# GAMMA OSCILLATIONS OF THE SURFACE OF AN ATOMIC NUCLEUS

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Submitted to JETP editor July 10, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) 36, 233-237 (January, 1959)

The energy and probability of excitation of $\gamma$ oscillations of the surface of an atomic nucleus are computed on the basis of the shell model assuming an anisotropic harmonic oscillator potential. It is shown that in order of magnitude the energy of the first excited state of this type is close to that of single-nucleon excitations, and that the probability of excitation of these states is much smaller than the probability of excitation of rotational states of the nucleus.

## INTRODUCTION

RECENTLY an opinion has been expressed in a number of papers that some of the excited energy levels of even-even nuclei should be interpreted as so-called $\gamma$ oscillations (reference 1, V, D, 1). However, Davydov and Filippov[2] have shown that these excited states may be regarded as rotational energy levels of non-axially-symmetric nuclei. It turned out that the sequence of energy levels, the values of their spins and the probabilities of electromagnetic transitions between them depend on only one parameter $\gamma_0$, which describes the deviation of nuclear shape from axial symmetry. The value of this parameter may be determined by measuring, for example, the ratio of the energies of the first two excited states of spin 2. In order to obtain agreement between experimental data and theory it was sufficient to assume that $\gamma_0 = 8.1°$ for $Pu^{238}$, $\gamma_0 = 13.5°$ for $Sm^{152}$, $\gamma_0 = 26.5°$ for $Te^{122}$, etc. The hypothesis of a non-axially-symmetric equilibrium shape of the nucleus is supported by the work of Gurskiĭ,[3] Geĭlikman,[4] and Zaikin[5] who have shown that the minimum energy of nucleons moving in an oscillator[3,4] or in a rectangular[5] potential corresponds to a non-axially-symmetric nuclear shape, if the number of nucleons (of the same type) outside a closed shell exceeds 2.

In this paper we investigate the question of the stability of a nucleus with respect to a change in the equilibrium value of $\gamma_0$ which corresponds to the minimum potential energy. By means of a simple model of an anisotropic harmonic oscillator field for individual nucleons it is shown that the energy of the first excited state corresponding to $\gamma$ oscillations in order of magnitude is close to the energy of single-nucleon excitation. Further it is shown that the reduced probability of electric quadrupole transitions to levels corresponding to $\gamma$ oscillations is smaller by a factor of approximately several hundred compared to the probability of transition to the first rotational level of an axially-symmetric nucleus. It appears to us that these results confirm the great stability of nuclear shape with respect to $\gamma$ oscillations.

## 1. POTENTIAL ENERGY OF SURFACE OSCILLATIONS OF AN ATOMIC NUCLEUS

We assume that nucleons of mass m move subject to a potential of the form

$$V = \frac{m\omega^2}{2}\sum_{k=1}^{3}(x_k / R_k)^2, \tag{1.1}$$

where

$$R_k = R\exp(\xi_k), \quad \xi_k = \sqrt{5/4\pi}\,\beta\cos\left(\gamma - \frac{2\pi}{3}k\right); \tag{1.2}$$

$\beta$ and $\gamma$ determine the shape of the nucleus; here $R_1R_2R_3 = R^3$, since $\Sigma\xi_k = 0$. By introducing the three frequencies

$$\omega_k = \omega R / R_k \approx \omega\left\{1 - \xi_k + \frac{1}{2}\xi_k^2 + \cdots\right\}, \tag{1.3}$$

we easily see that the energy of each nucleon will depend on three quantum numbers $n_k$, so that

$$E_s / \hbar\omega = n_s + {}^3/_2 - \sum_{k=1}^{3} n_{sk}\xi_k + \frac{1}{2}\sum_{k=1}^{3}(n_{sk} + {}^1/_2)\xi_k^2 + \cdots, \quad n_s = \sum_{k=1}^{3} n_{sk}. \tag{1.4}$$

If, in accordance with the Pauli exclusion principle, we fill the lowest energy levels (1.4) using Z protons and N neutrons, we obtain their total energy as a function of the parameters of the nuclear deformation ($\beta$ and $\gamma$). By treating this energy as the potential energy for surface oscillations about the equilibrium values $\beta_0$ and $\gamma_0$ corresponding to the minimum energy, we can determine the energy of excited states corresponding to $\beta$ and $\gamma$ oscillations.

It may be easily seen that for each filled shell

$$\sum_{sk} n_{sk}\xi_k = 0,$$

and that the total energy of the nucleons filling several shells (magic-number nucleus) may be written in the form

$$E_{M} = \hbar\omega(\varepsilon_0 + {}^1/_2 D\beta^2), \quad \varepsilon_0 = \sum_s (n_s + {}^3/_2), \tag{1.5}$$

where $D > 0$ and determines the stiffness of the nuclear surface with respect to $\beta$ oscillations.

It follows from (1.5) that for nuclei with filled shells the minimum energy corresponds to spherical shape of the nucleus, i.e., to $\beta_0 = 0$.

We now suppose that outside the filled shells the shell with $n_s = N$ contains $\nu$ nucleons of one kind, then the total energy of all the nucleons has the form

$$E(\nu) = E_{M} + \left[\nu(N + {}^3/_2) - \sum_{sk} n_{sk}\xi_k + \frac{1}{2}\sum_{sk}(n_{sk} + {}^1/_2)\xi_k^2\right]\hbar\omega, \tag{1.6}$$

where $\sum_k n_{sk} = N$; the summation $\sum_k$ here and in subsequent equations is taken over all the nucleons of the shell N.

If we introduce the notation

$$\tan\gamma_0 = \sqrt{3}\sum_s (n_{s1} - n_{s2}) \Big/ \sum_s (3n_{s3} - N), \tag{1.7}$$

$$\beta_0 = \sqrt{4\pi/45}\sum_s (3n_{s3} - N) / D\cos\gamma_0, \tag{1.8}$$

then (1.6) is brought into the form

$$E(\nu) = E_0(\nu) + {}^1/_2 C(\beta - \beta_0)^2 + C\beta\beta_0[1 - \cos(\gamma - \gamma_0)], \tag{1.9}$$

where

$$C = 15D\hbar\omega/8\pi, \quad E_0(\nu) = \hbar\omega[\varepsilon_0 + \nu(N + {}^3/_2) - {}^1/_2 C\beta_0^2].$$

It follows from (1.9) that when $\beta = \beta_0$ the potential energy regarded as a function of $\gamma$ has the form

$$V(\gamma) = E_0(\nu) + F[1 - \cos(\gamma - \gamma_0)], \tag{1.10}$$

where

$$F = (15D\hbar\omega/8\pi)\beta_0^2 = {}^1/_4\sqrt{5/\pi}\,\hbar\omega\beta_0\left\{\sum_s (3n_{s3} - N)\right\} \Big/ \cos\gamma_0. \tag{1.11}$$

In obtaining (1.11) we have utilized (1.8).

## 2. $\gamma$-OSCILLATIONS OF THE SURFACE OF AN ATOMIC NUCLEUS

According to A. Bohr,[6] the operator for the kinetic energy of the $\gamma$-oscillations may be written, when $\beta = \beta_0$, in the form

$$T_\gamma = -\frac{\hbar^2}{2B\beta_0^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}, \tag{2.1}$$

where B is the mass parameter. Taking into account the explicit form of the potential energy (1.10), we can write the Schrödinger equation, which determines the energy of the $\gamma$ oscillations, in the form

$$[T_\gamma - F\cos(\gamma - \gamma_0) - (E_\gamma - E_0(\nu) + F)]\,\Phi(\gamma) = 0. \tag{2.2}$$

By making use of (1.7) it is easily seen that the transformations $\gamma_0 \to -\gamma_0$, $\gamma_0 \to \gamma_0 + 2\pi/3$ correspond to an interchange of the axes 1, 2, 3. Since the shape of the nucleus, which is defined in terms of $\gamma$ by means of (1.2), is also invariant with respect to the transformations $\gamma \to -\gamma$, $\gamma \to \gamma + 2\pi/3$, it is necessary to seek the solutions of (2.2) in the form of periodic even functions of $\gamma$ (of period $2\pi/3$). For these conditions to be fulfilled, it is sufficient to set

$$\Phi(\gamma) = \Phi(\cos 3\gamma). \tag{2.3}$$

By introducing the variable $\zeta = 3\gamma$, we bring (2.2) into the form

$$[T_\zeta - M\cos(\zeta/3 - \gamma_0) - W]\,\Phi(\cos\zeta) = 0, \tag{2.4}$$

where

$$T_\zeta = -\frac{1}{\sin\zeta}\frac{\partial}{\partial\zeta}\sin\zeta\frac{\partial}{\partial\zeta}, \tag{2.5}$$

$$M = \frac{2B\beta_0^2}{9\hbar^2}F = \frac{1}{18}\sqrt{\frac{5}{\pi}}\frac{\hbar\omega}{(\hbar^2/B\beta_0^2)}\sum_s (3n_{s3} - N)\beta_0/\cos\gamma_0, \tag{2.6}$$

$$W = (2B\beta_0^2/9\hbar^2)(E_\gamma - E_0(\nu) + F). \tag{2.7}$$

To calculate W it is convenient to expand the function $\Phi$ in (2.4),

$$\Phi(\cos\zeta) = \sum_{\lambda=0}^{\infty} a_\lambda\psi_\lambda(\zeta), \tag{2.8}$$

where $\psi_\lambda(\zeta)$ are the eigenfunctions of the operator $T_\zeta$, i.e., they satisfy the equation

$$[T_\zeta - \lambda(\lambda + 1)]\,\psi_\lambda(\zeta) = 0. \tag{2.9}$$

The functions $\psi_\lambda(\zeta)$ normalized to unity may be directly expressed in terms of Legendre polynomials:

$$\psi_\lambda(\zeta) = \sqrt{(2\lambda + 1)/2}\,P_\lambda(\cos\zeta). \tag{2.10}$$

On substituting (2.8) into (2.9) we obtain the system of equations

$$\sum_{\lambda=0}^{\infty} \{[\lambda(\lambda+1) - W]\,\delta_{\lambda\mu} - V_{\lambda\mu}\}\, a_\lambda = 0, \qquad (2.11)$$

where

$$V_{\lambda\mu} = M \int_0^\pi \psi_\lambda(\zeta) \cos\left(\frac{\zeta}{3} - \gamma_0\right) \psi_\mu(\zeta) \sin\zeta\, d\zeta.$$

It may be easily shown that

$$V_{\lambda\mu} = \begin{cases} \frac{2}{\sqrt{3}} M I_{\lambda\mu} \cos\left(\frac{\pi}{6} - \gamma_0\right), & \text{if } \lambda - \mu \text{ is even} \\ 2 M I_{\lambda\mu} \sin\left(\frac{\pi}{6} - \gamma_0\right), & \text{if } \lambda - \mu \text{ is odd} \end{cases}$$

where

$$I_{\lambda\mu} = \frac{1}{2} \sqrt{(2\lambda+1)(2\mu+1)} \int_0^\pi P_\lambda(\cos\zeta) \cos\frac{\zeta}{3} P_\mu(\cos\zeta) \sin\zeta\, d\zeta.$$

We restrict ourselves to four terms in the expansion (2.8). Then (2.10) reduces to a system of four homogeneous equations in the coefficients $a_\lambda$. By setting the determinant of the system equal to zero we shall obtain the values for the energies of the $\gamma$ oscillations. Numerical results were obtained for $\gamma_0 = 0$, $\pi/12$, and $\pi/6$, and $M = 5$, 10, and 20. The result of this is that within the range of variation of $M$ that was investigated the difference between the ground state and the first excited $\gamma$-oscillation level may be approximately represented in the form

$$\Delta E_\gamma = (9\hbar^2/2B\beta_0^2)(2 + 0.11M), \quad \text{for } \gamma_0 = 0,$$
$$\Delta E_\gamma = (9\hbar^2/2B\beta_0^2)(2 + 0.052M), \quad \text{for } \gamma_0 = \pi/12,$$
$$\Delta E_\gamma = (9\hbar^2/2B\beta_0^2)(2 + 0.024M), \quad \text{for } \gamma_0 = \pi/6. \qquad (2.12)$$

From these expressions it may be seen that even if the contribution of $V = M\cos(\gamma - \gamma_0)$ to the difference $\Delta E_\gamma$ is not great, nevertheless the first excited $\gamma$ level lies at a value one order of magnitude higher than the first rotational level. If $\beta_0$ is small then the rotational levels lie very high and the energy of the $\gamma$ oscillations falls into the range of single particle excitations. In the case of large $\beta_0$ the value of $\gamma_0$ differs little from zero. On setting $\gamma_0 \approx 0$ and on substituting (2.6) into (2.12) we obtain

$$\Delta E_\gamma \approx 9\hbar^2/B\beta_0^2 + (0.9/36)\sqrt{5/\pi}\,\hbar\omega \sum_s (3n_{s3} - N). \qquad (2.13)$$

Large values of $\beta_0$ can be realized when a certain not very small number of nucleons exists outside a closed shell. We consider the filling of the shell with $N = 5$. For a number of nucleons exceeding four $\sum_s (3n_{s3} - N) > 40$. Consequently, for large $\beta_0$ the second term in (2.13) is of order $\hbar\omega$, i.e., of order of magnitude of single particle excitations.

The results obtained above are based on the assumption that the series (2.8) converges sufficiently rapidly, starting with the third term of the series. Calculations show that for not very large values of $M$ this assumption is satisfied. Thus, for $\gamma_0 = 0$ and $M = 5$:

$$\Phi_0 = 0.971\,\psi_0 + 0.239\,\psi_1 + 0.0022\,\psi_2 + \sim 10^{-4}\psi_3,$$
$$\Phi_1 = -0.239\,\psi_0 + 0.961\,\psi_1 + 0.198\,\psi_2 + 0.0039\,\psi_3.$$

For $\gamma_0 = 0$ and $M = 10$:

$$\Phi_0 = 0.923\,\psi_0 + 0.383\,\psi_1 + 0.029\,\psi_2 + 0.0018\,\psi_3.$$
$$\Phi_1 = -0.342\,\psi_0 + 0.894\,\psi_1 + 0.290\,\psi_2 + 0.0053\,\psi_3.$$

## 3. PROBABILITY OF EXCITATION OF $\gamma$ OSCILLATIONS

The probability given above for the transition of a nucleus from the ground state into the first excited $\gamma$-oscillation state under the action of an electromagnetic field is given by the expression

$$B_\gamma(E2) = (5e^2/16\pi)\, Q_0^2\, |\langle \Phi_1 \mid \cos\gamma \mid \Phi_0 \rangle|^2, \qquad (3.1)$$

where $Q_0$ is the internal nuclear quadrupole moment; $\Phi_0$ and $\Phi_1$ are wave functions of the ground and of the first $\gamma$-oscillation level respectively.

Calculations show that over a sufficiently wide range of variation of $M$ the ratio of (3.1) to the reduced probability of electric quadrupole transition to the first rotational level of an axially-symmetrical nucleus $(5e^2Q_0^2/16\pi)$ is very small. Thus, for $\gamma_0 = 0$ and $M = 0$ this ratio is equal to 0.01, for $M = 5$ and 20 this ratio respectively amounts to 0.009 and 0.003.

Thus, the reduced probability of excitation of the first $\gamma$-oscillation state is by a factor of several hundred smaller than the corresponding probability of excitation of the first rotational state of the nucleus.

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Translated by G. Volkoff
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