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ON THE SCATTERING MATRIX IN AN INDEFINITE METRIC

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A method is presented which, in theories with indefinite metric, excludes all nonphysical states from the initial and final states of the system. The method is applied to the Lee model and the scalar photon model.

1. In a paper by Heisenberg[1] it has been shown with the Lee model that a theory with an indefinite metric can give physically reasonable results if one adds to the initial states of the physical system a definite amplitude of the nonphysical states. This idea has been developed in a paper by Bogolyubov, Medvedev, and Polivanov,[2] who introduce the requirement that the amplitude of the nonphysical states be chosen in such a way that these states not be scattered (standing-wave condition). The essential defect of the recipe developed in references 1 and 2 is that it explicitly dispenses with macroscopic causality: the preparation of the initial state is dependent on the process to be studied. In the present paper a procedure is proposed which evidently does not have this disadvantage.

We can explain the idea of the paper by the example of rotations in three-dimensional Euclidean space. We shall call the xy plane the physical space (Hilbert space I), and shall call the space which supplements this to make the full three-dimensional space the nonphysical space (Hilbert space II). An arbitrary state is represented by a vector which starts at the origin. The manifold of vectors lying in the xy plane forms the manifold of physical states. The S matrix in the three-dimensional space is represented by a rotation of an arbitrary vector around a certain axis through a definite angle. If the axis is perpendicular to the xy plane, the action of the S matrix does not carry vectors that lie in the xy plane out of this plane. This situation illustrates a quantum theory with a definite metric. In order, however, to escape from divergences one must renounce the orthogonality of the axis of the S matrix to the physical space. But then (see Fig. 1) the action of the S matrix (S rotation) carries vectors out of the xy plane, and projections of vectors onto this plane are not conserved; the S matrix loses its physical meaning.

Following Bogolyubov et al.[2] we can restore the physical meaning of the S matrix (i.e., conserve the length of the projection of a vector onto the xy plane) by choosing for each initial physical state a projection along the z axis such that this projection is not changed in absolute value by the S rotation (Fig. 2).

The transformation $a \to a'$ can be called the S matrix for the physical states. In this example we can clearly see the impossibility of a further rotation $b' \to b''$ without a redefinition of the "ghost" component of the state vector.

The idea of the present work also consists of a transformation from S to another matrix S′, but of a different form. We call the matrix

$$S' = U^{-1}SU, \tag{1}$$

the S′ matrix for the physical states; the matrix U produces a rotation of the xy plane into the plane perpendicular to the axis of S. It is easy to see that the matrix S′ takes physical vectors into physical vectors with preservation of lengths (Fig. 3); the rotation U takes a into b, the rotation S takes b into b′, and finally U^{-1} takes b′ into a′.

In the general case of an indefinite metric the matrix U is a pseudounitary matrix, which transforms the physical space into an invariant subspace of the matrix S isomorphic to the physical space. In other words, the matrix U produces a transformation of the matrix S to a form in which the matrix elements between physical and nonphysical states are zero. Of course not every matrix S can be transformed to such a form. From the point of view used here, however, this means only that the only theories with indefinite metrics that can

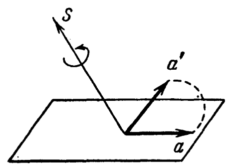

FIG. 1

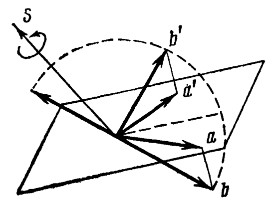

FIG. 2

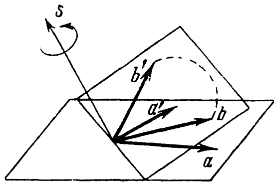

FIG. 3

describe actual processes are those that are such that the transformation U exists.

2. Let us find the equation determining the required pseudounitary transformation for a given form of the S matrix. We find a subspace of states invariant under the transformation S and consisting of vectors of the form

$$\Phi = \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} F \\ WF \end{pmatrix}, \tag{2}$$

where the upper row represents the projection onto the physical states, and the lower row the projection onto the space of states supplementing the physical space to the whole space. The transformation S takes the vectors (2) over into vectors of the same form:

$$F_+ = S_{11}F_- + S_{12}WF_-, \tag{3}$$

$$G_+ = WF_+ = S_{21}F_- + S_{22}WF_-, \tag{4}$$

where

$$S_{11} = P_1SP_1,\ S_{12} = P_1SP_2,\ S_{21} = P_2SP_1,\ S_{22} = P_2SP_2, \tag{5}$$

where P_1 and P_2 are operators of projection onto the state-spaces F and G.

Thus the invariant subspace exists if there exists a solution of the equation

$$WS_{11} + WS_{12}W = S_{21} + S_{22}W. \tag{6}$$

If besides this the invariant subspace (2) is a space of vectors with positive norm, then by the pseudounitary transformation U related to W by the equation

$$W = -P_2UP_1 / P_2UP_2 = -U_{21} / U_{22}, \tag{7}$$

it can be transformed into a physical space isomorphic to it, namely

$$U\begin{pmatrix} F \\ WF \end{pmatrix} = \begin{pmatrix} U_{11}F + U_{12}WF \\ 0 \end{pmatrix}. \tag{8}$$

Then it can be verified that the transformation

$$S' = USU^+ \tag{9}$$

takes physical states into physical states. We shall call this transformation the actual S matrix.

The equation (6) is a quadratic equation for the matrix W. It is hard to solve this equation, at least in the general case. In the present paper we succeed in finding the matrix elements of this matrix only for two simple examples — the scattering of a Θ particle by a V particle in the Lee model, and the scattering of a "scalar photon" by an "electron" near the threshold of production of "ghost" states.

3. Let us find the matrix U for the example of the Lee model. In the Lee model one has three particles interacting according to the scheme

$$V \rightleftarrows N + \Theta.$$

Furthermore all the states separate into sectors $(N + z\Theta,\ V + (z-1)\Theta)$, which do not make transitions to each other, so that they can be considered separately. We shall examine the case in which the V particle can exist in two different discrete states, one of which has a negative norm and is called a "ghost" state of the V particle. Repeating the calculation of Heisenberg (Sec. 3.1 of reference 1), we find that the function $\varphi(k)$ which describes the scattering of the Θ particle in the momentum representation obeys the equation [Eq. (68) of reference 1]

$$h^+(E-\omega)\,\varphi(\omega) = \int_{m_\theta}^{\infty} K(\omega, \omega')\,\varphi(\omega')\,d\omega + \varphi_{0i}(\omega), \tag{10}$$

where

$$K(\omega, \omega') = -\frac{1}{2}\,\frac{k'}{\omega + \omega' - E - i\gamma}\sqrt{\frac{\omega'}{\omega}},$$

and the function φ_{0i} depends on the initial amplitude of the state $N + 2\Theta$. In what follows we shall for simplicity set $\varphi_{0i} = 0$. The remaining notations are taken from reference 1. The general solution of Eq. (10) has the form (apart from an arbitrary common factor)

$$\varphi(\omega) = \delta(\omega - \omega_p) + x\delta(\omega - \omega_g) + \frac{\chi(\omega)}{h^+(E-\omega)}, \tag{11}$$

where $\omega_{p,g} = E - E_{p,g}$; E_p and E_g are the energies of the physical and ghost states of the V particle. We consider the case $E_p \neq E_g$.

In the stationary theory the S matrix gives the connection between the asymptotic values of converging and diverging waves:

$$\Phi_+ = S\Phi_-. \tag{12}$$

If Φ_+ and Φ_- can be represented in the form

$$\Phi_+ = UF_+,\ \Phi_- = UF_-, \tag{13}$$

where F_+ and F_- are purely physical states and U is a pseudounitary transformation, then, rewriting Eq. (12) in the form

$$F_+ = U^{-1}SUF_-, \tag{14}$$

we get the new scattering matrix

$$S' = U^{-1}SU, \tag{15}$$

which obviously takes physical states into physica states and preserves the norm. We emphasize that from the point of view of the present idea the physical scattering matrix is just the matrix (15), and the matrix (12) is an intermediate stage in the calculation, which is necessary if we wish to keep the old apparatus of the quantum field theory.

The condition (13) means that for the determination of the actual scattering we must choose from among all possible solutions of Eq. (10) those that are transformed by some pseudounitary transformation U^{-1} into expressions that asymptotically do not contain any "ghost" states.

Let us rewrite the function (11) in a form convenient for the pseudounitary transformation. For this purpose we must recall that for the passage to the coordinate representation we have the following correspondence:

$$\delta(\omega-\omega_0) \to \omega_0\left(\frac{1}{r}e^{ik_0r}-\frac{1}{r}e^{-ik_0r}\right),$$
$$\frac{1}{2\pi i}\,\frac{1}{\omega-\omega_0-i\gamma} \to \omega_0\left(\frac{1}{r}e^{ik_0r}\right). \tag{16}$$

With this in mind, we rewrite Eq. (11) in the form

$$\varphi(\omega) = a'\left[\frac{1}{\omega_p\sqrt{v_p}}\,\delta(\omega-\omega_p)\right] + \chi_p'(\omega)\left[\frac{1}{2\pi i\omega_p\sqrt{v_p}}\cdot\frac{1}{\omega-\omega_p-i\gamma}\right] + b'\left[\frac{1}{\omega_g\sqrt{v_g}}\,\delta(\omega-\omega_g)\right] + \chi_g'(\omega)\left[\frac{1}{2\pi i\omega_g\sqrt{v_g}}\cdot\frac{1}{\omega-\omega_g-i\gamma}\right], \tag{17}$$

$$a' = \omega_p\sqrt{v_p};\quad b' = x\omega_g\sqrt{v_g};\quad v_p = k_p/\omega_p;\quad v_g = k_g/\omega_g;$$
$$\chi_p' = \omega_p\sqrt{v_p}\,g(\omega)\,\chi(\omega);\quad \chi_g' = -\omega_g\sqrt{v_g}\,g(\omega)\,\chi(\omega);$$
$$g(\omega) = 2\pi i\,\frac{(\omega-\omega_p-i\gamma)(\omega-\omega_g-i\gamma)}{(\omega_p-\omega_g)\,h^+(E-\omega)};\quad \gamma\to 0. \tag{18}$$

The form (17) is convenient because the coefficients of the square brackets give a direct idea of the fluxes of the converging and diverging waves in the coordinate representation. Writing the expression in columns emphasizes the attribution to different states of the V particle (cf. reference 3). The desired pseudounitary transformation acts on the coefficients of the function (17) and can be represented in the form of a two-rowed pseudounitary matrix:

$$U = \begin{pmatrix} e^{i\alpha}\cosh\theta & e^{i\beta}\sinh\theta \\ e^{i\gamma}\sinh\theta & e^{i\delta}\cosh\theta \end{pmatrix},\quad \alpha-\beta-\gamma+\delta=0 \tag{19}$$

We now look for the class of functions (17) which are taken by a transformation of the form (19) into purely physical states at large distances from the origin. This imposes on the amplitudes of the converging and diverging waves the conditions

$$a'e^{i\gamma}\sinh\theta + b'e^{i\delta}\cosh\theta = 0, \tag{20}$$

$$(a'+\chi_p')\,e^{i\gamma}\sinh\theta + (b'+\chi_g')\,e^{i\delta}\cosh\theta = 0. \tag{21}$$

From this we get

$$\chi_p'(\omega_p)/a' = \chi_g'(\omega_g)/b'$$
$$\text{or}\quad xg(\omega_p)\,\chi(\omega_p) = -g(\omega_g)\,\chi(\omega_g). \tag{22}$$

Thus the class of solutions of Eq. (6) in which we are interested has the property that states with different states of the V particle are "scattered" in the same way. This still does not mean the absence of scattering, since there is possible the inelastic process

$$V+\theta \to N+2\theta.$$

The function (11) depends on one parameter x. Equation (22) can be regarded as an equation in the parameter x. Using the integral equation for $\chi(\omega)$,

$$\chi(\omega) = K(\omega,\omega_p) + xK(\omega,\omega_g) + \int K(\omega,\omega')\,\frac{\chi(\omega')}{h^+(E-\omega')}\,d\omega', \tag{23}$$

we rewrite Eq. (22) in the form

$$xg(\omega_p)\Big[K(\omega_p,\omega_p) + xK(\omega_p,\omega_g) + \int K(\omega_p,\omega')\,\frac{\chi(\omega';x)}{h^+(E-\omega')}\,d\omega'\Big]$$
$$= -g(\omega_g)\Big[K(\omega_g,\omega_p) + xK(\omega_g,\omega_g) + \int K(\omega_g,\omega')\,\frac{\chi(\omega';x)}{h^+(E-\omega')}\,d\omega'\Big]. \tag{24}$$

By writing $\chi(\omega', x)$ we emphasize the parametric dependence of the solution of Eq. (23) on x. Finding x from Eq. (24), we can recover the form (19) of the matrix, using the condition (20)

$$e^{i(\gamma-\delta)}\tanh\theta = -x\omega_g\sqrt{v_g}/\omega_p\sqrt{v_p}. \tag{25}$$

This equation is solvable if

$$|x| < \omega_p\sqrt{v_p}/\omega_g\sqrt{v_g}. \tag{26}$$

Thus we conclude that in the Lee model one can construct a scheme for calculating final physical states from initial physical states which preserves norms, if among the roots of the equation (24) there are values satisfying the condition (26).

Accordingly, we have shown for the example of the Lee model that under certain conditions the program proposed in Section 1 is feasible. The example considered is, however, a weak one, since in it we deal only with a spherically symmetric, i.e., a one-dimensional, problem. Therefore it is interesting to examine some other example closer to reality, for instance a model of the type of quantum electrodynamics. Such a model will be considered in the next section.

4. Let us now examine the model in which a spinor field ("electron") interacts with a real scalar field (the "scalar photon"):

$$L = g : \bar{\psi}(x)\varphi(x)\psi(x):, \tag{27}$$

where the interaction constant is taken to be small.

A Green's function that falls off sufficiently rapidly can be obtained[5] by supposing that the free field obeys an equation containing higher derivatives. For our illustrative purposes it is enough to consider the second-order equation

$$\left(-i\frac{\hat{\partial}}{\partial x} - m\right)\left(-i\frac{\hat{\partial}}{\partial x} - M\right)\psi(x) = 0; \quad M \gg m \tag{28}$$

(here $\frac{\hat{\partial}}{\partial x} \equiv \sum_\mu \gamma^\mu \frac{\partial}{\partial x^\mu}$), which leads to the Green's function

$$G(p) \sim 1/(\hat{p} - m)(\hat{p} - M). \tag{29}$$

The second-quantized function $\psi(x)$ that leads to the Green's function (29) can be represented as the sum of two fields, one of which satisfies the usual commutation relations, and the other, similar relations, but with the minus sign. This leads to the appearance of states with negative norm, which on physical grounds are not admissible. Thus this model can serve as a simple example of a theory with an indefinite metric, with which one can see the main features of any theory with an indefinite metric.

When a scalar photon is scattered by an electron at energy higher than the threshold, nonphysical states can arise, with the electron in the mass state M, which has a negative norm. Therefore we must consider a state of the form

$$\Phi = (2\pi)^3 \left\{ \begin{matrix} \sum\limits_{\nu=1,2} \int d\mathbf{p} d\mathbf{k} \varphi_\nu(\mathbf{p}, \mathbf{k})\, b_\nu^{*+}(\mathbf{p})\, a^+(\mathbf{k}) \\ \sum\limits_{\mu=1,2} \int d\mathbf{q} d\varkappa \psi_\mu(\mathbf{q}, \varkappa)\, c_\mu^{*+}(\mathbf{q})\, a^+(\varkappa) \end{matrix} \right\} |0\rangle, \tag{30}$$

where $|0\rangle$ is the state function of the vacuum, $a^+(\mathbf{k})$ is the operator for creation of a scalar photon,

$$[a^-(\mathbf{k}), a^+(\mathbf{k}')]_- = \delta(\mathbf{k} - \mathbf{k}'), \tag{31}$$

$b_\nu^{*+}(\mathbf{p})$ is the operator for creation of an electron in a state with rest mass m,

$$[b_\nu^-(\mathbf{p}), b_\mu^{*+}(\mathbf{p}')]_+ = \delta_{\mu\nu}\delta(\mathbf{p} - \mathbf{p}'). \tag{32}$$

$c_\mu^{*+}(\mathbf{q})$ is the operator for creation of an electron in a "ghost" state with rest mass M,

$$[c_\nu^-(\mathbf{q}), c_\mu^{*+}(\mathbf{q}')]_+ = -\delta_{\mu\nu}\delta(\mathbf{q} - \mathbf{q}'). \tag{33}$$

We choose the amplitude of the initial physical state in the form

$$\varphi_\nu(\mathbf{p}, \mathbf{k}) = \delta_{\nu\nu_0}\delta(\mathbf{p} - \mathbf{p}_0)\,\delta(\mathbf{k} - \mathbf{k}_0). \tag{34}$$

The amplitudes of the "ghost" states are related to the physical states by the required matrix W of Eq. (7):

$$\psi_\mu(\mathbf{q}, \varkappa) = \sum_{\nu=1,2} \int d\mathbf{p} d\mathbf{k} W_{\mu\nu}(\mathbf{q}, \varkappa; \mathbf{p}, \mathbf{k})\varphi_\nu(\mathbf{p}, \mathbf{k}). \tag{35}$$

Thus the initial state of the system has the form

$$\Phi_- = (2\pi)^3 \times \left\{ \begin{matrix} b_{\nu_0}^{*+}(\mathbf{p}_0)\, a^+(\mathbf{k}_0) \\ \sum\limits_{\mu_0=1,2} \int d\mathbf{q}_0 d\varkappa_0 W_{\mu_0\nu_0}(\mathbf{q}_0, \varkappa_0; \mathbf{p}_0, \mathbf{k}_0)\, c_{\mu_0}^{*+}(\mathbf{q}_0)\, a^+(\varkappa_0) \end{matrix} \right\} |0\rangle. \tag{36}$$

We shall examine the problem in the center-of-mass system, in which the total four-momentum of the system has the form

$$P = p_0 + k_0 = (E, 0). \tag{37}$$

Confining ourselves to the second order of perturbation theory, we write the ordinary S matrix in the form

$$S = 1 + S^{(2)} \tag{38}$$

so that the final states will have the form

$$\Phi_+ = \Phi_- + \begin{pmatrix} F'_+ \\ G'_+ \end{pmatrix}, \tag{39}$$

where

$$F'_+ = \frac{ig^2}{(2\pi)^2}(2\pi)^3 \sum_{\nu=1,2} \int d\mathbf{p}_2 d\mathbf{k}_2 a^+(\mathbf{k}_2)\, b_\nu^{*+}(\mathbf{k}_2) \times \Big[\frac{1}{\sqrt{2\omega_2}}\frac{1}{\sqrt{2\omega_0}}\bar{u}^{\nu(+)}(\mathbf{p}_2)\, Q(mm)\, u^{\nu_0(-)}(\mathbf{p}_0) - \sum_{\mu_0=1,2}\int d\mathbf{q}_0 d\varkappa_0 \frac{1}{\sqrt{2\omega_2 2\varkappa_0}}\bar{u}^{\nu(+)}(\mathbf{p}_2)\, Q(mM)\, U^{\mu_0(-)}(\mathbf{q}_0) \times W_{\mu_0\nu_0}(\mathbf{q}_0\varkappa_0; \mathbf{p}_0\mathbf{k}_0)\Big]\delta(p_2 + k_2 - P)\,|0\rangle; \tag{40}$$

$$G'_+ = \frac{ig^2}{(2\pi)^2}(2\pi)^3 \sum_{\mu=1,2} \int d\mathbf{q}_2 d\varkappa_2 a^+(\varkappa_2)\, c_\nu^{*+}(\mathbf{q}_2) \times \Big[\frac{1}{\sqrt{2\varkappa_2 2\omega_0}}\bar{U}^{\mu(+)}(\mathbf{q}_2)\, Q(Mm)\, u^{\nu_0}(\mathbf{p}_0) - \sum_{\mu_0=1,2}\int d\mathbf{q}_0 d\varkappa_0 \frac{1}{\sqrt{2\varkappa_2 2\varkappa_0}}\bar{U}^{\mu(+)}(\mathbf{q}_2)\, Q(MM)\, U^{\mu_0(-)}(\mathbf{q}_0) \times W_{\mu_0\nu_0}(\mathbf{q}_0, \varkappa_0; \mathbf{p}_0, \mathbf{k}_0)\Big]\delta(q_2 + \varkappa_2 - P)\,|0\rangle. \tag{41}$$

Here Q is defined as

$$Q = \frac{M\hat{f}_1(\hat{f}_1 + M)}{f_1^2(f_1^2 - M^2)} + \frac{M\hat{f}_2(\hat{f}_2 + M)}{f_2^2(f_2^2 - M^2)};$$

$$f_1 = p_1 + k_1 = p_2 + k_2; \qquad f_2 = p_1 - k_2 = p_2 - k_1, \tag{42}$$

where the indices 1, 2 denote the initial and final states of the system in states with rest mass m or M for the electron.

We look for a matrix W which has the property that simultaneously

$$G_- = WF_1 \tag{43}$$

and

$$G_+ = WF_+. \tag{44}$$

This gives us an equation for W:

$$\frac{1}{\sqrt{2\varkappa_2 2\omega_0}} \bar{U}^{\mu(+)} Q(Mm) u^{\nu_0}$$

$$- \sum_{\mu_0=1,2} \int dq_0 d\varkappa_0 \frac{1}{\sqrt{2\varkappa_2 2\varkappa_0}} \bar{U}^{\mu(+)} Q(MM) U^{\mu_0(-)} W_{\mu_0\nu_0}$$

$$= \sum_{\nu=1,2} \int dp_2 dk_2 W_{\mu\nu} \left[\frac{1}{2\omega_2 2\omega_0} \bar{u}^{\nu(+)} Q(mm) u^{\nu_0(-)} \right.$$

$$\left. - \sum_{\mu_0=1,2} \int dq_0 d\varkappa_0 \frac{1}{\sqrt{2\omega_2 2\varkappa_0}} \bar{u}^{\nu(+)} Q(mM) U^{\mu_0(-)} W_{\mu_0\nu_0} \right]. \tag{45}$$

We shall solve this quadratic equation for W near the threshold for creation of "ghosts," with the total energy of the system equal to

$$E = M(1+\varepsilon), \quad \varepsilon \ll 1. \tag{46}$$

We shall try to find W in the form

$$W_{\mu\nu}(q\varkappa; pk) = \bar{U}^{\mu(+)}(q) w u^{\nu(-)}(p) \delta(q + \varkappa - p - k). \tag{47}$$

We have here to use the facts that

$$\varkappa_2 = \varkappa_0 = \varkappa = M\varepsilon; \quad \omega_2 = \omega_0 = \omega = E/2,$$

$$\sum_{\nu=1,2} u^{\nu(-)} \bar{u}^{\nu(+)} = \frac{(\hat{p}+m)}{2p_0} \approx \frac{\hat{p}}{E};$$

$$\sum_{\mu=1,2} U^{\mu(-)} \bar{U}^{\mu(+)} = \frac{(\hat{q}+M)}{2q_0} \approx \frac{(\gamma_0+1)}{2},$$

$$Q(mm) \approx Q(mM) \approx Q(Mm)$$

$$\approx \frac{(1+\gamma_0)}{2M\varepsilon} \sim 1/\varepsilon; \quad Q(MM) \sim 1,$$

$$\int \delta(q_0 + \varkappa_0 - P) dq_0 d\varkappa_0 = \varkappa^2 dn_\varkappa,$$

$$\int \delta(p_2 + k_2 - P) dp_2 dk_2 = \frac{\omega^2}{2} dn_k. \tag{48}$$

We note that in Q(Mm) we have dropped the term proportional to $\hat{p}_1$, because by the Dirac equation $\hat{p}_1 u(p_1) = mu(p_1)$.

Equation (45) takes the form

$$Q(Mm) - \sqrt{\frac{\omega}{\varkappa}} \int dn_\varkappa \varkappa^2 Q(MM) \frac{\hat{q}_0 + M}{2q_0} w$$

$$- \sqrt{\frac{\varkappa}{\omega}} \int dn_{k_2} \frac{k_2^2}{2} w \frac{\hat{p}_2}{2} Q(mm)$$

$$+ \int dn_{k_2} dn_\varkappa \varkappa^2 \frac{k_2^2}{2} w \frac{\hat{p}_2}{E} Q(mM) \frac{\hat{q}_0 + M}{2q_0} w = 0. \tag{49}$$

In this equation the respective orders of magnitude of the terms are

$$\frac{1}{\varepsilon}, \quad \sqrt{\frac{1}{\varepsilon}} \varepsilon^2 w, \quad \sqrt{\varepsilon} \frac{1}{\varepsilon} w, \quad \frac{1}{\varepsilon} \varepsilon^2 w^2.$$

This means that $w \sim \epsilon^{-1/2}$. Keeping the main terms in Eq. (49), we get finally

$$Q(Mm) = \sqrt{\frac{\varkappa}{\omega}} 4\pi \frac{\omega^2}{2} w \frac{\gamma^2}{2} Q(mm).$$

From this we get the solution

$$w = 2\sqrt{2} / \pi M^2 \sqrt{\varepsilon}. \tag{50}$$

Thus the desired matrix W has the form

$$W_{\mu\nu}(q\varkappa; pk)$$

$$= \frac{2\sqrt{2}}{\pi M^2 \sqrt{\varepsilon}} \bar{U}^{\mu(+)}(q) u^{\nu(-)}(p) \delta(q + \varkappa - p - k). \tag{51}$$

This matrix produces a space of state vectors of the form

$$\Phi = (2\pi)^3$$

$$\times \left\{ \begin{array}{l} b_\nu^{*+}(p) a^+(k) \\ \sum_{\mu=1,2} \int dn_\varkappa \frac{2\sqrt{2}\varepsilon^2}{\pi \sqrt{\varepsilon}} \left(\bar{U}^{\mu(+)}(q) U^{\nu(-)}(p) \right) c_\mu^{*+}(q) a^+(\varkappa) \end{array} \right\} |0\rangle. \tag{52}$$

These states have positive norm, since the amplitude of the nonphysical states is small, $0(\epsilon^{3/2})$. Therefore the operator U that takes purely physical states over into states of the form (2) exists, and in the approximation in question has the form

$$U = \begin{pmatrix} 1 & W^+ \\ W & 1 \end{pmatrix}. \tag{53}$$

In the opposite limit of very large energies ($E \gg M$), when the difference of the rest masses of the two states of the "electron" can be neglected, the vectors of the required invariant subspace (2) will be vectors of the form

$$\Phi = \begin{pmatrix} F \\ F \end{pmatrix}, \tag{54}$$

i.e., vectors with zero norm. Therefore we may assume that in the general case $M < E < \infty$ these vectors will have positive norm, i.e., that for all energies the matrix U exists (it is obvious that for $E < M$ we have $U = 1$).

5. In conclusion, let us examine the question of the causal property of the matrix S′. Here it is convenient to use the concept of the switching-on function g(x). For the original S matrix

$$S(g) = T \exp \left\{ i \int L(x) g(x) dx \right\} \tag{55}$$

the causality condition can be written in the form[4]

$$S(g_1 + g_2) = S(g_1) S(g_2), \tag{56}$$

if $G_1 > G_2$, i.e., if the region in which g_1 is different from zero is located later in time than that where g_2 is different from zero. Equation (6), which defines the transformation U, was derived for the S′ matrix with the interaction completely switched on, i.e., for $g(x) = 1$ for all x. In the case of incomplete switching-on of the interaction two ways of defining the matrix S′ are possible:

$$S'(g) = U^{+} S(g) U, \tag{57}$$

$$S'(g) = U^{+}(g) S(g) U(g). \tag{58}$$

For the first S′ matrix the condition (56) is satisfied exactly, but this matrix does not contain transitions to nonphysical states only in the case in which $g(x) = 1$, at least in a macroscopically large region of four-space. For the second S′ matrix Eq. (6) is taken to be satisfied for arbitrary $g(x)$, i.e., it is exactly unitary with respect to physical transitions. On the other hand the causality condition (56) is satisfied only for macroscopically large regions with $g(x) = 1$, for which one can with arbitrary accuracy replace $U(g)$ by $U(1)$. Thus we come to the conclusion that it is necessary to give up one of the two fundamental properties of the scattering matrix; we must give up either microscopic causality or the exactly unitary character of the matrix. It is important, however, that in the large the matrix S′ is both causal and unitary in both cases.

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