BREAKUP OF A CHARGED LIQUID DROP AND NUCLEAR FISSION

B. T. GEĬLIKMAN

Moscow State Pedagogical Institute

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The disruption of a charged liquid drop is examined. The vibration energies of the parts of the drop are computed, as is the energy of their relative motion. The general results thus obtained are applied to nuclear fission.

In solving the problem of the breakup of a liquid drop into two parts, and in particular the problem of nuclear fission, the question arises of determining the generalized coordinates and velocities of the fragments of the drop after breakup from the equation of motion of the drop prior to breakup. We shall show below that knowledge of the internal coordinates and velocities of the fragments makes it possible to estimate the excitation energy of the fragments during nuclear fission.

Assume that the liquid is incompressible, that the motion of the liquid prior to and after breakup is potential, and that the radius vectors of the surface of the drop and of its fragments are of the following form (we confine ourselves to an axially-symmetrical case):

$$r(\vartheta)/R = 1 + \Sigma\alpha_n P_n(\cos\vartheta);$$
$$r_i(\vartheta_i)/R_i = 1 + \Sigma\alpha_n^{(i)} P_n(\cos\vartheta_i), \quad i = 1, 2.$$

In each case the origin is located at the center of gravity of the drop. From this condition and from the conservation of volume we obtain the coefficients α_0, α_1, $\alpha_0^{(i)}$, and $\alpha_1^{(i)}$ in terms of the remaining parameters α_n and $\alpha_n^{(i)}$.

Let us consider the change in speed of any mass element of the drop, Δm_d, during the time of breakup of the drop τ_b:

$$\Delta \mathbf{v}_d = \int_0^{\tau_p} \mathbf{f}_d dt/\Delta m_d \approx \mathbf{f}_d \tau_b/\Delta m_d.$$

We assume that τ_b is so small, that $\Delta v_d \ll v_d$ (this condition is satisfied in the case of nuclear fission). Then the velocities prior to and after breakup can be considered equal — this corresponds to the so-called shake-up method. Consequently,

$$\operatorname{grad}\varphi(x, y, z(z_i)) = \mathbf{v}_c^{(i)} + \operatorname{grad}\varphi_i(x, y, z_i), \quad i = 1, 2. \quad (1)$$

Here $\varphi(x, y, z(z_i))$ is the velocity potential prior to breakup of the drop, referred to the origin of the coordinates of the i-th part of the drop [i.e., z is assumed to be $\mp(z_i - d_c^{(i)})$; $d_c^{(i)}$ are the distances from the center of gravity of the drop to the centers of gravity of its parts]; $\varphi_i(x, y, z_i)$ is the velocity potential of the i-th part after breakup; $v_c^{(i)}$ is the velocity of the center of gravity of the i-th part after breakup:

$$\varphi = \sum_{n=2}^{\infty} \dot{\alpha}_n r^n P_n(\cos\vartheta)/nR^{n-2}; \quad \varphi_i = \sum_{n=2}^{\infty} \alpha_n^{(i)} r_i^n P_n(\cos\vartheta_i)/nR_i^{n-2}.$$

Changing to rectangular coordinates in the expressions for φ and φ_i, we obtain from (1):

$$\dot{\alpha}_2^{(i)} = \dot{\alpha}_2 \pm 2\dot{\alpha}_3 b_{ci} + 3\dot{\alpha}_4 b_{ci}^2 \pm \ldots,$$
$$\dot{\alpha}_3^{(i)} = -(\pm\dot{\alpha}_3 + 3b_{ci}\dot{\alpha}_4 \pm \ldots)(A_i/A)^{1/3}; \quad (2)$$
$$\dot{\alpha}_4^{(i)} = (\dot{\alpha}_4 \pm \ldots)(A_i/A)^{2/3}$$

(the minus corresponds to i = 1, the plus to i = 2); the relative velocities of the parts of the drop are

$$v_c = v_c^{(1)} + v_c^{(2)} = \dot{z}_c = [\dot{\alpha}_2(b_{c1} + b_{c2}) - \dot{\alpha}_3(b_{c1}^2 - b_{c2}^2)$$
$$+ \dot{\alpha}_4(b_{c1}^3 + b_{c2}^3) - \ldots]R; \quad b_{ci} = d_c^{(i)}/R.$$

These values of $\dot{\alpha}_n^{(i)}$ and $\dot{z}_c$ can be used as the initial conditions for the equations of motion of the parts of the drop. The initial values of the coordinates $\alpha_n^{(i)}$ and z_c can be readily calculated if the shape of the drop before breakup is known.[1]

In the case of nuclear fission, the potential energy U and the shape of the nucleus prior to fission were calculated for the states corresponding to minimum energy, within the framework of the Frankel and Metropolis[2] drop model (see also reference 3):

$$U(y) = {}^3/_{10}\, \varepsilon\xi(x, y) A^{2/3} e^2/r_0;$$
$$\xi(x, y) = 2.178\, y^2(1 - x) - 4.09\, y^3(1 - 0.645\, x)$$
$$+ 18.64\, y^4(1 - 0.894\, x) - 13.33\, y^5;$$
$$x = Z^2/A\varepsilon; \quad \varepsilon \equiv (Z^2/A)_{cr} = 48 \div 50;$$
$$r_0 = (1.2 \div 1.5)\cdot 10^{-13}\text{ cm}.$$
$$\alpha_0 = -y^2(1.06 + 9.76\cdot 10^{-4} f(y)); \quad (3)$$
$$\alpha_2 = y(2.3 + 5.42\cdot 10^{-4} f(y));$$
$$\alpha_4 = y^2[1.6 + y(3 + 2.84\cdot 10^{-3} f(y))];$$
$$\alpha_6 = -2.36\cdot 10^{-5} f(y); \quad \alpha_8 = 2\alpha_6; \quad f(y) = (0.49 - y)^{-4}.$$

The remaining $\alpha_n = 0$. In view of the smallness of α_6 and α_8, we first neglect them; y is a general deformation parameter. The fission barrier corresponds to $y = 1 - x$. The thickness of the neck d_n is close to zero ($d_n/R \sim 0.1$) when $y = y_k \approx 0.35$. If shell effects are taken into account $\alpha_3 \neq 0$. At the experimentally observed value ($A_1/A_2 \lessapprox 0.7$) of the fragment-mass ratio. $\alpha_3 \approx 0.06$.[4] The ratio $\dot{\alpha}_3/\dot{\alpha}_2$ at the break of the neck can be estimated in the following manner:

$$\dot{\alpha}_{3p}/\dot{\alpha}_{2p} \approx \alpha_{3p}/\Delta t_{\alpha_3}[(\alpha_{2p} - \alpha_{2c})/\Delta t_{\alpha_2}]^{-1}$$
$$\approx (1 \div 2)\,\alpha_{3p}\,(\alpha_{2p} - \alpha_{2c})^{-1} \approx 0.07 \div 0.12.$$

(Δt_{α_2} is the time of descent from the saddle point to the break in the neck; Δt_{α_3} is the time of descent from the point at which α_3 stops being equal to zero to the break in the neck; $\Delta t_{\alpha_2}/\Delta t_{\alpha_3} \sim 1$ or 2; α_{2b} is the value of α_2 at the point of the break of the neck, α_{2s} is the value at the saddle point); α_{2b} and α_{2s} were calculated from formula (3) for $x = 0.7$ to 0.8. The value of α_{3b} can thus be neglected (this introduces an error of 4 to 7% for $\dot{\alpha}_n^{(i)}$ and $\dot{z}_c$ when $\dot{\alpha}_3 = 0.1\,\dot{\alpha}_2$).

Since the deformation of the nucleus past the saddle point is slow,[5] one can assume that the nucleus passes through the values α_n corresponding to the minimum of the potential energy and which are consequently determined by formula (3). Then $\dot{\alpha}_n = y\,d\dot{\alpha}_n/dy$. In particular,

$$\dot{\alpha}_4 = \dot{\alpha}_2\,(d\alpha_4/dy)/(d\alpha_2/dy) = \beta(y)\,\dot{\alpha}_2.$$

At the point of breakup, i.e., at $y = y_k \approx 0.35$, we have $\beta \approx 0.8$. The kinetic energy of the drops after the breakup is of the form $T^{(i)} = \sum_{n,m} T_{nm}^{(i)}$.

Since $T_{nm} \gg T_{nm}$ ($n \neq m$), we have $T^{(i)} \approx \sum_{n=2}^{n_{max}} T_{nn}^{(i)}$.

In units of e^2/r_0 we have

$$T_{nn}^{(i)} = 3A_i^{5/3} I_n^{(i)}\,[\dot{\alpha}_n^{(i)}]^2/2n\,(2n+1);$$

$$I_n^{(i)} = \frac{n}{2}\int_{-1}^{1}[(P_n^{(1)}(\mu))^2/n^2 + (P_n(\mu))^2]\,[r_i(\mu)/R_i]^{2n+1}\,d\mu.$$

Let us estimate the ratio $T_{44}^{(i)}/T_{22}^{(i)}$ (for $A_1 \approx A_2$). If $I_4^{(i)} \approx I_2^{(i)}$ (see below), we have $T_{44}^{(i)}/T_{22}^{(i)} = 0.02$ to 0.04. One can therefore assume that $\alpha_4^{(i)} \approx 0$. It follows from (3) that $\alpha_4^{(i)} = 0$ when $y \approx 0.35$.[4] We can then confine ourselves to $n_{max} = 3$ in the expression for $r_i(\vartheta_i)$; $\alpha_2^{(i)}$ and $\alpha_3^{(i)}$ were found in reference 4 for various ratios A_1/A_2. When $A_1 = A_2$ we have $\alpha_2^{(i)} \approx 0.26$, $\alpha_3^{(i)} \approx 0.08$, and $2b_{c1} \approx 2.5$. For these values of $\alpha_n^{(i)}$ we have $I_2^{(i)} \approx 1.4$ and $I_3^{(i)} \approx 1.9$. (The values of $\alpha_n^{(i)}$, b_{ci}, and $I_n^{(i)}$ of reference 4a have been revised). Let us estimate the ratio of the kinetic energy $T^{(1)} + T^{(2)}$ of the internal motion of the fragments to the kinetic energy $T_c = v_c^2/2$ of the centers of gravity of the fragments (μ is the reduced mass of the fragments): when $A_1 = A_2$, we find from Eq. (2) ($\dot{\alpha}_4 \approx 0.8\,\alpha_2$) that $(T^{(1)} + T^{(2)})/(T_c + T^{(1)} + T^{(2)}) \sim 0.45$. This estimate is unfortunately not very accurate, for if we include the terms with $\dot{\alpha}_6$ and $\dot{\alpha}_8$ in the expressions for $\dot{\alpha}_n^{(i)}$ and $\dot{z}_c$ [see Eq. (3)], $T^{(1)} + T^{(2)}$ turns out to be on the order of not 45%, but of 90 to 100% of the total energy. In view of the fact that the terms with α_6 and α_8 in (3) have been determined very approximately, we disregard them for the time being. But for further calculations it is essential that formulas (2) be revised.

We note that to determine the absolute value of T_c and $T^{(i)}$ it is essential to replace $\dot{\alpha}_2$ by a certain effective value. As is known, the law of conservation of energy is not obeyed rigorously in the "shake-up" method. Therefore, to satisfy the law of conservation of energy it is necessary to equate the energy $T_c + T^{(1)} + T^{(2)}$, expressed with the aid of formulas (2) in terms of $\dot{\alpha}_2 = (\dot{\alpha}_2)_{eff}$, to the energy of the nucleus prior to the breakup of the neck. If the force of friction for the degrees of freedom α_n in the descent from the saddle point is small, (this corresponds to spontaneous and threshold fission, see reference 1), the nuclear energy prior to the breakup is equal to the difference in the potential energy $U(y)$ at the saddle point and at the point of breakup (for threshold fission), or to the difference in $U(y)$ at the ground state ($y = 0$) and the point of breakup (for spontaneous fission).

To determine the excitation energy of the fragments it is necessary to solve the equations of motion (quantum or classical) for the degrees of freedom $\alpha_n^{(i)}$ and z_c.[1] However, an approximate estimate of the excitation energy can be obtained by considering the sum of the internal kinetic energies $T^{(1)} + T^{(2)}$ and the energy of deformation of both fragments at the point of breakup of the neck: $E_0 = U_d^{(1)} + U_d^{(2)} + T^{(1)} + T^{(2)} = E_0^{(1)} + E_0^{(2)}$. As shown above,

$$T^{(1)} + T^{(2)} \approx \gamma\,[U(y_0) - U(y_d)];$$
$$y_0 = 1 - x \quad \text{or} \quad y_0 = 0; \quad \gamma > 0.45.$$

Bohr and Wheeler[6] calculated $U_d^{(i)}$ in an approxi-

Element	U^{236} Threshold fission	U^{236} Spontaneous fission	Pu^{240} Threshold fission	Pu^{240} Spontaneous fission	Cm^{242} Threshold fission	Cm^{242} Spontaneous fission	Cf^{242} Threshold fission	Cf^{242} Spontaneous fission
$U_d^{(1)} + U_d^{(2)}$, Mev	18.0		17.9		17.7		18.1	
$T^{(1)} + T^{(2)}$, Mev, $\gamma=0.45$	2.0	0	3.2	0	5.2	1.9	5.4	2.0
$T^{(1)} + T^{(2)}$, Mev, $\gamma=0.75$	3.4	0	5.4	0	8.8	3.2	9.2	3.4
E_0, Mev, $\gamma=0.45$	20.0	18.0	21.1	17.9	22.9	19.6	23.5	20.1
E_0, Mev, $\gamma=0.75$	21.4	18.0	23.3	17.9	26.5	20.9	27.3	21.5

mation that is quadratic in $\alpha_2^{(i)}$ and $\alpha_3^{(i)}$. Correction terms of higher order in $\alpha_n^{(i)}$ were obtained by Present and Knipp.[7] The table lists the values of E_0 calculated for four nuclei with $A_1 = A_2$ for spontaneous and threshold fission, using new values of r_0 and ϵ:[8] $r_0 = 1.22 \times 10^{-13}$ cm and $\epsilon = 50.1$ (and in the quadratic approximation with respect to $\alpha_n^{(i)}$ for $U_d^{(i)}$). These data are also more accurate than those of reference 4a.

As can be seen from the table, $U_d^{(1)} + U_d^{(2)}$ change very little with increasing Z^2/A (they decrease slowly). The increase in E_0 is thus due only to the increase in $T^{(1)} + T^{(2)}$, i.e., $U(y_0) - U(y_d)$. The rapid increase in $|U(y_d)|$ is clearly seen in Fig. 3 of reference 3. From the experimental data on the number of secondary neutrons, ν, it follows that the increase in the fragment excitation energy with increasing Z^2/A corresponds to $\gamma \sim 1$ than to $\gamma = 0.45$ (see reference 9), but $\gamma = 0.45$ is only the lower estimate for γ (see above). Furthermore, E_0 is only approximately equal to the excitation energy of the fragments. At any rate it is clear that the increase in the excitation energy with increasing Z^2/A is due entirely to $T^{(1)} + T^{(2)}$, i.e., to $U(y_0) - U(y_d)$, and not to U_d (if a larger value is assumed for y_d $|U(y_d)|$ will be greater).

Let us now attempt to estimate, for a specified Z^2A, the ratio of the excitation energies of two fragments, or the ratio $E_0^{(1)}/E_0^{(2)}$, in the same approximation, as functions of A_1/A_2. Numerical calculations show that $U_d^{(1)}/U_d^{(2)}$ depends little on A_1/A_2 (it diminishes slowly with diminishing A_1/A_2 when $A_1/A_2 \lessgtr 1$). But since $b_{c1}/b_{c2} = A_2/A_1$ (the distance between the center of gravity of the fragment and the common center of gravity is greater for a lighter fragment than for a heavy one), we have $\dot{\alpha}_n^{(1)} > \dot{\alpha}_n^{(2)}$ for $A_1 < A_2$. According to numerical calculations given in reference 4, $\alpha_n^{(1)}$ is also somewhat greater than $\alpha_n^{(2)}$. Consequently, in spite of the fact that $A_1^{5/3} < A_2^{5/3}$, we nevertheless find that $T^{(1)} > T^{(2)}$. Thanks to this, $E_0^{(1)}$ may turn out to greater than $E_0^{(2)}$ when $A_1 \lessgtr A_2$ (this agrees with the experimental data of reference 10). $E_0^{(1)}/E_0^{(2)}$ diminishes with further increase in A_1/A_2, owing to faster decrease in $U_d^{(1)}/U_d^{(2)}$.

Consequently in this case, too, the main reason for the principal effect ($E_0^{(1)} > E_0^{(2)}$ when $A_1 < A_2$) is $T^{(i)}$ and not $U_d^{(i)}$. We remark that allowance for $\dot{\alpha}_4$ and $\dot{\alpha}_6$ would increase the ratio $E_0^{(1)}/E_0^{(2)}$ substantially, since $\dot{\alpha}_4$ and $\dot{\alpha}_6$ contain high powers of b_{c1} and b_{c2}.

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