MULTIPLE SCATTERING OF POLARIZED ELECTRONS

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Multiple elastic scattering of polarized spin-$\frac{1}{2}$ particles in an isotropic homogeneous medium is considered. The kinetic equations determining the distribution function and polarization vector of scattered particles are solved approximately. A solution which is valid for both small and large scattering angles has been obtained as a series expansion in spherical harmonics and spherical vectors.

1. INTRODUCTION

In connection with the testing of parity conservation, electrons and other particles produced in the course of various transformations have recently been found to be polarized. In experiments with polarized particles it is necessary to take into account the effect of multiple scattering on both the magnitude (depolarization) and direction of the polarization vector. The latter effect has been used in a number of investigations[1,2] to transform longitudinal polarization into transverse polarization, which can then be detected from the azimuthal asymmetry of electron scattering in nuclear Coulomb fields.

Bethe and Rose[3] were the first to estimate electron depolarization through multiple scattering. Mühlschlegel and Koppe[4] recently determine the distribution function and polarization of multiply scattered particles for only small angles. However, polarization effects are most significant at large angles, although the scattering cross section is greatly reduced with increase of the angle. In the present work we considered both large and small angles to obtain the angular distribution and polarization of particles passing through a scatterer of limited thickness. Inelastic collisions were disregarded. The equations that were derived agreed with the results obtained by Mühlschlegel and Koppe for small angles.

2. SINGLE SCATTERING

The scattering matrix of electrons in a centrally symmetrical field can, as we know, be represented by

$$\Omega = f(\theta) - ig(\theta)\boldsymbol{\nu}\cdot\boldsymbol{\sigma}, \quad \boldsymbol{\nu} = [\mathbf{n}\times\mathbf{n}']/\sin\theta. \qquad (1)$$

Here θ is the scattering angle; $\mathbf{n}$ and $\mathbf{n}'$ are the unit vectors of electron momentum before and after scattering; σ is the Pauli spin operator. The functions f and g for a Coulomb field have been given by Mott.[5] Approximate expressions for these functions up to terms containing $(\alpha Z)^2$ are given in the Appendix.

Before scattering let the state of polarization of the electron beam be given (in its rest frame) by the density matrix $\rho(\zeta) = (1 + \zeta\sigma)/2$, where $\zeta = \mathrm{Sp}(\sigma\rho)$ is the polarization vector. Then the differential cross section for scattering from the state with momentum $\mathbf{p} = p\mathbf{n}$ and polarization ζ to the state with momentum $\mathbf{p}' = p\mathbf{n}'$ and arbitrary polarization is given by

$$S(\mathbf{n}, \boldsymbol{\zeta}; \mathbf{n}') = \mathrm{Sp}(\Omega\rho\Omega^+) = B(\theta) + D(\theta)\boldsymbol{\nu}\cdot\boldsymbol{\zeta}. \qquad (2)$$

The polarization vector ζ' of the scattered beam is given by

$$\boldsymbol{\zeta}' = \mathrm{Sp}(\sigma\Omega\rho\Omega^+)/\mathrm{Sp}(\Omega\rho\Omega^+). \qquad (3)$$

Using (2) and calculating Sp, we obtain

$$\boldsymbol{\zeta}' S(\mathbf{n}, \boldsymbol{\zeta}; \mathbf{n}') = B\boldsymbol{\zeta} + F[\boldsymbol{\nu}\times\boldsymbol{\zeta}] + C\,\boldsymbol{\nu}\times[\boldsymbol{\nu}\times\boldsymbol{\zeta}] + D\boldsymbol{\nu}. \qquad (4)$$

B, C, D and F are given in terms of f and g as follows:

$$B = |f|^2 + |g|^2; \quad C = 2|g|^2,$$
$$D = i(fg^* - f^*g); \quad F = fg^* + f^*g. \qquad (5)$$

It is evident from (4) that scattering can change ζ in both magnitude and direction. Polarization effects in both single and double scattering have been considered in detail in many papers (see the review by Tolhoek, reference 6) and will not be considered here.

3. THE KINETIC EQUATIONS

An electron beam scattered in an isotropic homogeneous medium will be characterized by the intensity $I(\mathbf{n}, \mathbf{r})$ and polarization vector $\zeta(\mathbf{n}, \mathbf{r})$. The equations that determine I and ζ are easily obtained from elementary considerations. Let us consider the product $I\zeta = \mathbf{G}$ which gives the intensity of spin transport. The variation of this quantity per unit length of path is given by the derivative $(\mathbf{n}\cdot\nabla)\mathbf{G}$ and consists of the two parts

$$-N\mathbf{G}(\mathbf{n}, \mathbf{r})\int S(\mathbf{n}, \zeta; \mathbf{n}')\,d\Omega'$$

and $N\int \mathbf{G}(\mathbf{n}', \mathbf{r})\,S(\mathbf{n}', \zeta(\mathbf{n}', \mathbf{r}); \mathbf{n})\,d\Omega'$,

where N is the number of scattering centers per unit volume. Using (4) and denoting the total scattering cross section by σ, we obtain

$$\mathbf{n}\cdot\nabla\,\mathbf{G} = -N\sigma\mathbf{G} + N\int (A\mathbf{G}' + DI'\boldsymbol{\nu})\,d\Omega'. \qquad (6)$$

Here $\mathbf{G}' = \mathbf{G}(\mathbf{n}', \mathbf{r})$, $I' = I(\mathbf{n}', \mathbf{r})$, and A denotes the operator

$$A = B + F[\boldsymbol{\nu}\times\ldots] + C[\boldsymbol{\nu}\times[\boldsymbol{\nu}\times\ldots]]. \qquad (7)$$

An equation for I is obtained analogously:

$$(\mathbf{n}\cdot\nabla)I = -N\sigma I + N\int (BI' + D\boldsymbol{\nu}\cdot\mathbf{G}')\,d\Omega'. \qquad (8)$$

We note that in a paper by Waldmann[7] the kinetic equations for a particle with spin are given in matrix form using the scattering matrix. For the case of electrons these equations can easily be put into the form of (6) and (8).

We shall consider a scatterer in the form of a plane-parallel layer bounded by the planes z = 0 and z = d. For a beam of finite width I and G will depend on all three coordinates x, y, and z. Both sides of (6) and (8) will be integrated with respect to x and y; the x and y derivatives vanish and ∇ is replaced by $\partial/\partial z$. $I(\mathbf{n}, z)\cos\vartheta\,d\Omega$ will represent the number of particles passing through the plane z = const in the $\mathbf{n}$ direction per unit time, and $\zeta = \mathbf{G}(\mathbf{n}, z)/I(\mathbf{n}, z)$ will represent the polarization of these particles. Introducing the dimensionless variable $\tau = N\sigma z$, Eqs. (6) and (8) become

$$\cos\vartheta\,\partial I/\partial\tau = -I + \int (BI' + D\boldsymbol{\nu}\mathbf{G}')\,d\Omega',$$

$$\cos\vartheta\,\partial\mathbf{G}/\partial\tau = -\mathbf{G} + \int (A\mathbf{G}' + DI'\boldsymbol{\nu})\,d\Omega'; \qquad (9)$$

B, C, D and F in these equations differ from the expressions in (5) by the factor $1/\sigma$.

The boundary conditions are

$$I(\mathbf{n}, 0) = I^{(0)}\delta(\mathbf{n}-\mathbf{n}_0),\quad \mathbf{G}(\mathbf{n}, 0) = I^{(0)}\zeta^{(0)}\delta(\mathbf{n}-\mathbf{n}_0) \quad \text{for } \cos\vartheta > 0,$$

$$I(\mathbf{n}, t) = \mathbf{G}(\mathbf{n}, t) = 0 \quad \text{for } \cos\vartheta < 0; \qquad (10)$$

$\mathbf{n}_0(\vartheta_0, \varphi_0)$ gives the direction of the incident beam; $t = N\sigma d$; the δ function is normalized by $\int \delta(\mathbf{n}-\mathbf{n}_0)\,d\Omega = 1$.

Hereinafter we shall limit ourselves to a beam of normal incidence ($\cos\vartheta_0 = 1$). Instead of (9) with the boundary conditions (10) it is more convenient to consider two system of equations with corresponding boundary conditions:

$$\partial I_0/\partial\tau = -I_0 + \int (BI_0' + D\boldsymbol{\nu}\cdot\mathbf{G}_0')\,d\Omega',$$

$$\partial\mathbf{G}_0/\partial\tau = -\mathbf{G}_0 + \int (A\mathbf{G}_0' + DI_0'\boldsymbol{\nu})\,d\Omega',$$

$$I_0(\mathbf{n}, 0) = I^{(0)}\delta(\mathbf{n}-\mathbf{n}_0);\quad \mathbf{G}_0(\mathbf{n}, 0) = I^{(0)}\zeta^{(0)}\delta(\mathbf{n}-\mathbf{n}_0); \qquad (\mathrm{I})$$

$$\cos\vartheta\,(\partial I_1/\partial\tau) = -I_1 + \int (BI_1' + D\boldsymbol{\nu}G_1')\,d\Omega' + (1-\cos\vartheta)\,\partial I_0/\partial\tau,$$

$$\cos\vartheta\,\partial\mathbf{G}_1/\partial\tau = -\mathbf{G}_1 + \int (A\mathbf{G}_1' + DI_1'\boldsymbol{\nu})\,d\Omega' + (1-\cos\vartheta)\,\partial\mathbf{G}_0/\partial\tau,$$

$$I_1(\mathbf{n}, 0) = \mathbf{G}_1(\mathbf{n}, 0) = 0 \quad \text{for } \cos\vartheta > 0,$$

$$I_1(\mathbf{n}, t) = -I_0(\mathbf{n}, t);\ \mathbf{G}_1(\mathbf{n}, t) = -\mathbf{G}_0(\mathbf{n}, t) \quad \text{for } \cos\vartheta < 0. \qquad (\mathrm{II})$$

When $I = I_0 + I_1$ and $\mathbf{G} = \mathbf{G}_0 + \mathbf{G}_1$ the equations (I) and (II) are equivalent to the original equation (9) and boundary conditions (10).

4. SOLUTION OF SYSTEM (I)

The equations of (I) differ from the exact equations by the fact that $\cos\vartheta$ is replaced by 1 in the left-hand members, which means that the true path $d\tau/\cos\vartheta$ traversed by a particle in the layer $d\tau$ is replaced by a segment $d\tau$ representing its path in the original direction of motion. The distribution function for unpolarized electrons was obtained in this approximation by Goudsmit and Saunderson.[8]

In the small-angle approximation the right-hand members in (I) give equations for I and G in small-angle scattering. However, it is our intention to solve the exact system (9) for all angles, so that we shall first obtain an exact solution of (I).

I_0 and $\mathbf{G}_0$ will be given as series of spherical harmonics and spherical vectors:*

$$I_0(\mathbf{n}, \tau) = \sum_{lm} I_{lm}(\tau)\,Y_{lm}(\mathbf{n}),$$

$$\mathbf{G}_0(\mathbf{n}, \tau) = \sum_{JLM} G_{JM}^{L}(\tau)\,\mathbf{Y}_{JM}^{L}(\mathbf{n}). \qquad (11)$$

In order to transform the integrals in the right-hand members we expand the integrands in series of spherical harmonics, as follows:

*We shall use Bethe's definition[9] of spherical harmonics. The definition and properties of spherical vectors have been given by Berestetskii, Dolginov and Ter-Martirosyan.[10]

$$B(\theta) = \Sigma B_l Y^*_{lm}(\mathbf{n}') Y_{lm}(\mathbf{n});$$

$$B_l = 2\pi \int_0^\pi B(\theta) P_l(\cos\theta) \sin\theta \, d\theta. \quad (12)$$

Considering also that

$$[\mathbf{Y}^L_{JM}]_\mu = (-1)^{1-\mu} C^{JM}_{LM+\mu 1-\mu} Y_{LM+\mu},$$

where $C^{lm}_{l_1 m_1 l_2 m_2}$ is a Clebsch-Gordan coefficient and the index $\mu = 0, \pm 1$ denotes cyclical components of a vector, which are related to its Cartesian components by

$$a_0 = a_z, \; a_{\pm 1} = \pm (a_x \pm i a_y)/\sqrt{2},$$

we obtain

$$\int B(\theta) I_0' d\Omega' = \sum B_l I_{lm}(\tau) Y_{lm}(\mathbf{n}), \; \int B(\theta) \mathbf{G}_0' d\Omega'$$

$$= \sum B_L G^L_{JM}(\tau) \mathbf{Y}^L_{JM}(\mathbf{n}). \quad (13)$$

For the expansion of $D(\theta)\nu_\mu = D(\theta)[\mathbf{nn}']_\mu/\sin\theta$ in spherical harmonics we note that $D(\theta)/\sin\theta$ transforms as a scalar while $[\mathbf{n}\times\mathbf{n}']_\mu$ transforms as a vector component. We can therefore write

$$D(\theta)[\mathbf{n}\times\mathbf{n}']_\mu/\sin\theta = i \sum D_l' C^{1\mu}_{lmlm'} Y_{lm}(\mathbf{n}) Y_{lm'}(\mathbf{n}'). \quad (14)$$

To determine D_l' we let $\vartheta' = \varphi' = 0$ in this equation. In virtue of the orthogonality of the spherical harmonics and of the Clebsch-Gordan coefficients, we obtain for D_l':

$$D_l' = (\sqrt{8\pi(2l+1)}/3) \sum C^{1\mu}_{l\mu l0} C^{1\mu}_{101\mu} \int \frac{D(\vartheta)}{\sin\vartheta} n_\mu Y^*_{l\mu}(\mathbf{n}) \, d\Omega$$

$$= (-1)^l [(2l+1)/3l(l+1)]^{1/2} 2\pi \int_0^\pi D(\theta) P_l^1(\theta) \sin\theta \, d\theta,$$

where P_l' is an unnormalized associated Legendre polynomial.

Making use of (14), we obtain for the integrals contain $D(\theta)$:

$$\int D(\theta) I_0' \boldsymbol{\nu} \, d\Omega' = -i \sum D_l I_{lm} \mathbf{Y}^l_{lm},$$

$$\int D(\theta) \mathbf{G}_0' \boldsymbol{\nu} \, d\Omega' = -i \sum D_l G^l_{lm} Y_{lm}; \quad (15)$$

$$D_l = [2\pi/\sqrt{l(l+1)}] \int_0^\pi D(\theta) P_l^1(\theta) \sin\theta \, d\theta. \quad (16)$$

The integrals containing $F(\theta)$ and $C(\theta)$ are calculated similarly, using Racah's formula for the summation of Clebsch-Gordan coefficients. We obtain

$$\int F(\theta) [\boldsymbol{\nu}\times\mathbf{G}_0'] \, d\Omega' = \sum_{JLM} F^J_L G^L_{JM} \mathbf{Y}^L_{JM}; \quad (17)$$

$$F^J_L = -\frac{L(L+1) - J(J+1) + 2}{2L(L+1)} 2\pi \int_0^\pi F(\theta) P_l^1(\theta) \sin\theta \, d\theta; \quad (18)$$

$$\int C(\theta) \boldsymbol{\nu}\times[\boldsymbol{\nu}\times\mathbf{G}_0'] \, d\Omega' = \sum_{JLL'M} C^J_{LL'} G^{L'}_{JM} \mathbf{Y}^L_{JM}, \quad (19)$$

$$C^l_{ll} = -\pi C_{l0} - \pi C_{l2}/l(l+1),$$

$$C^{l+1}_{ll} = -\pi(3l+4) C_{l0}/(2l+3) + \pi C_{l2}/(l+1)(2l+3), \quad (20)$$

$$C^{l-1}_{ll} = -\pi(3l-1) C_{l0}/(2l-1) + \pi C_{l2}/l(2l-1),$$

$$C^l_{l-1\,l+1} = C^l_{l+1\,l-1} = [2\pi/(2l+1)\sqrt{l(l+1)}] \int_0^\pi C(\theta) P_l^1(\theta) \, d\theta,$$

$$C_{lm} = \int_0^\pi C(\theta) P_l^m(\theta) \sin\theta \, d\theta.$$

Substituting these values of the integrals into (I), we obtain four first-order differential equations associated in pairs:

$$dI_{LM}/d\tau = -b_L I_{LM} - i D_L G^L_{LM},$$
$$dG^L_{LM}/d\tau = -a^L_L G^L_{LM} - i D_L I_{LM}; \quad (21)$$

$$dG^{L+1}_{LM}/d\tau = -a^{L+1}_L G^{L+1}_{LM} + c_L G^{L-1}_{LM},$$
$$dG^{L-1}_{LM}/d\tau = -a^{L-1}_L G^{L-1}_{LM} + c_L G^{L+1}_{LM}. \quad (22)$$

The following notation has been used:

$$b_L = 1 - B_L; \; a^J_L = 1 - B_L - F^J_L - C^J_{LL};$$

$$c_L = C^L_{L+1\,L-1}. \quad (23)$$

Equations (21) and (22) are easily integrated, the integration constants being determined by the boundary conditions

$$I_{LM}(0) = I^{(0)} \sqrt{(2L+1)/4\pi} \delta_{M0},$$

$$G^L_{JM}(0) = I^{(0)} \sqrt{(2l+1)/4\pi} C^{L0}_{1-M\,JM} \zeta^{(0)}_{-M} (J = L, \; L \pm 1). \quad (24)$$

We finally obtain the distribution function of scattered electrons:

$$I_0(\mathbf{n}, \tau) = \frac{I^{(0)}}{4\pi} \sum_l \frac{(2l+1)}{1-k_l^2} (e^{-\alpha_1\tau} - k_l^2 e^{-\alpha_2\tau}) P_l(\cos\vartheta)$$

$$- \frac{I^{(0)} P_0}{4\pi} \sin\chi \sin\varphi \sum_l \frac{(2l+1) k_l}{\sqrt{l(l+1)}(1-k_l^2)} (e^{-\alpha_1\tau} - e^{-\alpha_2\tau}) P_l^1(\vartheta),$$

$$2\alpha_{1,2} = b_l + a^l_l \pm [(b_l - a^l_l)^2 - 4D_l^2]^{1/2},$$

$$k_l = 2D_l/(b_l - a^l_l + \sqrt{(b_l - a^l_l)^2 - 4D_l^2}), \quad (25)$$

where $P_0 = |\zeta^{(0)}|$ is the degree of polarization of the initial beam, χ is the angle between $\zeta^{(0)}$ and $\mathbf{n}_0$, and the x axis is in the $\zeta^{(0)}$, $\mathbf{n}_0$ plane.

The second term in (25) gives the azimuthal asymmetry due to the transverse polarization component of the incident beam. For $D = 0$, which represents the absence of spin-orbit coupling, α_2 and k_l vanish, while α_1 becomes equal to b_l.

Equation (25) is thus converted into the Goudsmit-Saunderson formula

$$I_0(\vartheta, \tau) = (I^{(0)}/4\pi) \sum (2l+1)\, e^{-b_l\tau} P_l(\cos\vartheta). \quad (26)$$

For the vector $\mathbf{G}_0$ in the case of initial longitudinal polarization ($\zeta_z^{(0)} = \zeta^{(0)}$, $\zeta_x^{(0)} = \zeta_y^{(0)} = 0$), we obtain

$$\mathbf{G}_0(\mathbf{n}, \tau) = \frac{I^{(0)}}{\sqrt{4\pi}} \sum \Big\{ \frac{ik_l (2l+1)^{1/2}}{1-k_l^2} (e^{-\alpha_1\tau} - e^{-\alpha_2\tau})\, \mathbf{Y}_{l0}^l$$

$$+ \frac{\zeta^{(0)}}{1+s_l^2} \big[\{(l+1)^{1/2} (e^{-\beta_1\tau} + s_l^2 e^{-\beta_2\tau}) + s_l l^{1/2} (e^{-\beta_1\tau} - e^{-\beta_2\tau})\}\, \mathbf{Y}_{l0}^{l+1}$$

$$- \{l^{1/2} (s_l^2 e^{-\beta_1\tau} + e^{-\beta_2\tau}) + s_l (l+1)^{1/2} (e^{-\beta_1\tau} - e^{-\beta_2\tau})\}\, \mathbf{Y}_{l0}^{l-1}\big]\Big\},$$

$$2\beta_{1,2} = a_l^{l+1} + a_l^{l-1} \pm [(a_l^{l+1} - a_l^{l-1})^2 + 4c_l^2]^{1/2},$$

$$s_l = 2c_l/(a_l^{l+1} - a_l^{l-1} + \sqrt{(a_l^{l+1} - a_l^{l-1})^2 + 4c_l^2}\,. \quad (27)$$

We also write the projections of $\mathbf{G}_0$ on the rectangular axes χ, η, ζ, which are in the directions of the vectors $\mathbf{n}\times\mathbf{n}_0$, $\mathbf{n}\times[\mathbf{n}\times\mathbf{n}_0]$ and $\mathbf{n}$, as follows:

$$G_\chi = -\frac{I^{(0)}}{4\pi} \sum \frac{k_l(2l+1)}{(1-k_l^2)\sqrt{l(l+1)}} (e^{-\alpha_1\tau} - e^{-\alpha_2\tau})\, P_l^1(\vartheta),$$

$$G_\eta = \frac{I^{(0)}\zeta^{(0)}}{4\pi} \sum \frac{1 - s_l^2 - s_l/\sqrt{l(l+1)}}{1+s_l^2} (e^{-\beta_1\tau} - e^{-\beta_2\tau})\, P_l^1(\vartheta),$$

$$G_\zeta = (I^{(0)}\zeta^{(0)}/4\pi) \sum (1+s_l^2)^{-1} \{(l+1)(e^{-\beta_1\tau} + s_l^2 e^{-\beta_2\tau})$$

$$+ l(s_l^2 e^{-\beta_1\tau} + e^{-\beta_2\tau})$$

$$+ 2s_l \sqrt{l(l+1)}/(e^{-\beta_1\tau} - e^{-\beta_2\tau})\}\, P_l(\cos\vartheta). \quad (28)$$

The projection G_χ is independent of the initial polarization, but gives the polarization of the initially unpolarized electron beam due to multiple scattering (the Mott effect). However, this effect disappears when $D = 0$, which occurs in the first Born approximation.

5. SOLUTION OF SYSTEM (II)

When the scatterer is not very thick each of the functions I_0 and $\mathbf{G}_0$ possesses a very sharp peak in the direction of initial electron motion, since the Coulomb scattering occurs predominantly in the forward direction. I_1 and $\mathbf{G}_1$, which are determined by (II), will have a considerably smoother form, because the functions $(1-\cos\vartheta)\,\partial I_0/\partial\tau$ and $(1-\cos\vartheta)\,\partial\mathbf{G}_0/\partial\tau$ in (II) as well as $I_0(\mathbf{n}, t)$ and $\mathbf{G}_0(\mathbf{n}, t)$ for $\cos\vartheta < 0$, which determine I_1 and $\mathbf{G}_1$ on the boundary, are smooth functions. Successive approximations can therefore be used to solve (II), as in reference 11 for the first equation of (29) with $D = 0$.

We now rewrite (II) in the form

$$\cos\vartheta\, \partial I_1/\partial\tau = J + (1-\cos\vartheta)\, \partial I_0/\partial\tau,$$
$$\cos\vartheta\, \partial\mathbf{G}_1/\partial\tau = \mathbf{Y} + (1-\cos\vartheta)\, \partial\mathbf{G}_0/\partial\tau, \quad (29)$$

where J and $\mathbf{Y}$ denote the differences between the respective integrals and I_1 and $\mathbf{G}_1$. In first approximation we set $J = \mathbf{Y} = 0$. The error thus introduced is the smaller, the smoother the functions I_1 and $\mathbf{G}_1$ and the sharper the forward peak of the single-scattering cross section. A more precise criterion for the applicability of this approximation will be given below.

The equations now have the simple form

$$\cos\vartheta\, \partial I_1/\partial\tau = (1-\cos\vartheta)\, \partial I_0/\partial\tau,$$
$$\cos\vartheta\, \partial\mathbf{G}_1/\partial\tau = (1-\cos\vartheta)\, \partial\mathbf{G}_0/\partial\tau, \quad (30)$$

and their solutions which satisfy the required boundary conditions are as follows:

$$\left.\begin{aligned} I_1^{(1)}(\mathbf{n},\tau) &= (\sec\vartheta - 1)\, I_0(\mathbf{n}, \tau), \\ \mathbf{G}_1^{(1)}(\mathbf{n}, \tau) &= (\sec\vartheta - 1)\, \mathbf{G}_0(\mathbf{n}, \tau), \end{aligned}\right\} \text{ for } \cos\vartheta > 0; \quad (31)$$

$$\left.\begin{aligned} I_1^{(1)}(\mathbf{n}, \tau) &= (\sec\vartheta - 1)\, I_0(\mathbf{n}, \tau) - \sec\vartheta\, I_0(\mathbf{n}, t), \\ \mathbf{G}_1^{(1)}(\mathbf{n}, \tau) &= (\sec\vartheta - 1)\, \mathbf{G}_0(\mathbf{n}, \tau) - \sec\vartheta\, \mathbf{G}_0(\mathbf{n}, t) \end{aligned}\right\} \cos\vartheta < 0. \quad (32)$$

These solutions do not apply to the vicinity of the point $\vartheta = \pi/2$, where they possess a singularity; the approximate equations (30) themselves cease to be valid near $\vartheta = \pi/2$. However, the angular region near $\pi/2$ is of least interest since the number of particles moving at such angles is very small

To obtain the next approximation we substitute the values found for I_1 and $\mathbf{G}_1$ into J and $\mathbf{Y}$, which we shall regard as known functions. Then from (29) with the corresponding boundary conditions we obtain for $\cos\vartheta > 0$:

$$I_1^{(2)}(\mathbf{n}, \tau) = (\sec\vartheta - 1)\, I_0(\mathbf{n}, \tau) + \sec\vartheta \int_0^\tau J(\mathbf{n}, \tau')\, d\tau',$$

$$\mathbf{G}_1^{(2)}(\mathbf{n}, \tau) = (\sec\vartheta - 1)\, \mathbf{G}_0(\mathbf{n}, \tau) + \sec\vartheta \int_0^\tau \mathbf{Y}(\mathbf{n}, \tau')\, d\tau'; \quad (33)$$

and for $\cos\vartheta < 0$:

$$I_1^{(2)}(\mathbf{n}, \tau) = (\sec\vartheta - 1)\, I_0(\mathbf{n}, \tau)$$

$$- \sec\vartheta\, I_0(\mathbf{n}, t) - \sec\vartheta \int_\tau^t J(\mathbf{n}, \tau')\, d\tau',$$

$$\mathbf{G}_1^{(2)}(\mathbf{n}, \tau) = (\sec\vartheta - 1)\, \mathbf{G}_0(\mathbf{n}, \tau)$$

$$- \sec\vartheta\, \mathbf{G}_0(\mathbf{n}, t) - \sec\vartheta \int_\tau^t \mathbf{Y}(\mathbf{n}, \tau')\, d\tau'. \quad (34)$$

The method of estimating corrections containing J and $\mathbf{Y}$ is similar to that by which the Fokker-Planck equation is obtained from the exact kinetic equation. We shall make use of the sharply anisotropic character of the Coulomb cross section and shall consider only small-angle scattering. In this case the first Born approximation can also be applied to heavy nuclei. Replacing $\sin\theta/2$ by $\theta/2$

and cos $\theta/2$ by 1 and retaining the terms with the largest values when θ is small, we obtain from the formulas given in the Appendix:

$$B(\theta) = 16\,\sigma_0 q(\theta)/\theta^4;\ F(\theta) = 16\,\sigma_0 bq(\theta)/\theta^3,$$
$$C(\theta) = 8\,\sigma_0 b^2 q(\theta)/\theta^2,\ D(\theta) = 0. \qquad (35)$$

Here $q(\theta)$ takes into account the screening of the nuclear field by atomic electrons and removes the divergence at $\theta = 0$.

$I_1^{(1)}(n', \tau)$ and $G_1^{(1)}(n', \tau)$ near the point $n' = n$ are expanded in series, keeping terms up to the second order inclusively. We thus obtain

$$G_1^{(1)}(n', \tau) = G_1^{(1)}(n, \tau) + (\theta\nabla)\, G_1^{(1)}(n, \tau) + \frac{1}{2}\sum_{ik}\theta_i\theta_k\nabla_i\nabla_k G_1^{(1)}(n, \tau) \qquad (36)$$

and a similar expansion for $I_1^{(1)}$. In this expression the difference $n' - n$ is replaced by θ; $|\theta| = \theta$, the vector θ lying in a plane perpendicular to n. i and k take two values; $\theta_1 = \theta\cos\Phi$, $\theta_2 = \theta\sin\Phi$, $0 \le \Phi \le 2\pi$. ∇ is the portion of the gradient which operates on the angles. When integrating over angles we must replace $d\Omega'$ by $\theta\, d\theta\, d\Phi$. Using (35) and integrating, we obtain

$$J(n, \tau) = \varkappa\nabla^2 I_1^{(1)}(n, \tau),$$
$$Y(n, \tau) = \varkappa\{\nabla^2 G_1^{(1)} + 2b\times[n\times\nabla]\times G_1^{(1)} - b^2[G_1^{(1)} + (n\cdot G_1^{(1)})\,n]\}. \qquad (37)$$

Here κ denotes $8\pi(\sigma_0/\sigma)\int_0^{\pi} q(\theta)\,d\theta/\theta$.

The approximation under discussion can be used when the terms containing J and Y in (33) and (34) are much smaller than $I_1^{(1)}$ and $G_1^{(1)}$. It follows from (33), (34), and (37) [see also (41) and (42)] that this condition is satisfied when $\kappa t \ll |\cos^3\vartheta|$, excluding the angular region in the vicinity of $\vartheta = \pi/2$ and limiting the thickness of the scatterer. At small angles $|\vartheta^2 \le \kappa t|$ the solution of (II) need not be considered at all, since in this region a good approximation is given by (25) and (27) or by the equations in reference 4.

We can estimate the order of κ by setting $q(\theta) = 0$ for $\theta < \chi_0$, $q(\theta) = 1$ for $\theta > \chi_0$, where $\chi_0 = \lambda\!\!\!^-/a$, $\hbar = \lambda/p$ is the de Broglie electron wavelength, $a = 0.885\, a_0 Z^{1/3}$ is the Thomas-Fermi atomic radius and a_0 is the Bohr radius. $\chi_0 < 10^{-2} Z^{1/3}$ for electrons of all energies beginning with 150 kev. With σ and κ calculated in the same approximation, we obtain

$$2\varkappa = \chi_0^2 \ln(\pi/\chi_0). \qquad (38)$$

Finally, for I and G we obtain for forward scattering ($\cos\vartheta > 0$):

$$I(n, t) = \sec\vartheta I_0(n, t) + \sec\vartheta\int_0^t J(n, \tau)\,d\tau,$$
$$G(n, t) = \sec\vartheta G_0(n, t) + \sec\vartheta\int_0^t Y(n, \tau)\,d\tau, \qquad (39)$$

and for backward scattering ($\cos\vartheta < 0$):

$$I(n, 0) = -\sec\vartheta I_0(n, t) - \sec\vartheta\int_0^t J(n, \tau)\,d\tau,$$
$$G(n, 0) = -\sec\vartheta G_0(n, t) - \sec\vartheta\int_0^t Y(n, \tau)\,d\tau. \qquad (40)$$

It is evident from (39) and (40) that, neglecting terms proportional to κ, the vector ζ equals the ratio G_0/I_0, i.e., G_0 and I_0 are good zeroth approximations for the determinations of ζ.

Integration over τ in (39) and (40) can easily be performed when the explicit forms of I_0 and G_0 are used. In obtaining explicit expressions for J and Y from (37) it is convenient to use the following relations. When $G = \sec\vartheta G_0$, where $G_0 = \Sigma G_{JM}^{L} Y_{JM}^{L}(n)$, we have

$$[n\times\nabla]\times G = \sec^2\vartheta\,[n_0\times n]\times G_0 + \sec\vartheta\sum B_L^J G_{JM}^L Y_{JM}^L, \qquad (41)$$

where n_0 is a unit vector in the z direction;

$$B_L^L = 1,\quad B_L^{L+1} = L+1,\quad B_L^{L-1} = -L,$$
$$\nabla^2 G = 2\sec^3\vartheta G_0 - 2\sec^2\vartheta\,(n_0\cdot\nabla)\,G_0 + \sec\vartheta\nabla^2 G_0. \qquad (42)$$

Operation by ∇^2 on a spherical vector or spherical harmonic is equivalent to multiplication by $-L(L+1)$. Use of the operator $(n_0\cdot\nabla)$ is equivalent to calculating the z-component of the gradient of a spherical harmonic, as was done in reference 9.

6. EVALUATION OF INTEGRALS

We do not know the exact functions f and g which determine the amplitude of electron scattering by an atom. In order to evaluate the integrals in (12), (16), (18), and (20) we take the Coulomb functions f and g as series in (αZ) and consider terms up to $(\alpha Z)^3$ in the scattering cross section. This is the second Born approximation, which is apparently sufficiently good for light and intermediate nuclei, and can be used for approximate evaluation in the case of heavy nuclei.

To allow for screening we shall regard the scattering cross section as vanishing for $\theta < \chi_0$, the angle χ_0 being given in the preceding section.

In calculating the integrals it is convenient to use the representation of associated Legendre polynomials as sums of powers of $\sin(\theta/2)$.[12]

As a result of integration terms of the form

$\ln \sin(\chi_0/2) \approx \ln(\chi_0/2)$ and $1-\sin^n(\chi_0/2)$, $n \geq 1$, appear after the summation signs. The second of these (the difference) can be replaced by unity, after which the corresponding sums will depend only on l and $\ln(\chi_0/2)$ and can easily be computed directly. We finally obtain the following expressions for the integrals:

$$B_l = 1 - 8\pi(\sigma_0/\sigma)\, l(l+1)[\ln(2/\chi_0) - s_0(l) + 1] + 8\pi\beta^2(\sigma_0/\sigma)s_0(l) + 8\pi^2\beta\alpha Z(\sigma_0/\sigma)[s_0(l) - 2l]; \quad (43)$$

$$\int_0^\pi F(\theta)P_l^1(\theta)\sin\theta\, d\theta = -(4\sigma_0/\sigma)b^2 + 8b(\sigma_0/\sigma)\{l(l+1)[\ln(2/\chi_0) - s_0(l) + 1] - {}^1/_2(l+{}^1/_2)^2\} - 4\pi\beta\alpha Z(\sigma_0/\sigma)(2l+1)^{-1}\{(1-\beta^2)^{1/2} \times [(l-1)(2l+3)+2] - 2(l-1)(2l+3) - 5\}; \quad (44)$$

$$C_{l0} = 4b^2(\sigma_0/\sigma)\{2\ln(2/\chi_0) - 2s_0(l) - \delta_{l0}\} - 4\pi\beta\alpha Z(\sigma_0/\sigma)b\{\delta_{l0} - 2/(2l+1)\}; \quad (45)$$

$$C_{l2} = 4b^2(\sigma_0/\sigma)(l-1)(l+2) + 8\pi\beta\alpha Z(\sigma_0/\sigma)b(l-1)(l+2)/(2l+1); \quad (46)$$

$$\int_0^\pi C(\theta)P_l^1(\theta)\,d\theta = 2b^2(\sigma_0/\sigma) \times \{2l(l+1)[\ln(2/\chi_0) - s_0(l) + 1] - (l+{}^1/_2)^2\} + 2\pi\beta\alpha Z(\sigma_0/\sigma)b\{2l - (l+{}^1/_2)^2(\chi_0/2)\}; \quad (47)$$

$$D_l = 8\pi\beta\alpha Z(\sigma_0/\sigma)[(1-\beta^2)/l(l+1)]^{1/2} \times \{{}^1/_4(l+{}^1/_2)^2\chi_0^2\ln(2/\chi_0) - s_0(l)\}; \quad (48)$$

$$\sigma = 8\pi\sigma_0\{(2/\chi_0^2) - \beta^2\ln(2/\chi_0) + \pi\beta\alpha Z[2/\chi_0 - \ln(2/\chi_0)]\}; \quad (49)$$

$$s_0(l) = \sum_{k=1}^{l} 1/k = \Psi(l+1) + C. \quad (50)$$

$\Psi(l)$ is the logarithmic derivative of the γ function and C is the Euler constant.

Equations (43) – (48) are valid for $l\chi_0 \ll 1$. They contain one parameter which is partly arbitrary – the screening angle χ_0.

7. COMPARISON WITH THE THEORY OF MÜHLSCHLEGEL AND KOPPE

We shall show that when only small-angle scattering is considered our own Eqs. (25) and (28) go over into the corresponding equations in reference 4. Following Bethe, we replace the Legendre polynomials with Bessel functions through the following familiar formula,[12] which holds true for $\theta \ll 1$:

$$P_l^{-m}(\cos\theta) = (-1)^m (l+{}^1/_2)^{-m} J_m((l+{}^1/_2)\theta). \quad (51)$$

We shall integrate over θ from 0 to ∞. Also, in the expansion coefficients (16), (18), and (20) we shall neglect all terms except those containing the highest power of l, since terms with large l make the principal contributions to the sums in (25) – (28). These simplifications lead to the following expressions for the expansion coefficients:

$$B_l = 2\pi\int_0^\infty B(\theta)J_0(l\theta)\theta\, d\theta,$$

$$C_{ll}^l = -\pi\int_0^\infty C(\theta)J_0(l\theta)\theta\, d\theta - \pi\int_0^\infty C(\theta)J_2(l\theta)\theta\, d\theta,$$

$$C_{ll}^{l+1} = C_{ll}^{l-1} = -\frac{3\pi}{2}\int_0^\infty C(\theta)J_0(l\theta)\theta\, d\theta' + \frac{\pi}{2}\int_0^\infty C(\theta)J_2(l\theta)\theta\, d\theta,$$

$$C_{l+1\,l-1}^l = C_{l-1\,l+1}^l = \frac{\pi}{2}\int_0^\infty C(\theta)\{J_0(l\theta) + J_2(l\theta)\}\theta\, d\theta,$$

$$F_l^l = 0,\ F_l^{l+1} = -F_l^{l-1} = 2\pi\int_0^\infty F(\theta)J_1(l\theta)\theta\, d\theta,$$

$$D_l = 2\pi\int_0^\infty D(\theta)J_1(l\theta)\theta\, d\theta. \quad (52)$$

By comparing (52) with the equations (24) of reference 4 we easily obtain the relation between the notation of reference 4 and our notation:

$$\omega(l) = B_l\tau;\quad \omega_0 = \tau;\quad C_{ll}^{l+1}\tau = -2(\gamma+\gamma_1);\quad D_l\tau = \delta(l),$$
$$F_l^{l+1}\tau = \varphi(l);\quad C_{ll}^l\tau = -2\gamma_1(l);\quad c_l\tau = \gamma_1(l). \quad (53)$$

In (25) the summation over l is replaced by an integral and the use of (53) leads to

$$I(\mathbf{n},\tau) = \frac{I_0}{2\pi}\int dl\cdot l\left\{\cosh\sqrt{\gamma_1^2-\delta^2} + \frac{\gamma_1}{\sqrt{\gamma_1^2-\delta^2}}\sinh\sqrt{\gamma_1^2-\delta^2}\right\}e^{\omega-\omega_0-\gamma_1}J_0(l\vartheta) + \frac{I^{(0)}P_0}{2\pi}\sin\chi\sin\varphi\int dl\cdot l \times \frac{\delta}{\sqrt{\gamma_1^2-\delta^2}}\sinh\sqrt{\gamma_1^2-\delta^2}\, e^{\omega-\omega_0-\gamma_1}J_1(l\vartheta), \quad (54)$$

which agrees with (30) of reference 4.

The equations in (32) can be transformed in similar fashion.

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APPENDIX

Up to terms containing $(\alpha Z)^2$ the functions f and g are given by

$$f = -F' + G;\ g = F'\cot(\theta/2) + G\tan(\theta/2),$$

$$F' = -(Ze^2/2pv)(1-\beta^2)^{1/2}\left[1 - (2i\alpha Z/\beta)\left(C + \ln\sin\frac{\theta}{2}\right)\right],$$

$$G = (Ze^2/2pv)\left\{\cot^2\frac{\theta}{2} + (\pi\beta\alpha Z/2)\left(\operatorname{cosec}\frac{\theta}{2} - 1\right)\right.$$

$$\left. + i\alpha Z\left[(2/\beta)\cot^2\frac{\theta}{2}\left(C + \ln\sin\frac{\theta}{2}\right) - (\beta/2)\ln\operatorname{cosec}^2\frac{\theta}{2}\right]\right\},$$

where C is the Euler constant.

Up to $(\alpha Z)^3$ we have $B(\theta)$, $C(\theta)$, $D(\theta)$, and $F(\theta)$ given by

$$B(\theta) = \sigma_0\operatorname{cosec}^4\frac{\theta}{2}\left[1 - \beta^2\sin^2\frac{\theta}{2} + \pi\beta\alpha Z\sin\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\right)\right],$$

$$C(\theta) = 2b^2\sigma_0\cot^2\frac{\theta}{2} + 2\pi\beta b\sigma_0\alpha Z\left(\operatorname{cosec}\frac{\theta}{2} - 1\right),$$

$$D(\theta) = 4\beta\sigma_0\alpha Z(1-\beta^2)^{1/2}\operatorname{cosec}\theta\ln\sin\frac{\theta}{2},$$

$$F(\theta) = 2\sigma_0\cot\frac{\theta}{2}\cdot b\left[(1-\beta^2)^{1/2} + \cot^2\frac{\theta}{2}\right]$$

$$+ 2\pi\beta\sigma_0\alpha Z\tan\frac{\theta}{2}\left\{\left(\operatorname{cosec}^2\frac{\theta}{2} - 1\right)\right.$$

$$\left. + (1-\beta^2)^{1/2}\left(1 - \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\right)\right\}\left(\operatorname{cosec}\frac{\theta}{2} - 1\right),$$

$$b = 1 - (1-\beta^2)^{1/2};\ \sigma_0 = (Ze^2/2pv)^2.$$

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