of energy transformed directly into heat will be determined by the damping rate of such waves.

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