SOVIET PHYSICS JETP VOLUME 36 (9), NUMBER 3 SEPTEMBER, 1959

ON THE APPLICATION OF QUANTUM-FIELD-THEORY METHODS TO PROBLEMS OF QUANTUM STATISTICS AT FINITE TEMPERATURES

A. A. ABRIKOSOV, L. P. GOR' KOV, and I. E. DZYALOSHINSKIĬ

Institute of Physical Problems, Academy of Sciences, U.S.S.R.

Submitted to JETP editor December 4, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 900-908 (March, 1959)

A formulation of thermodynamic perturbation theory is proposed which makes it possible to make full use of quantum-field-theory methods in quantum statistics at finite temperatures. The method is a generalization of the Matsubara technique and is based on the expansion of the Green's functions in Fourier series in the "imaginary time" variable. The technique so obtained differs from the usual diagram technique for T = 0 by the replacement of integration over the frequencies by summation over discrete values of the imaginary frequency. The analytic properties of the Fourier components of the Green's functions are examined. It is shown that owing to the possibility of analytic continuation a knowledge of the corresponding equilibrium Green's functions is sufficient for the solution of various kinetic and nonstationary processes.

1. INTRODUCTION

THE methods of quantum field theory have recently been successfully applied to problems of statistics.[1-4] The application is based on the fact that the apparatus of quantum field theory is developed through the wide use of the "diagram technique," which provides a very intuitive representation of the structure and character of any approximation. Earlier methods of quantum statistics, which started from ideal-gas approximations, could scarcely take the interaction between the particles into account beyond one or two approximations because of the complexity of the older quantum perturbation theory. This is utterly inadequate for the actual many-body problem; to obtain any approximation of physical significance it is necessary to sum over an infinite set of different terms of the perturbation-theory series. The diagram technique is extraordinarily useful in such problems, since it formulates simple rules by which any term of the perturbation theory can be written down.

The basis of the quantum-field-theory methods is the calculation of the so-called Green's function of a particle, which for the case of temperatures different from zero is defined as

$$G(1,2) = -i\,\mathrm{Sp}\,\{\exp[(\Omega+\mu N-\hat{H})/T]\cdot T(\tilde{\psi}(x_1)\tilde{\psi}^+(x_2))\}, \qquad (1)$$

where $\tilde{\psi}$, $\tilde{\psi}^+$ are second-quantization operators in the Heisenberg representation. Knowledge of the Green's function makes it possible easily to calculate all the thermodynamic quantities of the system. For example, the particle number density as a function of the chemical potential μ and the temperature is connected with G by the relation

$$N(\mu, T) = \mp iG(x, x)|_{t'=t+0}$$

(the minus sign is for Fermi statistics, the plus for Bose statistics).

At the absolute zero of temperature the Green's function is calculated by going over to the so-called "interaction representation." In this representation G(1, 2) has the form

$$G(1,2) = -i\langle T(\psi_1\psi_2^+ S)\rangle / \langle S\rangle, \qquad (1a)$$

where $<\dots>$ means the average over the ground state of the system, and S is the well known S matrix of the quantum field theory. Expanding S in powers of the interaction constant, we obtain the usual Feynman diagram technique.

For temperatures other than zero it is already impossible to represent G in a form like Eq. (1a), and consequently it is impossible to calculate G by means of the diagram technique.

Several years ago Matsubara[4] proposed a new formulation of the thermodynamic perturbation theory which is almost completely analogous to the diagram technique of field theory. The present paper is devoted to the description of a technique which is a further development of the Matsubara method. This technique makes it possible to calculate the thermodynamic quantities at finite temperatures with almost the same simplicity with which this is done for T = 0. It will further be shown that by establishing the connection of the quantities that figure in this technique with the corresponding quantities in the usual technique one can determine all the kinetic characteristics of the system.*

2. THE MATSUBARA METHOD

The basic idea of the Matsubara method is the passage from the time t to a certain imaginary "time" τ. We note that the ordinary Green's function G(1, 2) can be represented in the form

$$G(1,2)=\left\{\begin{array}{c} -i\,\mathrm{Sp}\left\{\exp\left[\dfrac{\Omega+\mu N-\hat{H}}{T}\right]\right. \\ \times\exp\left[\dfrac{i(\hat{H}-\mu N)}{\hbar}(t_1-t_2)\right]\psi(\mathbf{r}_1) \\ \left.\times\exp\left[-\dfrac{i(\hat{H}-\mu N)}{\hbar}(t_1-t_2)\right]\psi^+(\mathbf{r}_2)\right\},\quad t_1>t_2, \\ \pm i\,\mathrm{Sp}\left\{\exp\left[\dfrac{\Omega+\mu N-\hat{H}}{T}\right]\right. \\ \times\exp\left[-\dfrac{i(\hat{H}-\mu N)}{\hbar}(t_1-t_2)\right]\psi^+(\mathbf{r}_2) \\ \left.\times\exp\left[\dfrac{i(\hat{H}-\mu N)}{\hbar}(t_1-t_2)\right]\psi(\mathbf{r}_1)\right\},\quad t_1<t_2, \end{array}\right\} \tag{2}$$

where ψ, ψ^+ are the second-quantization operators in the Schrödinger representation.

In the Matsubara method one introduces a temperature-dependent Green's function $\mathfrak{G}$, which is obtained from G by the replacement $t \to -i\tau\hbar$:

$$\mathfrak{G}(1,2)=\left\{\begin{array}{c} -\mathrm{Sp}\left\{\exp\left[\dfrac{\Omega+\mu N-\hat{H}}{T}\right]\right. \\ \times\exp\left[(\hat{H}-\mu N)(\tau_1-\tau_2)\right]\psi(\mathbf{r}_1) \\ \left.\times\exp\left[-(\hat{H}-\mu N)(\tau_1-\tau_2)\right]\psi^+(\mathbf{r}_2)\right\},\tau_1>\tau_2, \\ \pm\mathrm{Sp}\left\{\exp\left[\dfrac{\Omega+\mu N-\hat{H}}{T}\right]\right. \\ \times\exp\left[-(\hat{H}-\mu N)(\tau_1-\tau_2)\right]\psi^+(\mathbf{r}_2) \\ \left.\times\exp\left[(\hat{H}-\mu N)(\tau_1-\tau_2)\right]\psi(\mathbf{r}_1)\right\},\quad \tau_1<\tau_2. \end{array}\right\} \tag{3}$$

Matsubara showed that in the formula (3) one can go over to a sort of "interaction representation". Namely, let us represent the statistical matrix $\hat{\rho} = \exp\{(\mu N - \hat{H})/T\}$ in the form

$$\exp\left[\frac{\mu N-\hat{H}}{T}\right]=\exp\left[\frac{\mu N-\hat{H}_0}{T}\right]\cdot\hat{S}(1/T) \tag{4}$$

and introduce the new operators $\bar{\psi}(\mathbf{r}, \tau)$, $\psi(\mathbf{r}, \tau)$ in the "interaction representation" by the formulas

$$\psi(\mathbf{r}, \tau)=\exp\left[\tau(\hat{H}_0-\mu N)\right]\psi(\mathbf{r})\exp\left[-\tau(\hat{H}_0-\mu N)\right],$$

$$\bar{\psi}(\mathbf{r}, \tau)=\exp\left[\tau(\hat{H}_0-\mu N)\right]\psi^+(\mathbf{r})\exp\left[-\tau(\hat{H}_0-\mu N)\right]. \tag{5}$$

Then, using the formal resemblance of the equation satisfied by the matrix $\hat{S}(1/T)$, $-\partial\hat{S}(\tau)/\partial\tau = H(\tau)\hat{S}(\tau)$, with the Schrödinger equation, we write $\hat{S}(1/T)$, in complete analogy with the usual S matrix, in the form

$$\hat{S}(1/T)=T_\tau\exp\left\{-\int\limits_0^{1/T}\hat{H}_{\mathrm{int}}(\tau)\,d\tau\right\},$$

where T_τ is the time-ordering operator with respect to the imaginary "time" τ. The interaction Hamiltonian $H_{\mathrm{int}}(\tau)$ now depends on the operators $\psi(\mathbf{r}, \tau)$ and $\bar{\psi}(\mathbf{r}, \tau)$ in the same way that it formerly depended on $\psi(\mathbf{r})$ and $\psi^+(\mathbf{r})$.

Now, using the possibility of cyclic permutation of the operators under the sign Sp in Eq. (2), we can show that $\mathfrak{G}$ can be written in a form analogous to that used in the field theory:

$$\mathfrak{G}(1, 2)=-\frac{\mathrm{Sp}\{\exp[(\mu N-\hat{H}_0)/T]T_\tau[\psi_1\bar{\psi}_2\hat{S}(1/T)]\}}{\mathrm{Sp}\{\exp[(\mu N-\hat{H}_0)/T]\hat{S}(1/T)\}}, \tag{6}$$

It is not hard to see that for the potential Ω we can write the following formula:

$$\Omega=\Omega_0-T\ln\left[\frac{\mathrm{Sp}\left\{\exp\left[\dfrac{\mu N-\hat{H}_0}{T}\right]\hat{S}(1/T)\right\}}{\mathrm{Sp}\left\{\exp\left[\dfrac{\mu N-\hat{H}_0}{T}\right]\right\}}\right]. \tag{6a}$$

The expressions (6) and (6a) have the forms of the corresponding formulas of field theory:

$$\mathfrak{G}(1,2)=-\langle T(\psi_1\bar{\psi}_2S)\rangle/\langle S\rangle,\quad \Omega=\Omega_0-T\ln\langle S\rangle, \tag{7}$$

if we understand the averaging symbol $<\ldots>$ to mean the operation

$$\mathrm{Sp}\left\{\exp\left[\frac{\mu N-\hat{H}_0}{T}\right]\ldots\right\}\Big/\,\mathrm{Sp}\left\{\exp\left[\frac{\mu N-\hat{H}_0}{T}\right]\right\}.$$

Matsubara showed that for such averages of T-products of several ψ-operators there exists a "Wick's theorem," according to which the average in question breaks up into a sum of products of averages by pairs of operators:

---

*We have learned that similar results have been obtained by E. S. Fradkin (J. Exptl. Theoret. Phys. (U.S.S.R.), in press).

$$\mathfrak{G}_0(1,2) = -\langle T_\tau(\psi_1\bar{\psi}_2)\rangle.$$

Of course this does not apply to Bose systems below the point of the Bose condensation. Similarly, the Wick's theorem is not applicable to Fermi systems in which superconductivity exists. For the construction of a thermodynamic perturbation theory in this range of temperatures one must use in the first case an obvious generalization of some work of Belyaev,[5] and in the second case, work of Gor'kov.[6]

Expanding the expression for $S(1/T)$ appearing in Eq. (7) in a series of powers of the interaction Hamiltonian and using Wick's theorem, we get the usual diagram technique, in which each line corresponds to a zeroth-order Green's function $\mathfrak{G}_0(1, 2)$. For example, in the case of the four-fermion interaction with

$$\hat{H}_{\text{int}} = \frac{g}{2}\Lambda_{\alpha\beta;\,\gamma\delta}\bar{\psi}_\alpha(x)\bar{\psi}_\beta(x)\psi_\gamma(x)\psi_\delta(x),$$

where $x = \{\mathbf{r}, \tau\}$ and $\Lambda_{\alpha\beta;\gamma\delta}$ is some function of the spin indices antisymmetric in $\alpha\beta$ and $\gamma\delta$,

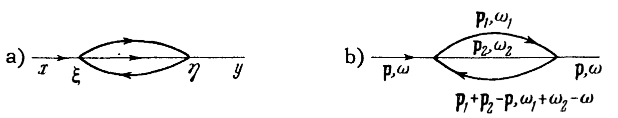

FIG. 1

we find for the second-approximation correction to the Green's function, corresponding to the diagram of Fig. 1a

$$2g^2\Lambda_{\lambda\mu\nu\tau}\Lambda_{\rho\pi\varkappa\omega}\int_0^{1/T}d\tau_\xi\int_0^{1/T}d\tau_\eta\int d\xi\int d\eta\,\mathfrak{G}^0_{\alpha\lambda}(x-\xi)$$
$$\times\,\mathfrak{G}^0_{\nu\rho}(\xi-\eta)\,\mathfrak{G}^0_{\tau\pi}(\xi-\eta)\mathfrak{G}^0_{\varkappa\mu}(\eta-\xi)\,\mathfrak{G}^0_{\omega\beta}(\eta-y). \qquad (8)$$

In the case of the electron-phonon interaction[1,4]

$$\hat{H}_{\text{int}} = g\bar{\psi}\psi\varphi,$$

the first correction to the electron Green's function, corresponding to the diagram of Fig. 2a, has

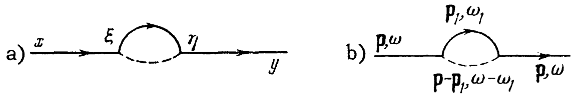

FIG. 2

the form (the dotted line indicates the phonon Green's function $\mathfrak{D}_0 = -\langle T_T(\varphi_1\varphi_2)\rangle$, which is the analogue of the ordinary D function of the phonon):

$$-g^2\int_0^{1/T}d\tau_\xi\int_0^{1/T}d\tau_\eta\int d\xi\int d\eta\,\mathfrak{G}^0$$
$$\times\,(x-\xi)\,\mathfrak{G}^0(\xi-\eta)\,\mathfrak{D}^0(\xi-\eta)\,\mathfrak{G}^0(\eta-y). \qquad (9)$$

The actual calculation of expressions of types (8), (9) and of the more complicated expressions in the higher approximations is, however, very laborious in the Matsubara technique. The point is that the success of field-theory methods in statistics for $T = 0$ is due to the highly automatic nature of the calculations, which is achieved by the use of Fourier expansions of all the quantities with respect to all four coordinates. It is obvious that in the Matsubara method the automatic feature is lacking because of the finiteness of the range of τ, from zero to $1/T$. Indeed in the coordinate representations (as regards τ) $\mathfrak{G}^0$ (and $\mathfrak{G}$) is a discontinuous function of the variable τ; therefore in fact all the integrals over τ break up into integrals over a very large number of regions. The number of such regions obviously increases rapidly with increase of the order of the approximation.

3. THE EXPANSION IN FOURIER SERIES

The Matsubara technique can be decidedly improved by the exploitation of certain general properties of the thermodynamic Green's functions.

As we have shown, by Eq. (3) the Green's function $\mathfrak{G}(1, 2)$ is a function of the difference $\tau_1 - \tau_2$, specified in the interval from $-1/T$ to $1/T$. It is therefore expedient to continue it periodically and expand it in Fourier series in the variable τ:*

$$\mathfrak{G}(\tau) = T\sum_n e^{-i\omega_n\tau}\mathfrak{G}(\omega_n),\quad \omega_n = \pi T n,$$
$$\mathfrak{G}(\omega_n) = \frac{1}{2}\int_{-1/T}^{1/T}e^{i\omega_n\tau}\mathfrak{G}(\tau)\,d\tau. \qquad (10)$$

One major property of the function $\mathfrak{G}$ is essential for the transformation of the perturbation-theory series. It follows from Eq. (3) that the function $\mathfrak{G}$ for negative values of τ is simply related to the $\mathfrak{G}$ for $\tau > 0$, namely we have the relation

$$\mathfrak{G}(\tau) = \mp\,\mathfrak{G}(\tau + 1/T),\quad \tau < 0, \qquad (11)$$

where the minus sign is for Fermi statistics and the plus for Bose statistics. The formula (11) is easily derived if we use the fact that one may cyclically permute the operators under the sign of the trace. The relation (11) is of course also valid for the free Green's functions. For the functions as periodically continued this relation is valid for any value of τ.

---

*Recently I. M. Khalatnikov called our attention to the fact that the idea of expanding the temperature-dependent Green's functions of Matsubara in Fourier series is also contained in a paper by Ezawa, Tomozawa, and Umezawa.[9] These writers, however, being interested only in the application of the Matsurbara method to the problem of multiple production, confined themselves to the case of zero chemical potention. Meanwhile the construction of the technique is nontrivial just in the case $\mu \neq 0$.

If we note further that at each vertex of a Feynman diagram an even number of fermion lines come together, it is easy to see that all the integrals $\int_0^{1/T} \ldots d\tau$ in formulas of the type of Eqs. (8) and (9) can be replaced by $\frac{1}{2}\int_{-1/T}^{1/T} \ldots d\tau$. When this is done the Fourier transformation (10) is easily carried out in all terms of the perturbation-theory series.

The relation (11) also has the consequence that in the Fourier expansion of a fermion Green's function only the components with $\omega_n = \pi T(2n+1)$ are different from zero, and the expansion of a boson Green's function has only components with $\omega_n = \pi T \cdot 2n$. We shall present the expressions for the zeroth-order Green's functions $\mathfrak{G}_0(\omega_n, \mathbf{p})$ (here the Fourier integral transformation for the space coordinates has also been carried out).

Fermions:

$$\mathfrak{G}_0(\omega_n, \mathbf{p}) = [i\omega_n + \mu - \varepsilon(\mathbf{p})]^{-1}, \qquad \omega_n = (2n+1)\pi T$$

(here $\epsilon(\mathbf{p}) = p^2/2m$, and μ is the chemical potential); bosons above the Bose condensation point:

$$\mathfrak{G}_0(\omega_n, \mathbf{p}) = [i\omega_n + \mu - \varepsilon(\mathbf{p})]^{-1}, \qquad \omega_n = 2n\pi T;$$

phonons:

$$\mathfrak{D}_0(\omega_n, \mathbf{k}) = -\omega^2(\mathbf{k}) / [\omega^2(\mathbf{k}) + \omega_n^2], \quad \omega_n = 2n\pi T$$

($\omega(\mathbf{k})$ is the energy of a phonon).

Carrying out the Fourier transformation in all the terms of the expansion of the Green's function in the perturbation-theory series, we can verify that the technique so obtained is entirely equivalent to the diagram technique in the momentum space at absolute zero. To each line of a diagram there corresponds a zeroth-order Green's function $\mathfrak{G}_0(\omega_n, \mathbf{p})$, and to each vertex a δ function expressing the conservation laws $\Sigma p = 0$, $\Sigma\omega_n = 0$. The only difference is that in our case we have instead of an integration over frequency a summation over discrete imaginary "frequencies" $i\omega_n$. For example, the formula (8) for the correction to the electron Green's function from the four-fermion interaction is transformed into the form

$$\Lambda_{\alpha\mu\nu\tau}\Lambda_{\nu\tau\mu\beta}\frac{2g^2T^2}{(2\pi)^6}\sum_{\omega_1\omega_2}\int d^3\mathbf{p}_1 d^3\mathbf{p}_2\,[i\omega_1 + \mu - \varepsilon(\mathbf{p}_1)]^{-1}$$
$$\times [i\omega_2 + \mu - \varepsilon(\mathbf{p}_2)]^{-1}[i(\omega_1 + \omega_2 - \omega) + \mu - \varepsilon(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p})]^{-1}$$
$$\times [i\omega + \mu - \varepsilon(\mathbf{p})]^{-2},$$

corresponding to the diagram of Fig. 1b.

The expression (9) takes the form

$$\delta\mathfrak{G}(\omega, \mathbf{p}) = \frac{g^2T}{(2\pi)^3}\Big\{\sum_{\omega'}\int d^3\mathbf{p}'\,[i\omega' + \mu - \varepsilon(\mathbf{p}')]^{-1}\omega^2(\mathbf{p} - \mathbf{p}')$$
$$\times [\omega^2(\mathbf{p} - \mathbf{p}') + (\omega - \omega')^2]^{-1}\Big\}[i\omega + \mu - \varepsilon(\mathbf{p})]^{-2},$$

which corresponds to the diagram of Fig. 2b.

On passage to the absolute zero of temperature, large values of n play the main part in the sums. Therefore in the perturbation-theory formulas we must make the replacement

$$T\sum_{\omega_n} \to \frac{1}{2\pi}\int_{-\infty}^{\infty} \ldots d\omega.$$

Furthermore, it is obvious that when one replaces all frequencies by imaginary values, $\omega \to -i\omega'$, integrals with respect to $d\omega$ from $-\infty$ to $+\infty$ of products of several factors of the form

$$(i\omega + \mu - \varepsilon(\mathbf{p}))^{-1}$$

go over into integrals of products of factors

$$(\omega' + \mu - \varepsilon(\mathbf{p}))^{-1}, \tag{12}$$

which corresponds to a rotation of the path of integration to real ω. In the case of Fermi Green's functions $\epsilon(\mathbf{p}) - \mu$ can be either positive or negative. Therefore after the indicated transformation of the contour of integration the path around the poles is chosen as shown in Fig. 3a. For Bose particles the chemical potential μ of an ideal gas is always negative; therefore the poles in Eq. (11) lie in the lower half-plane of ω' (Fig. 3b). Conversely, it can be seen from this how from the well known rules of the diagram technique for T = 0 one can write down the corresponding term of the perturbation theory for T ≠ 0. To do this we first make the replacement $\omega' \to i\omega_n$ $(n = (2m+1)$ for Fermi particles and $n = 2m$ for Bose systems) in the corresponding expressions for T = 0, and must then go from integration over the frequencies to summation:

$$\int_{-\infty}^{+\infty} d\omega \to -2\pi T i \sum_{\omega_n}.$$

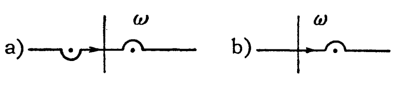

FIG. 3

Everything said up to now has referred to calculations by perturbation theory. In quite a number of problems, however, calculations by perturbation theory have been found to be inadequate. For these purposes it is essential to know certain exact equations connecting the quantities with each other. One such relation is the Dyson equation. As is well known, in field theory one can write an equation connecting the Green's function with the so-called vertex part (the Dyson equation). The analogue of this equation can of course be obtained in the technique proposed here. For the case of the four-fermion interaction it is shown schematically, in the language of the diagrams, in Fig. 4. Here heavy lines mean the complete Green's function $\mathfrak{G}$, and the rhomb means the complete vertex part, i.e., the sum of all connected diagrams having four external fermion lines. In the momentum representation the Dyson equation corresponding to this diagram has the form:

FIG. 4

$$\mathfrak{G}^{-1}_{\alpha\beta}(\omega, \mathbf{p}) = \mathfrak{G}^{0-1}_{\alpha\beta}(\omega, \mathbf{p}) - \frac{2gT}{(2\pi)^3}\sum_{\omega'}\int d^3\mathbf{p}\,\mathfrak{G}_{\nu\mu}(\omega', \mathbf{p}')\,\Lambda_{\alpha\mu;\nu\beta}$$
$$-\frac{2gT^2}{(2\pi)^6}\Lambda_{\alpha\lambda,\mu\nu}\sum_{\omega_1\omega_2}\int d^3\mathbf{p}_1 d^3\mathbf{p}_2\,\mathfrak{G}_{\mu\tau}(\omega_1, \mathbf{p}_1)\,\mathfrak{G}_{\nu\rho}(\omega_2, \mathbf{p}_2)$$
$$\times\mathfrak{G}_{\varkappa\lambda}(\omega_1 + \omega_2 - \omega; \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p})$$
$$\times\mathfrak{T}_{\tau\rho,\varkappa\beta}(\omega_1, \mathbf{p}; \omega_2, \mathbf{p}_2; \omega_1 + \omega_2 - \omega, \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}; \omega, \mathbf{p}).$$

Here $\mathfrak{T}$ is the complete vertex part already mentioned. It is connected with the Fourier components of the two-particle Green's function

$$\mathfrak{G}_{\alpha\beta,\gamma\delta}(x_1, x_2; x_3, x_4) = \langle T(\psi_\alpha(x_1)\,\psi_\beta(x_2)\bar{\psi}_\gamma(x_3)\bar{\psi}_\delta(x_4)\hat{S})\rangle/\langle\hat{S}\rangle$$

by the relation

$$\mathfrak{G}_{\alpha\beta,\gamma\delta}(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2; \omega_3, \mathbf{p}_3, \omega_4, \mathbf{p}_4)$$
$$= \frac{(2\pi)^3}{T}\left\{\frac{(2\pi)^3}{T}\Big[\mathfrak{G}_{\alpha\delta}(\omega_1, \mathbf{p}_1)\mathfrak{G}_{\beta\gamma}(\omega_2, \mathbf{p}_2)\right.$$
$$\times\delta_{\omega_1\omega_4}\delta(\mathbf{p}_1 - \mathbf{p}_4) - \mathfrak{G}_{\alpha\gamma}(\omega_1, \mathbf{p}_1)\,\mathfrak{G}_{\beta\delta}(\omega_2, \mathbf{p}_2)\,\delta_{\omega_1\omega_3}\delta(\mathbf{p}_1 - \mathbf{p}_3)\Big]$$
$$-\frac{1}{2}\mathfrak{G}_{\alpha\lambda}(\omega_1, \mathbf{p}_1)\,\mathfrak{G}_{\beta\mu}(\omega_2, \mathbf{p}_2)\,\mathfrak{T}_{\lambda\mu,\nu\tau}(\omega_1, \mathbf{p}_1; \omega_2, \mathbf{p}_2; \omega_3, \mathbf{p}_3; \omega_4, \mathbf{p}_4)$$
$$\left.\times\mathfrak{G}_{\nu\gamma}(\omega_3, \mathbf{p}_3)\mathfrak{G}_{\tau\delta}(\mathbf{p}_4, \omega_4)\right\}\delta_{\omega_1+\omega_2-\omega_3-\omega_4}\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4).$$

We shall not write out the analogous Dyson equation for the case of the electron-phonon interaction.

Knowing the thermodynamic Green's function $\mathfrak{G}$ one can find thermodynamic functions of the system. For this purpose one can use, for example, the following simple relation

$$\frac{\partial\Omega}{\partial g} = T\frac{n}{g}\sum_{\omega}\int\frac{d^3\mathbf{p}}{(2\pi)^3}[\mathfrak{G}^{0-1}_{\alpha\beta}(\mathfrak{G}_{\beta\alpha} - \mathfrak{G}_{\beta\alpha}{}^0)].$$

Here n is a coefficient depending on the form of the interaction. For the interaction with phonons n = 1; for the four-fermion interaction n = ½.

4. THE ANALYTICAL PROPERTIES OF THE GREEN'S FUNCTIONS AND THE TRANSITION TO TIME-DEPENDENT QUANTITIES

As has already been mentioned, a knowledge of the Green's function is sufficient for the determination of the thermodynamic characteristics of a system. But for the study of various kinetic phenomena, such as scattering or absorption of light or sound, diffusion, and so on, one must know the ordinary time-dependent functions. We shall show how, knowing the thermodynamic one-particle function $\mathfrak{G}(\omega_n, \mathbf{p})$, one can find the time-dependent $G(\omega, \mathbf{p})$.

Beginning with the coordinate space, and comparing Eqs. (2) and (3), we see that in the range of τ from $-1/T$ to $1/T$ the transition from $\mathfrak{G}$ to G is accomplished by replacing τ by it. It is much more interesting, however, to find the connection between the Fourier components $G(\omega, \mathbf{p})$ and $\mathfrak{G}(\omega_n, \mathbf{p})$.

As has been shown by Landau,[7] when the function $G(\omega, \mathbf{p})$ is expanded as a Fourier integral, its Fourier transform is a nonanalytic function of ω. Its real and imaginary parts are connected by the relation

$$G'(\omega, \mathbf{p}) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\coth\frac{x}{2T}\,\frac{G''(x, \mathbf{p})}{x - \omega}\,dx$$

for the Fermi statistics, and the relation

$$G'(\omega, \mathbf{p}) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\tanh\frac{x}{2T}\,\frac{G''(x, \mathbf{p})}{x - \omega}\,dx$$

for Bose statistics.

It can be shown that the Fourier transform of the so-called retarded Green's function $G^R(\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2)$,

$$G^R(1,2) = \begin{cases} -i\,\mathrm{Sp}\left\{\exp\left[\dfrac{\Omega + \mu N - \hat{H}}{T}\right]\right. & \\ \left.\times[\tilde{\psi}(x_1)\tilde{\psi}^+(x_2) \pm \tilde{\psi}^+(x_2)\tilde{\psi}(x_1)]\right\}, & t_1 > t_2, \\ 0, & t_1 < t_2. \end{cases}$$

is analytic in the upper half-plane. Here the plus sign is for Fermi statistics, the minus for Bose statistics. It turns out that the Fourier transform $G^R(\omega, \mathbf{p})$ of this function is simply related to $G(\omega, \mathbf{p})$. Namely, their real parts are identical, and

$$G''^R(\omega, \mathbf{p}) = \coth(\omega/2T)\,G''(\omega, \mathbf{p})$$

for the Fermi statistics, and

$$G''^R(\omega, \mathbf{p}) = \tanh(\omega/2T)\,G''(\omega, \mathbf{p})$$

for the Bose statistics. For $G^R(\omega, \mathbf{p})$ we can write an expansion of the type used by Lehmann[8]

$$G^R(\omega, \mathbf{p}) = \int\limits_{-\infty}^{+\infty} \frac{\rho(x, \mathbf{p})}{x - \omega - i\delta} dx, \qquad (13)$$

where

$$\rho(\omega, \mathbf{p}) = -(2\pi)^3 \sum_{n,m} \exp\left[\frac{\Omega + \mu N_n - E_n}{T}\right] \times |\psi_{mn}(0)|^2 \left(1 \pm \exp\left[-\frac{\omega_{mn}}{T}\right]\right) \times \delta(\mathbf{p} - \mathbf{p}_{mn})\, \delta(\omega - \omega_{mn}). \qquad (14)$$

Here $\omega_{mn} = E_m - E_n - \mu$, $\mathbf{p}_{mn} = \mathbf{p}_m - \mathbf{p}_n$; E_n, $\mathbf{p}_n$ are the energy and momentum in the n-th state of the system. It can be seen from Eq. (14) that $\rho(\omega, \mathbf{p})$ is a real function, for which the integral $\int_{-\infty}^{+\infty} \rho(\omega, \mathbf{p})\, d\omega$ is finite.

Let us write the analogous Lehmann expansion for $G(\omega_n, \mathbf{p})$. By the use of Eq. (6) it is easy to show, by a method analogous to that of reference 7, that

$$\mathfrak{G}(\omega_n, \mathbf{p}) = \int\limits_{-\infty}^{\infty} \frac{\rho(x)\, dx}{x - i\omega_n}. \qquad (15)$$

Comparing Eqs. (13) and (15), we see that

$$\mathfrak{G}(\omega_n) = G^R(i\omega_n), \quad \omega_n > 0. \qquad (16)$$

On the other hand, let us use the integral (15), regarded formally as a function of the complex variable $i\omega_n$, to determine a function $\mathfrak{G}(\omega)$ analytic in the upper half-plane. In virtue of Eq. (16) and the well known theorem of the theory of functions of a complex variable regarding the analytic continuation of a function given at an infinite set of discrete points possessing a point of concentration, we find from the above that G^R is, apart from a constant factor, the analytic continuation of the function $\mathfrak{G}(-i\omega_n)$ into the upper half-plane. Namely, we have

$$G^R(\omega) = \mathfrak{G}(-i\omega). \qquad (17)$$

The converse problem of finding $\mathfrak{G}(\omega_n)$ from a given $G^R(\omega)$ is solved by Eq. (16) and the relation

$$\mathfrak{G}(\omega_n < 0) = \mathfrak{G}^*(|\omega_n|), \qquad (16a)$$

which follows from Eq. (15).

Relations of the types (16) and (17) can in principle be obtained also for the vertex parts Γ and $\mathfrak{T}$. The general relations will, however, be more complicated. For physical applications one ordinarily needs not Γ and $\mathfrak{T}$ themselves, but the correlation functions, i.e., certain integrals of these quantities, and the practical procedure is to establish the connections between them separately for each concrete case.

In conclusion we express our gratitude to Academician L. D. Landau and L. P. Pitaevskiĭ for helpful discussions of the results of this work.

---

[1] A. Salam, Prog. Theor. Phys. **9**, 550, 1953.

[2] A. B. Migdal, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 1438 (1958), Soviet Phys. JETP **7**, 996 (1958).

[3] V. M. Galitskiĭ and A. B. Migdal, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 139 (1958), Soviet Phys. JETP **7**, 96 (1958).

[4] T. Matsubara, Prog. Theor. Phys. **14**, 351 (1955).

[5] C. T. Belyaev, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 517 (1958), Soviet Phys. JETP **7**, 289 (1958).

[6] L. P. Gor'kov, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 735 (1958), Soviet Phys. JETP **7**, 505 (1958).

[7] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 262 (1958), Soviet Phys. JETP **7**, 182 (1958).

[8] H. Lehmann, Nuovo cimento **11**, 342 (1954).

[9] Ezawa, Tomozawa, and Umezawa, Nuovo cimento **5**, 810 (1957).

Translated by W. H. Furry
162