THE INFLUENCE OF THE ELECTRON-PHONON INTERACTION ON THE CYCLOTRON RESONANCE FREQUENCY

A. V. TULUB

Leningrad State University

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The electron-phonon interaction in polar crystals leads to a non-linear dependence of the cyclotron resonance frequency on the magnetic field strength. An evaluation of the non-linear terms shows that they are small for the magnetic fields used in practice. The polaron effect also gives a correction to the diamagnetic susceptibility. In this paper we perform a mass renormalization in a magnetic field without assuming that the coupling constant is small.

THE cyclotron-resonance phenomenon takes place in a uniform magnetic field, provided the frequency of the additional, variable field is equal to twice the Larmor frequency $2\omega_0 = eH/m^*$. From experiments on cyclotron resonance one can find the magnitude of the effective mass m^*, or, in the general case, of the mass tensor. In the present paper the influence of the electron-phonon interaction on the cyclotron resonance frequency is considered. If the periodic field of the lattice is taken into account, as is usually done, by the introduction of an effective mass, the interaction with the lattice leads to a change (renormalization) of that mass. If there is an external magnetic field, this renormalized mass will in that case depend on the field strength H. The frequency ω_0 will thus be a nonlinear function of H. The problem consists in evaluating the nonlinear terms. These terms will also influence the magnetic properties of the electrons.

In the following we shall consider the case of polar crystals, and for the sake of simplicity we shall assume the effective mass to be isotropic.

The interaction of the electrons with the lattice cannot be considered as a perturbation in the case of polar crystals. For large values of the coupling constant α the adiabatic approximation[1] is valid, and for $\alpha = 1$ to 4 the intermediate coupling approximation. We shall consider the latter case below.

The method of mass renormalization when a magnetic field is present which is developed in the present paper is based upon the approximate elimination of the variables of the phonon field from the energy operator. One can find the action of the phonon absorption operator a_k on the eigenfunctional Ω of the energy operator by evaluating the commutator of a_k with the Hamiltonian $\mathcal{H}$. This method was proposed in meson theory by Chew, Low, and Wick.[2] The result of the action of a_k on Ω is the multiplication of the functional Ω by some function which will depend on the electron momentum as a parameter. Thus in the evaluation of the average value of the energy the phonon field operators give additional terms depending on the electron momentum. The terms quadratic in the momentum will determine the magnitude of the renormalized mass.

In the evaluation of $a_k\Omega$ there arises a difficulty connected with the fact that the action of a_k on Ω is expressed in terms of the momentum operator of the phonon field which arises when recoil is taken into account. This operator is replaced in the expression for $a_k\Omega$ by its average value which is evaluated below.

This method to take the momentum into account is in the case of a free polaron equivalent to the following model. It is assumed that all phonons which form a cloud around the electron possess identical "radial functions" which can be found from a variational principle.[3] For the case of a polaron at rest a more exact variational principle[4] was proposed.

THE ELIMINATION OF THE PHONON FIELD VARIABLES FROM THE ENERGY OPERATOR

We consider the electron energy operator in a magnetic field, taking into account the interaction between the electron and the field of the longitudinal optical phonons, which is known from the polaron theory[1,3]

$$\mathcal{H} = \frac{\pi^2}{m} + \sum_k \omega a_k^+ a_k + \sum_k (V_k a_k e^{i\mathbf{k}\mathbf{x}} + V_k^* a_k^+ e^{-i\mathbf{k}\mathbf{x}}), \quad (1.1)$$

a_k are the second-quantization operators,

$$V_k = -\frac{i\omega}{k}\left(\frac{4\pi\alpha}{L^3}\right)^{1/2}(2m\omega)^{-1/4}, \quad \alpha = e^2\left(\frac{m}{2\omega}\right)^{1/2}\left(\frac{1}{n^2} - \frac{1}{\epsilon}\right),$$
$$\pi = -i\nabla - eA, \quad (1.2)$$

ω is the limiting frequency of the longitudinal optical vibrations, ϵ the static dielectric constant, and n the refractive index for light.*

We assume that the magnetic field is directed along the 3-axis. The commutation relations for the operators π_i will have the form

$$[\pi_1, \pi_2] = ieH; \quad (1.3)$$

the other commutators being equal to zero.

We shall write the eigenfunctional Ψ of the energy operator (1.1) in the form

$$\Psi\{a\} = \exp\left\{i\left(\pi_3 x_3 - \sum_k (\mathbf{k}\cdot\mathbf{x})\, a_k^+ a_k\right)\right\}\Omega\{a\}. \quad (1.4)$$

Substituting (1.4) into (1.1) we get the following equation for the functional Ω:

$$\Big\{\pi^2/2m + \sum_k (\omega + k^2/2m - (\mathbf{k}\pi)/m)\, a_k^+ a_k + \sum_k (V_k a_k + V_k^* a_k^+)$$
$$+ (1/2m)\sum_{k,k'} (\mathbf{k}\cdot\mathbf{k}')\, a_k^+ a_{k'}^+ a_k a_{k'}\Big\}\Omega = E\Omega. \quad (1.5)$$

In Eq. (1.5) π_3 is a number, and not an operator, since the quantity

$$\pi_3 = -i\nabla_3 + \sum_k k_3 a_k^+ a_k$$

is an integral of motion.

To eliminate the phonon field variables from the energy operator we shall evaluate the commutator of the energy operator with the operator a_k:

$$[\mathcal{H}, a_k] = -V_k^* - (k^2 + c^2 - 2\mathbf{k}\cdot\boldsymbol{\pi})/2m - (\mathbf{k}\cdot\mathbf{q})\, a_k/m,$$

$$\mathbf{q} = \sum_k \mathbf{k} a_k^+ a_k, \quad c^2 = 2m\omega. \quad (1.6)$$

Letting the operators in the left hand and the right hand side of (1.6) act on the functional Ω we get

$$a_k\Omega = -2mV_k^*\{2m(\mathcal{H} - E)$$
$$+ k^2 + c^2 - 2\mathbf{k}\cdot\boldsymbol{\pi} + 2\mathbf{k}\cdot\mathbf{q}\}^{-1}\Omega. \quad (1.7)$$

Since V_k^* is a function it is immaterial in what order V_k^* is written down, and this is taken into account in (1.7).

For the following it is convenient to use the following integral transform for the inverse operator L^{-1}:

$$L^{-1} = i\int_0^\infty e^{-isL} ds. \quad (1.8)$$

*We use a system of units for which $\hbar = 1$, $c = 1$.

The transform (1.8) is analogous to V. A. Fock's eigen time method in relativistic quantum mechanics. The integral on the right hand side of (1.8) must be considered to be the limiting value of the expression

$$\lim_{\delta\to 0} i\int_0^\infty e^{-isL-\delta s} ds.$$

We shall rewrite (1.7) in the form

$$a_k\Omega = -2miV_k^*\int_0^\infty ds\, \exp\{-is(k^2 + c^2)$$
$$- is[2m(\mathcal{H} - E) - 2\mathbf{k}\cdot\boldsymbol{\pi} + 2\mathbf{k}\cdot\mathbf{q}]\}\,\Omega \quad (1.7a)$$

taking (1.8) into account. The next transformations raises the problem of "disentangling" the operator $\exp\{-is\, 2m(\mathcal{H} - E)\}$ from the exponent in (1.7a) in the form of a factor.

To ascertain the character of the next approximation we shall consider to begin with that case where there is no magnetic field and where one can thus manipulate the operators π_i as numbers. The intermediate coupling method used by Lee, Low, and Pines[3] is equivalent to the following simplification of Eq. (1.7a). If we neglect the commutator $[q, \mathcal{H}]$, the action of the absorption operator a_k on the state functional is expressed by the equation

$$a_k\Omega = -2mi V_k^*\int_0^\infty ds$$
$$\times \exp\{-is(k^2 + c^2 - 2\mathbf{k}\cdot\boldsymbol{\pi} + 2\mathbf{k}\cdot\mathbf{q})\}\,\Omega. \quad (1.9)$$

Replacing the operator q in (1.9) by its average value, evaluated with the functionals Ω, and taking into account that because of symmetry the average value of the phonon field momentum will be directed solely along the only vector which occurs in the problem, π ($\langle q\rangle = \eta\pi$), we obtain from (1.9) an approximate expression for $a_k\Omega$, where the parameter η is evaluated from the above-mentioned relation $\langle q\rangle = \eta\pi$. If a variational method is used for the functional of the phonon field, this approximation corresponds, according to reference 3, to the assumption that all phonons are in the same state.

The renormalization of the mass when there is a magnetic field proceeds in the same approximation: we assume that after disentangling the exponent in (1.9) we can assume $[q, \mathcal{H}] = 0$ and after that replace q by its average value.

We shall now return to Eq. (1.7a). We first of all single out the operator $\exp\{2is\mathbf{k}\cdot\pi\}$, taking into account that the operators π_1 and π_2 do not commute with one another. We shall use the equations (see Appendix A).

$$\exp\{-is(k^2+\pi^2-2\mathbf{k}\cdot\boldsymbol{\pi}-2\mathbf{q}\cdot\boldsymbol{\pi})\}$$
$$=\exp\{-is[k_3^2+\xi(k_1^2+k_2^2)]\}\exp\{2is(\mathbf{k},\boldsymbol{\Pi}-\mathbf{Q}+\mathbf{q})\}$$
$$\times\exp\{-is(\pi^2-2\boldsymbol{\pi}\cdot\mathbf{q})\}, \quad (1.10)$$

where

$$\xi(x)=(2x)^{-1}\sin 2x, \quad (1.11)$$
$$\Pi_1(x)=x^{-1}(\sin x)(\pi_1\cos x-\pi_2\sin x), \quad (1.12)$$
$$\Pi_2(x)=x^{-1}(\sin x)(\pi_2\cos x+\pi_1\sin x),\ \Pi_3=\pi_3,$$
$$[\Pi_1,\Pi_2]=ieH(x^{-1}\sin x)^2. \quad (1.13)$$

The quantities Q_i are expressed in terms of the q_i in the same way as the Π_i are expressed in terms of the π_i in (1.12) and (1.13). In Eq. (1.10) and in the following equations Π_1 and Π_2 are functions of the argument $x = eHs$ which we shall drop for the sake of simplicity.

Neglecting the commutator of q with $\mathcal{H}$ we obtain from (1.7a) the following expression for $a_k\Omega$:

$$a_k\Omega=-2miV_k^*\int_0^\infty ds$$
$$\times\exp\{-is(c^2+k_3^2+\xi(k_1^2+k_2^2)-2\mathbf{k}(\boldsymbol{\Pi}-\mathbf{Q}))\}\Omega. \quad (1.14)$$

As was pointed out above, the quantity Q will in the following be replaced by its average value where the average value is understood to be the average over the phonon field variables. If we write the functional Ω in the form $\Omega = \Phi\{a\}\,\psi(x)$, after this averaging the quantity $\langle q_i\rangle = \langle\Phi, q_i\Phi\rangle$ is still an operator, acting upon the function ψ.

There is in this way a connection between the average values of the phonon field operator and the operators π_i arising from taking the properties of the state functional Ω into account. By themselves, of course, the operators a_k and π are dynamically independent.

We shall now turn to the evaluation of the average value of the phonon field momentum operator.

EVALUATION OF THE CONSTANTS DETERMINING THE MASS RENORMALIZATION

We shall assume that $\langle q_i\rangle$ and π_i are also in the case of a free polaron proportional to one another:

$$\langle q_1\rangle=\eta\pi_1,\quad \langle q_2\rangle=\eta\pi_2,\quad \langle q_3\rangle=\eta_3\pi_3. \quad (2.1)$$

The validity of this assumption is confirmed by the following consideration: if we substitute (2.1) into (1.14) and evaluate the average value of q_i with the $a_k\Omega$ defined in that way, it turns out that, indeed, the average value of q_i is proportional to π_i.

From (1.7a) we have

$$\langle\Omega, q_i\Omega\rangle=-4m^2\sum_k k_i|V_k|^2\int_0^\infty s\,ds$$
$$\times\langle\Omega,\ \exp\{-is[c^2+k^2-2\mathbf{k}\cdot\boldsymbol{\pi}$$
$$+2\mathbf{k}\cdot\mathbf{q}+2m(\mathcal{H}-E)]\}\,\Omega\rangle. \quad (2.2)$$

The operator $\exp\{-2is(\mathcal{H}-E)\}$ in (2.2) can with equal convenience be taken out as a factor to the right as to the left of the other operators. Taking half of the sum of these two expressions and using (1.2) and the relation

$$k^{-2}=i\int_0^\infty e^{-i\tau k^2}d\tau \quad (2.3)$$

we get the following expression

$$\langle\Omega, q_i\Omega\rangle=-i\frac{4\pi\alpha}{(2\pi)^3}c^3\int k_i d^3k\int_0^\infty d\tau$$
$$\times\int_0^\infty s\,ds\,\langle\Omega, \exp\{-isc^2-i(\tau+s)k_3^2-i(\tau+\xi s)(k_1^2+k_2^2)\}$$
$$\times\frac{1}{2}[\exp\{2is\mathbf{k}\tilde{\boldsymbol{\Pi}}(eHs)\}+\exp\{2is\mathbf{k}\tilde{\boldsymbol{\Pi}}(-eHs)\}]\,\Omega\rangle, \quad (2.4)$$

where

$$\tilde{\Pi}_i=(1-\eta)\Pi_i \text{ for } i=1,2;\ \tilde{\Pi}_3=(1-\eta_3)\Pi_3.$$

It is clear from (2.4) that $\langle q_i\rangle$ is an even function of H. In evaluating the integral over d^3k (see Appendix B) we used the formula

$$e^Ae^B=e^{A+B}e^{[A,B]/2}; \quad (2.5)$$

where $[A, B]$ is a number

In the integration over $d\tau\,ds$ we expanded the integrand in powers of H, after which the integrals were evaluated using

$$\int_0^\infty d\tau\int_0^\infty ds\,s^n\frac{e^{-isc^2}}{(\tau+s)^{m/2}}=\frac{\sqrt{\pi}\,(2k-1)!!}{2^kc^{2k+1}(m/2-1)}e^{-i\pi(2k+1)/4}; \quad (2.6)$$

where n and m are integers, $2k = 2n - m + 3$.

In the asymptotic expansion so obtained we retain the quadratic terms in $(\omega_0/\omega)^2$. As a result we get the following equation to determine η

$$\eta=(1-\eta)\frac{\alpha}{6}\left\{1-\left(\frac{\omega_0}{\omega}\right)^2\left[\frac{1}{2}-\frac{45}{56}(1-\eta)^4\right]\right\}. \quad (2.7)$$

Assuming moreover that

$$\eta=\eta^{(0)}+\eta^{(1)},\qquad \eta^{(0)}=\frac{\alpha}{6}\Big/\left(1+\frac{\alpha}{6}\right),$$

we find

$$\eta=\eta^{(0)}\left\{1-\left(1+\frac{\alpha}{6}\right)^{-1}\left(\frac{\omega_0}{\omega}\right)^2\left[\frac{1}{2}-\frac{45}{56}\left(1+\frac{\alpha}{6}\right)^{-4}\right]\right\}. \quad (2.8)$$

The function η_3 turns out to be equal to

$$\eta_3 = \eta^{(0)}\left\{1 - 3\left(1+\frac{\alpha}{6}\right)^{-1}\left(\frac{\omega_0}{\omega}\right)^2\left[\frac{1}{2} - \frac{15}{56}\left(1+\frac{\alpha}{6}\right)^{-4}\right]\right\}. \quad (2.9)$$

The action of a_k upon Ω is thus in the approximation under consideration completed determined.

The problem now consists in evaluating the average values of the phonon field operators in the Hamiltonian (1.5).

We shall consider the expression $\langle\sum_k a_k^+(\mathbf{k}\cdot\boldsymbol{\pi})a_k\rangle$.

From (1.7) and (1.8) we have

$$\langle\sum_k a_k^+(\mathbf{k}\cdot\boldsymbol{\pi})\, a_k\rangle = -4m^2\sum_k |V_k|^2 \int_0^\infty ds \int_0^\infty ds' \langle e^{-isL}(\mathbf{k}\cdot\boldsymbol{\pi})\, e^{-is'L}\rangle. \quad (2.10)$$

Expression (2.10) can conveniently be evaluated as follows. We shift the operator $(\mathbf{k}\cdot\boldsymbol{\pi})$ first to the right of $\exp(-is'L)$ and then to the left of $\exp(-isL)$ and take half the sum of the expressions thus obtained which will be an even function of H as can easily be ascertained.

Using the formula

$$(\mathbf{k}\cdot\boldsymbol{\pi})\, e^{-is(\pi^2-2\mathbf{p}\cdot\boldsymbol{\pi})} = e^{-is(\pi^2-2\mathbf{p}\cdot\boldsymbol{\pi})}(\mathbf{k}\cdot\boldsymbol{\pi}'(2eHs) - eHs\,[\mathbf{k}\times\mathbf{p}]_3),$$

$$\pi_1'(x) = \pi_1\cos x + \pi_2 \sin x,$$

$$\pi_2'(x) = \pi_2\cos x - \pi_1\sin x,\ \ \pi_3'(x) = \pi_3, \quad (2.11)$$

we get after some transformations the following expression for the terms quadratic in the momenta π_1^2 and π_2^2

$$-i\frac{4\pi\alpha}{(2\pi)^3}c^3\pi^{3/2}e^{-3i\pi/4}\int_0^1 d\nu\int_0^\infty d\tau\int_0^\infty ds\cdot s e^{-isc^2}\frac{s}{(\tau+s)^{1/2}(\tau+\xi s)^2}$$
$$\times\left(1+\frac{s^2}{(\tau+\xi s)^2}\frac{\sin^4 eHs}{(eHs)^2}(1-\eta)^4\right)^{-1/2}$$
$$\times\frac{1}{2}\{\pi_1(-\nu sH)\,\tilde{\Pi}_1(esH) + \tilde{\Pi}_1(-esH)\,\pi_1'(\nu sH) + (1\to 2)\}. \quad (2.12)$$

Integration over $d\nu$ gives

$$\int_0^1 d\nu\,\frac{1}{2}\{\dots\} = (1-\eta)\left(\frac{\sin eHs}{eHs}\right)^2(\pi_1^2+\pi_2^2). \quad (2.13)$$

The remaining integration is carried out by means of (2.6).

After evaluating the average values of the field operators we obtained the following values for the renormalized mass:

$$m_{11}^* = m_{22}^* = m_\perp = m_0\left\{1+\left(\frac{\omega_0}{\omega}\right)^2\left[\frac{29}{24}\alpha\left(1+\frac{\alpha}{6}\right)^{-1} - \frac{5}{12}\alpha - \frac{15}{14}\alpha\left(1+\frac{\alpha}{6}\right)^{-5}\right]\right\}; \quad (2.14)$$

$$m_{33}^* = m_\parallel = m_0\left\{1+\left(\frac{\omega_0}{\omega}\right)^2\left[\frac{\alpha}{2}\left(1+\frac{\alpha}{6}\right)^{-1} - \frac{45}{112}\alpha\left(1+\frac{\alpha}{6}\right)^{-5}\right]\right\}; \quad (2.15)$$

$$m_0 = m\left(1+\frac{\alpha}{6}\right).$$

For H = 0 we get the renormalized mass value obtained in reference 3.

The ratio of the unrenormalized cyclotron resonance frequency to the renormalized one is, according to (2.14),

$$(\omega_0/\omega_0^*) = [1 + d_\perp(\alpha)(\omega_0/\omega)^2]. \quad (2.16)$$

In the intermediate coupling region the function $d_\perp(\alpha)$ is, apart from a factor of the order of unity, equal to

$$d(1) = 0.12;\ d(2) = 0.46;\ d(3) = 0.74;\ d(4) = 0.90.$$

Since the energies of the longitudinal optical phonons are about 0.01 ev the correction term in (2.16) is for all practically used fields vanishingly small, i.e., cyclotron resonance experiments give $m|_{H=0}$.

CORRECTION TO THE DIAMAGNETIC SUSCEPTIBILITY

As the result of the elimination of the phonon field variables, the energy operator takes the following form

$$\mathcal{H} = (\pi_1^2+\pi_2^2)/2m_\perp + \pi_3^2/2m_\parallel. \quad (3.1)$$

We consider the problem of evaluating the density matrix $\rho(x, x', \beta)$ for an ideal gas, when the Hamiltonian of the system is given by Eq. (3.1). We write the matrix $\rho(x, x', \beta)$ in the form

$$\rho(x, x', \beta) = \langle x|\exp(-\beta\mathcal{H})|x'\rangle, \quad (3.2)$$

where x stands for the three coordinates x_1, x_2, x_3. If we introduce the variable s, equal to $s = -i\beta$, instead of β, we can formally consider expression (3.2) as the transformation function $\langle x(s)|x'(0)\rangle$. The latter can be found from the following equations:[5,6]

$$i\frac{\partial}{\partial s}\langle x(s)|x'(0)\rangle = \langle x(s)|\mathcal{H}|x'(0)\rangle; \quad (3.3)$$

$$(-i\partial/\partial x - eA(x))\langle x(s)|x'(0)\rangle = \langle x(s)|\pi(s)|x'(0)\rangle;$$

$$(i\partial/\partial x' - eA(x')\langle x(s)|x'(0)\rangle = \langle x(s)|\pi(0)|x'(0)\rangle \quad (3.4)$$

and the boundary condition

$$\lim_{s\to 0}\langle x(s)|x'(0)\rangle = \delta(x-x'). \quad (3.5)$$

To evaluate the transformation function it is necessary to solve the Heisenberg equations of motion for the operators π and x. These equations have in the case considered the following form

$$dx/ds = P\pi;\ d\pi/ds = 2\omega_0 Q\pi; \quad (3.6)$$

$$P = \begin{pmatrix} m_\perp^{-1} & 0 & 0 \\ 0 & m_\perp^{-1} & 0 \\ 0 & 0 & m_\parallel^{-1} \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.7)$$

The solution of equations (3.6) can be written in the form

$$\pi(s) = e^{2\omega_0 Qs}\pi(0);$$

$$x(s) = x(0) + (2\omega_0 Q)^{-1}(e^{2\omega_0 Qs} - 1)P\pi(0). \quad (3.8)$$

Using expression (3.8) we can express the energy operator as a function of the operators x(s) and x(0). Shifting x(s) to the left and x(0) to the right, we get after some transformations (see reference 5) the following equation

$$i\frac{\partial}{\partial s}\langle x(s)\,|\,x'(0)\rangle = \Big\{\frac{1}{2}\omega_0^2(x-x')P^{-1}\frac{Q^2}{\sinh^2\omega_0 Qs}(x-x')$$
$$- i\omega_0 \operatorname{Sp} Q\,(1-e^{-2\omega_0 Qs})^{-1}\Big\}\langle x(s)\,|\,x'(0)\rangle. \quad (3.9)$$

Taking the properties of the matrix Q into account we find

$$2\operatorname{Sp} Q\,(1-e^{-2\omega_0 Qs})^{-1} = \operatorname{Sp} Q \coth \omega_0 Qs$$
$$= 2\cot\omega_0 s + 1/\omega_0 s. \quad (3.10)$$

Integration of Eq. (3.9) gives the following value for the partition function Z

$$Z = \operatorname{Sp}\rho(x, x', \beta) = \frac{2V}{h^3}(2\pi kT)^{1/2} m_\perp m_\parallel^{1/2}\frac{\beta\hbar\omega_0}{\sinh\beta\hbar\omega_0}, \quad (3.11)$$

where V is the volume of the system.

Using (2.14) and (2.15), the diamagnetic susceptibility turns out to be equal to

$$\chi = -N\mu^2/3kT$$
$$+ NkT(2d_\perp + d_\parallel)\mu^2/\hbar^2\omega^2 = \chi_1 + \chi_2, \quad (3.12)$$

where $\mu = e\hbar/2mc$.

The relative order of magnitude is given by

$$|\chi_2/\chi_1| \sim (kT/\hbar\omega)^2.$$

The polaron effect leads thus to a change in the diamagnetic susceptibility. The correction term is appreciable at temperatures of the order of $kT \sim \hbar\omega$.

In conclusion I wish to express my gratitude to Academician V. A. Fock for discussing the present paper.

APPENDIX A

Introducing the operators

$$c = (2eH)^{-1/2}(\pi_1 + i\pi_2),\ c^+ = (2eH)^{-1/2}(\pi_1 - i\pi_2) \quad (A.1)$$

and using the notation

$$q' = (2eH)^{-1/2}(q_1 + iq_2),\ k' = (2eH)^{-1/2}(k_1 + ik_2), \quad (A.2)$$

we can rewrite $\exp\{-is(\pi^2 - 2\mathbf{k}\cdot\pi - 2\mathbf{q}\cdot\pi)\}$ in the form

$$\exp\{-is(\pi_3^2 - 2k_3\pi_3 - 2q_3\pi_3 + eH)\}$$
$$\times \exp\{-2ieHs(c^+c - q'c^+ - q'^+c - k'c^+ - k'^+c)\}.$$

We consider moreover the function

$$\Phi(\varepsilon) = \exp\varphi(\varepsilon) = \exp\{-2ieHs\ (c^+c - q'c^+$$
$$- q'^+c - \varepsilon(k'c^+ + k'^+c))\}. \quad (A.3)$$

To disentangle the operator $\exp\{2ieHs(k'c^+ + k'^+c)\}$ we write down the derivative[7]

$$\frac{d\Phi}{d\varepsilon} = \Phi(\varepsilon)\Big\{\varphi' + \frac{1}{2!}[\varphi', \varphi] + \frac{1}{3!}[[\varphi', \varphi]\varphi] + \ldots\Big\}. \quad (A.4)$$

After evaluating the commutators, one can sum the series. Integrating the equation thus obtained over ϵ from zero to unity we get the result given in the text.

APPENDIX B

We consider the evaluation of the integral

$$I = \int_{-\infty}^{+\infty} k_1 dk_1 \int_{-\infty}^{+\infty} dk_2 \exp\{-is\xi(k_1^2 + k_2^2)$$
$$+ 2is(k_1\Pi_1 + k_2\Pi_2)\}, \quad (B.1)$$

where $[\Pi_1, \Pi_2] = ieH\lambda^2$.

Using (2.5) the integral over dk_2 is equal to

$$\exp(2isk_1\Pi_1)\int_{-\infty}^{+\infty} dk_2 \exp\{-is\xi k_2^2 + 2isk_2\ (\Pi_2 + k_1 eHs\lambda^2)\}$$
$$= \exp(2isk_1\Pi_1)\,e^{-i\pi/4}$$
$$\times (\pi/s\xi)^{1/2}\exp\{is\xi^{-1}(\Pi_2 + k_1 eHs\lambda^2)^2\},$$

In integrating over dk_1 we must use (2.5) again.

After averaging the operators over the phonon field variables we must pick out the terms quadratic in the momenta. It is advisable first to symmetrize the exact values of the integrals. In (B.1), for instance, one can in the integration over dk_2 place the operator $\exp\{2isk_1\Pi_1\}$ both to the left and to the right of the terms with the operator Π_2.

One can, finally, evaluate first of all the integral over dk_1 and then the integral over dk_2. The expressions obtained by that means differ by the arrangement of the operators Π_1 and Π_2 with respect to one another. As a result the value of the integral (B.1) turns out to be equal to

$$I = -i\pi(\xi')^{-1}(s\xi)^{-1}(\Pi_1 T + T\Pi_1)/2, \quad (B.2)$$

$$T = \exp(is\Pi_1^2/2\xi)\exp(is\xi\Pi_2^2/\xi'^2)\exp(is\,\Pi_1^2/2\xi) + (1 \to 2),$$

$$(\xi')^2 = \xi^2 + (eHs)^2\lambda^4. \quad (B.3)$$

up to terms of the order $(eH)^3$. To simplify the expressions obtained we used the formula[7]

$$\exp(-ap^2)\exp(bq^2)\exp(ap^2)$$

$$= \exp\{b(q + 2iap)^2\}, \ [q, p] = i. \quad (B.4)$$

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