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ON THE THEORY OF RELAXATION PROCESSES IN FERRODIELECTRICS AT LOW TEMPERATURES

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A theory of relaxation of the magnetic moment of ferrodielectrics is given. It is shown that exchange interaction between the spin waves causes first a Bose distribution to be established, with given non-equilibrium values of the square and projection of the magnetic moment on the axis of preferred magnetization. Because of magnetic interaction and interaction due to the anisotropy energy, equilibrium values of these quantities are gradually attained. The relaxation times are computed.

1. The kinetic and relaxation phenomena in ferrodielectrics are determined by various processes of interaction of spin waves with each other and with phonons. The strongest interaction between spin waves in the temperature region $\Theta_c \gg \Theta \gg \Theta_c(\mu M_0/\Theta_c)^{4/7}$ (Θ_c is the Curie temperature, μ the Bohr magneton, and M_0 the magnetic moment of such a region) is the exchange interaction that establishes a Bose distribution of the spin waves. This distribution, however, does not correspond to the equilibrium value of the magnetic moment. To the contrary, since the Hamiltonian of the exchange interaction commutes with the total magnetic moment of the body $\mathfrak{M}$ and with its projection of the axis of the lightest magnetization $\mathfrak{M}_z$, the latter quantities can be arbitrary.

The transition to equilibrium values of these quantities, together with the equalization of the spin and lattice temperatures, is due to the magnetic dipole interaction between the spin waves, to the interaction caused by the anisotropy energy, and to the interaction between the spin waves and the phonons. All these interactions are weak compared with the exchange interaction between the spin waves, and the relaxation of the magnetic moment and equalization of the temperatures are therefore slow compared with the establishment of the Bose distribution for spin waves with specified values of magnetic moment.

The first to be established is the equilibrium value of the absolute magnetic moment. This process is caused principally by magnetic dipole interaction. The equalization of the spin-wave and lattic temperatures and the establishment of the equilibrium value of the projection of the magnetic moment on the axis of the least magnetization are slower.

2. The Hamiltonian of interaction of the spin waves with each other or with phonons can be represented in the following form

$$\mathcal{H}_{\text{int}} = \mathcal{H}_e + \mathcal{H}_w + \mathcal{H}_a + \mathcal{H}_p,$$

where $\mathcal{H}_e$ and $\mathcal{H}_w$ are the Hamiltonians of the exchange and magnetic interactions, $\mathcal{H}_a$ is the anisotropy energy, and $\mathcal{H}_p$ is the Hamiltonian that describes the interaction between the spin waves and the phonons.

To find $\mathcal{H}_e$, we start with the expression for the exchange energy of a ferromagnet:[1]

$$\mathcal{H}_e = \frac{\alpha}{2}\int \frac{\partial M_l}{\partial x_i}\frac{\partial M_l}{\partial x_i}\,dV,$$

where M is the magnetic moment per unit volume and α is the exchange integral ($\alpha = \Theta_c a^2/2\mu M_0$, and a is the lattice constant). Let us put[2]

$$\begin{aligned} M^+ &= M_x + iM_y = (4\mu M_0)^{1/2} a^+ [1 - (\mu/M_0)\, a^+a]^{1/2}, \\ M^- &= M_x - iM_y = (4\mu M_0)^{1/2} [1 - (\mu/M_0)\, a^+a]^{1/2} a, \\ M_z &= M_0 - 2\mu a^+ a, \end{aligned} \quad (1)$$

where a^+ and a are the spin-wave creation and absorption operators, satisfying the commutation conditions

$$[a(\mathbf{r}),\ a^+(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}').$$

Expanding the operators a in a Fourier series

$$a(\mathbf{r}) = V^{-1/2}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}},\quad a^+(\mathbf{r}) = V^{-1/2}\sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}}^+, \quad (1')$$

we obtain the following expression for the exchange-interaction Hamiltonian

$$\mathcal{H}_e = \frac{\mu^2\alpha}{V}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} \{(\mathbf{k}_2 - \mathbf{k}_1)^2 + (\mathbf{k}_3 - \mathbf{k}_4)^2 - \mathbf{k}_1^2 - \mathbf{k}_4^2\} \times a_{\mathbf{k}_1}^+ a_{\mathbf{k}_2} a_{\mathbf{k}_3}^+ a_{\mathbf{k}_4} \Delta(\mathbf{k}_1 + \mathbf{k}_3 - \mathbf{k}_2 - \mathbf{k}_4), \quad (2)$$

where

$$\Delta(\mathbf{k}) = \begin{cases} 1, & \mathbf{k} = 0 \\ 0, & \mathbf{k} \neq 0 \end{cases}$$

and $a_{\mathbf{k}}^{+}$, $a_{\mathbf{k}}$ are the operators of creation and annihilation of a spin wave of momentum **k**. These operators satisfy the commutation condition

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^{+}] = \Delta(\mathbf{k} - \mathbf{k}').$$

The number of spin waves with momentum **k** is $n_{\mathbf{k}} = a_{\mathbf{k}}^{+} a_{\mathbf{k}}$.

The operator $\mathcal{H}_e$ is responsible for the scattering of the spin waves by the spin waves, i.e., for transitions of the type $n_1, n_2, n_3, n_4 \rightarrow n_1+1, n_2-1, n_3+1, n_4-1$.

The Hamiltonians $\mathcal{H}_w$ and $\mathcal{H}_e$ are defined by the formulas

$$\mathcal{H}_w = -8\pi\mu\sqrt{\mu M_0/V} \times \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \frac{k_1^{+}k_1^{z}}{k_1^2} a_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3}^{+} \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) + \text{compl. conj.}, \tag{3}$$

$$\mathcal{H}_a = -\frac{2\mu^2\beta}{V} \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} a_{\mathbf{k}_1}^{+} a_{\mathbf{k}_2} a_{\mathbf{k}_3}^{+} a_{\mathbf{k}_4} \Delta(\mathbf{k}_1 + \mathbf{k}_3 - \mathbf{k}_2 - \mathbf{k}_4), \tag{4}$$

where β is the anisotropy constant [the latter expression can be obtained from the anisotropy energy $\frac{1}{2}\beta \int (M_x^2 + M_y^2)\,dV$]. It is easy to see that the operator $\mathcal{H}_w$ is responsible for the processes of the merging of two spin waves into one and for the splitting of one spin wave into two, i.e., for processes of the type $n_1, n_2, n_3 \rightarrow n_1+1, n_2-1, n_3-1$ and $n_1, n_2, n_3 \rightarrow n_1-1, n_2+1, n_3+1$. The Hamiltonian $\mathcal{H}_a$, like $\mathcal{H}_e$, is responsible for the scattering of the spin waves.

The energy of the interaction between the spin waves and the lattice can be represented in the following form[3,4]

$$\mathcal{H}_p = \delta_1 \int \frac{\partial M_l}{\partial x_i} \frac{\partial M_l}{\partial x_k} u_{ik}\,dV + \delta_2 \int \frac{\partial M_l}{\partial x_i} \frac{\partial M_l}{\partial x_i} u_{kk}\,dV,$$

where u_{ik} is the deformation tensor; δ_1 and δ_2 are constants connected with the exchange integral:

$$\delta_1 = \beta_1 (\Theta_c/\mu M_0)\, a^2, \quad \delta_2 = {}^1/_2 \beta_2 (\Theta_c/\mu M_0)\, a^2$$

(β_1 and β_2 are quantities on the order of unity). We have written in $\mathcal{H}_p$ only the energy connected with the exchange interaction, and did not take into account the magnetostriction energy, which necessitates small corrections.

Introducing the creation and annihilation operators $b_{\mathbf{f}s}^{+}$ and $b_{\mathbf{f}s}$ of a phonon with momentum **f** and polarization s, in accordance with the formula

$$u = \sqrt{\frac{\hbar}{\rho V}} \sum_{\mathbf{f}s} \frac{\mathbf{e}_{\mathbf{f}s}}{\sqrt{2\omega_{\mathbf{f}s}}} \{b_{\mathbf{f}s} e^{i\mathbf{f}\cdot\mathbf{r}} + b_{\mathbf{f}s}^{+} e^{-i\mathbf{f}\cdot\mathbf{r}}\}$$

(ρ is the density of the matter, $\mathbf{e}_{\mathbf{f}s}$ is the vector of polarization of a phonon with momentum **f**; the number of phonons with momentum **f** and polarization s is $N_{\mathbf{f}s} = b_{\mathbf{f}s}^{+} b_{\mathbf{f}s}$), and using (1) and (1′) we obtain the following expression for $\mathcal{H}_p$:

$$\mathcal{H}_p = i\Theta_c a^2 \sqrt{\frac{2\hbar}{\rho V}} \sum_{\mathbf{k}\mathbf{k}'\mathbf{f}s} \{\beta_1 [(\mathbf{e}_{\mathbf{f}s}\cdot\mathbf{k})(\mathbf{f}\cdot\mathbf{k}') + (\mathbf{e}_{\mathbf{f}s}\cdot\mathbf{k}')(\mathbf{f}\cdot\mathbf{k})] + \beta_2 (\mathbf{k}\cdot\mathbf{k}')(\mathbf{e}_{\mathbf{f}s}\cdot\mathbf{f})\}\, a_{\mathbf{k}'}^{+} a_{\mathbf{k}} b_{\mathbf{f}s} \frac{\Delta(\mathbf{k}' - \mathbf{k} - \mathbf{f})}{\sqrt{\omega_{\mathbf{f}s}}} + \text{compl. conj.} \tag{5}$$

This Hamiltonian is responsible for the emission and absorption of a phonon by a spin wave, i.e., for transitions of the type

$$n_1, n_2, N_{\mathbf{f}s} \rightarrow n_1 - 1, n_2 + 1, N_{\mathbf{f}s} \pm 1.$$

3. The change per unit time in the number of spin waves with momentum **k**, due to the above interactions, is determined by the following formulas.

$$\dot{n}_{\mathbf{k}} = \dot{n}_{\mathbf{k}}^{\text{col}} \equiv L\{n_{\mathbf{k}}, N_{\mathbf{f}s}\}, \tag{6}$$

$$L\{n_{\mathbf{k}}, N_{\mathbf{f}s}\} = L_e\{n_{\mathbf{k}}\} + L_w\{n_{\mathbf{k}}\} + L_a\{n_{\mathbf{k}}\} + L_p\{n_{\mathbf{k}}, N_{\mathbf{f}s}\},$$

where L_e, L_w, L_a, and L_p are collision operators connected with the Hamiltonians $\mathcal{H}_e$, $\mathcal{H}_w$, $\mathcal{H}_a$, and $\mathcal{H}_p$ as follows:

$$L_e\{n_{\mathbf{k}}\} = (24\pi\mu^2\Theta_c^2 a^4/\hbar M_0^2 V^2) \times \sum_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} (\mathbf{k}_2\cdot\mathbf{k}_4 + \mathbf{k}_1\cdot\mathbf{k}_3)(\mathbf{k}_2\cdot\mathbf{k}_4 + \mathbf{k}_1\cdot\mathbf{k}_3 + 8\beta\mu M_0/\Theta_c a^2) \times \{(n_{\mathbf{k}_3}+1) n_{\mathbf{k}_2} n_{\mathbf{k}_4} (n_{\mathbf{k}_1}+1) - n_{\mathbf{k}_1} n_{\mathbf{k}_3} (n_{\mathbf{k}_2}+1)(n_{\mathbf{k}_4}+1)\} \times \delta(\varepsilon_1 + \varepsilon_3 - \varepsilon_2 - \varepsilon_4)\,\Delta(\mathbf{k}_1 + \mathbf{k}_3 - \mathbf{k}_2 - \mathbf{k}_4); \tag{7}$$

$$L_w\{n_{\mathbf{k}}\} = (2\pi/\hbar) \sum_{\mathbf{k}_2\mathbf{k}_3} \{A_{\mathbf{k}_3\mathbf{k}_2\mathbf{k}_1} [n_{\mathbf{k}_3} n_{\mathbf{k}_2} (n_{\mathbf{k}_1}+1) - (n_{\mathbf{k}_3}+1)(n_{\mathbf{k}_2}+1) n_{\mathbf{k}_1}]\, \delta(\varepsilon_3 + \varepsilon_2 - \varepsilon_1)\,\Delta(\mathbf{k}_3 + \mathbf{k}_2 - \mathbf{k}_1) + A_{\mathbf{k}_3\mathbf{k}_1\mathbf{k}_2} [(n_{\mathbf{k}_3}+1) n_{\mathbf{k}_2} (n_{\mathbf{k}_1}+1) - n_{\mathbf{k}_3} (n_{\mathbf{k}_2}+1) n_{\mathbf{k}_1}] \times \delta(\varepsilon_3 - \varepsilon_2 + \varepsilon_1)\,\Delta(\mathbf{k}_3 - \mathbf{k}_2 - \mathbf{k}_1) + A_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} [n_{\mathbf{k}_3} (n_{\mathbf{k}_2}+1) \times (n_{\mathbf{k}_1}+1) - (n_{\mathbf{k}_3}+1) n_{\mathbf{k}_2} n_{\mathbf{k}_1}]\, \delta(\varepsilon_3 - \varepsilon_2 - \varepsilon_1)\Delta(\mathbf{k}_3 - \mathbf{k}_2 - \mathbf{k}_1)\};$$

$$A_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = (64\pi^2\mu^3 M_0/V)\, |\sin\vartheta_1 \cos\vartheta_1 \exp i\varphi_1 + \sin\vartheta_2 \cos\vartheta_2 \exp i\varphi_2|^2; \tag{8}$$

$$L_a\{n_{\mathbf{k}}\} = (24\pi\beta^2\mu^4/\hbar V^2) \sum_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} \{(1 + n_{\mathbf{k}_1} + n_{\mathbf{k}_3}) n_{\mathbf{k}_2} n_{\mathbf{k}_4} - (1 + n_{\mathbf{k}_2} + n_{\mathbf{k}_4}) n_{\mathbf{k}_1} n_{\mathbf{k}_3}\} \times \delta(\varepsilon_1 + \varepsilon_3 - \varepsilon_2 - \varepsilon_4)\,\Delta(\mathbf{k}_1 + \mathbf{k}_3 - \mathbf{k}_2 - \mathbf{k}_4); \tag{9}$$

$$L_p\{n_{\mathbf{k}}, N_{\mathbf{f}s}\} = (4\pi a^4\Theta_c^2/\rho V) \sum_{\mathbf{k}_2\mathbf{f}s} \omega_{\mathbf{f}s}^{-1} \{\beta_1 [(\mathbf{e}_{\mathbf{f}s}\cdot\mathbf{k}_1)(\mathbf{f}\cdot\mathbf{k}_2) + (\mathbf{e}_{\mathbf{f}s}\cdot\mathbf{k}_2)(\mathbf{f}\cdot\mathbf{k}_1)] + \beta_2 (\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{e}_{\mathbf{f}s}\cdot\mathbf{f})\}^2 \{(N_{\mathbf{f}s} n_{\mathbf{k}_2} - n_{\mathbf{k}_1} n_{\mathbf{k}_2} - n_{\mathbf{k}_1} - N_{\mathbf{f}s} n_{\mathbf{k}_1})\, \delta(\varepsilon_1 - \varepsilon_2 - \hbar\omega_{\mathbf{f}s})\,\Delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{f}) + (n_{\mathbf{k}_1} n_{\mathbf{k}_2} + n_{\mathbf{k}_2} N_{\mathbf{f}s} + n_{\mathbf{k}_2} - n_{\mathbf{k}_1} N_{\mathbf{f}s}) \times \delta(\varepsilon_2 - \varepsilon_1 - \hbar\omega_{\mathbf{f}s})\,\Delta(\mathbf{k}_2 - \mathbf{k}_1 - \mathbf{f})\}. \tag{10}$$

By using the expression for the collision operators it is possible to obtain the mean probabilities of various interactions of spin waves and phonons.

The mean probability of scattering of a spin wave by a spin wave, due to exchange interaction, is:[4]

$$w_e \approx (\Theta_c / \hbar)(\Theta / \Theta_c)^4. \quad (11)$$

The mean probability of splitting of a spin wave into two and of merging of two spin waves into one[3] is

$$w_w \approx 4\pi^{1/2}(\Theta_c / \hbar)(\mu M_0 / \Theta_c)^2 (\Theta / \Theta_c)^{1/2} \ln^2 (\Theta / \mu M_0). \quad (12)$$

The probabilities of the remaining processes are less than w_e or w_w. In the temperature range $\Theta_c(\mu M_0/\Theta_c)^{4/7} < \Theta \ll \Theta_c$ the inequality $w_e \gg w_w$ holds, and therefore when $\Theta > \Theta_c(\mu M_0/\Theta_c)^{4/7}$ the largest term in $\dot{n}_{\mathbf{k}}^{col}$ will be $L_e\{n_{\mathbf{k}}\}$. The remainder of L, i.e., $L' = L_w + L_p + L_a$, can be considered in this temperature region as a small perturbation.

It is easy to see that the general solution of the equation

$$L_e\{n_{\mathbf{k}}\} = 0$$

has the form

$$n_k = \begin{cases} [e^{(\varepsilon_k - \gamma)/\Theta_s} - 1]^{-1}, & k \neq 0, \\ n_0, & k = 0, \end{cases} \quad (13)$$

where γ and n_0 are arbitrary constants. They can be connected with the initial values of the square of the magnetic moment $\mathfrak{M}^2$ and the projection of the magnetic moment $\mathfrak{M}_z$ on the axis of least magnetization. According to (1), these values are

$$\mathfrak{M}_z = \int_V M_z dV = M_0 V - 2\mu n_0 - 2\mu \sum_{\mathbf{k}\neq 0} n_{\mathbf{k}}$$
$$\mathfrak{M}^2 = \left\{\int_V \mathbf{M} dV\right\}^2 = (M_0 V)^2 - 4\mu V M_0 \sum_{\mathbf{k}\neq 0} n_{\mathbf{k}}. \quad (14)$$

We note that $\mathfrak{M}^2$ and $\mathfrak{M}_z$ commute with the Hamiltonian of exchange interaction $\mathcal{H}_e$ ($\mathfrak{M}_z$ commutes also with $\mathcal{H}_a$, but $\mathfrak{M}^2$ does not).

The time required to establish the Bose distribution (13) is of the following order of magnitude

$$\tau_e = 1 / w_e \approx (\hbar / \Theta_c)(\Theta_c / \Theta)^4.$$

We now take into consideration the weak interactions $\mathcal{H}_a$, $\mathcal{H}_w$, and $\mathcal{H}_p$. Then the distribution (13), which satisfies the equation $L_e\{n_{\mathbf{k}}\} = 0$, will no longer satisfy the equation $L\{n_{\mathbf{k}}, N_{\mathbf{f}s}\} = 0$. Since, however, the mean probability of the exchange interaction of the spin waves w_e is considerably greater than the probabilities of all the remaining processes, the distribution (13) with slowly varying parameters γ, n_0, and Θ_s, can satisfy approximately the equation

$$\dot{n}_{\mathbf{k}} = L\{n_{\mathbf{k}}, N_{\mathbf{f}s}\}, \quad N_{\mathbf{f}s} = [\exp\{\hbar\omega_{\mathbf{f}s}/\Theta_p\} - 1]^{-1}$$

(here the phonon temperature Θ_p also changes slowly with time). By finding the form of the functions $\gamma(t)$, $n_0(t)$, $\Theta_s(t)$, and $\Theta_p(t)$ we can, in accordance with formulas (14), determine the variation of $\mathfrak{M}^2$ and $\mathfrak{M}_z$ with time, and also find the equalization time of the spin-wave and the phonon temperatures.

To determine the time derivatives of γ, n_0, Θ_s, and Θ_p we insert the distribution (13) into the expressions

$$\sum_{\mathbf{k}\neq 0} \dot{n}_{\mathbf{k}} = \sum_{\mathbf{k}\neq 0} \dot{n}_{\mathbf{k}}^{col}, \quad \dot{n}_0 = \dot{n}_0^{col}$$
$$\sum_{\mathbf{f}s} \hbar\omega_{\mathbf{f}s}\dot{N}_{\mathbf{f}s} = -\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}\dot{n}_{\mathbf{k}} = -\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}\dot{n}_{\mathbf{k}}^{col}, \quad (15)$$
$$\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} n_{\mathbf{k}} + \sum_{\mathbf{f}s} \hbar\omega_{\mathbf{f}s} N_{\mathbf{f}s} = E_0.$$

The third equality determines the amount of heat transferred to the lattice from the spin system, while the last relation is the law of conservation of energy.

The change in γ and n_0 can be found from the first two equations. Since γ, being the "chemical potential" of the spin waves, is determined by the total number of the spin waves, which does not change if only the strong interaction $\mathcal{H}_e$ is taken into account, it is possible to determine $\dot{\gamma}$ by finding the change in $\sum_{\mathbf{k}\neq 0} n_{\mathbf{k}}$. Inserting the expression for the collision operators into (15) and linearizing over the quantities γ, $\Delta\Theta = \Theta_s - \Theta_p$, and $\eta = \epsilon_0 n_0/N$ (N is the total number of atoms in the body, and the quantity n_0/N is considered small but finite when $N \to \infty$), we obtain after simple transformations:

$$\Delta\dot{\Theta} + \Gamma_1\dot{\gamma} - \dot{\eta}/c_p = B_{\gamma\gamma}\gamma + B_{\gamma\eta}\eta,$$
$$\Delta\dot{\Theta} + \Gamma_2\dot{\gamma} + \dot{\eta}/c_s = B_{\Theta\Theta}\Delta\Theta, \quad \dot{\eta} = B_{\eta\eta}\eta, \quad (16)$$

where c_s and c_p are the specific heats of the phonons, referred to one atom:

$$c_s = \frac{15\zeta(5/2)}{32\pi^{3/2}}\left(\frac{\Theta}{\Theta_c}\right)^{3/2}, \quad c_p = \frac{2\pi^2}{5}\left(\frac{\Theta}{\Theta_D}\right)^3 = \frac{2\pi^2}{5}\frac{\Theta^3}{3}\left(\frac{1}{\Theta_l^3} + \frac{2}{\Theta_t^3}\right)$$

($\Theta_l = \hbar s_l/a$, $\Theta_t = \hbar s_t/a$ are the Debye temperatures for the longitudinal and transverse sound); Γ_1 and Γ_2 are defined as

$$\Gamma_1 = -\frac{1}{c_p}\left\{\frac{\Theta}{N}\sum_{\mathbf{k}\neq 0}\frac{\partial n_{\mathbf{k}}}{\partial\Theta} + (c_s + c_p)\sum_{\mathbf{k}\neq 0}\frac{\partial n_{\mathbf{k}}}{\partial\varepsilon_{\mathbf{k}}} \Big/ \sum_{\mathbf{k}\neq 0}\frac{\partial n_{\mathbf{k}}}{\partial\Theta}\right\},$$
$$\Gamma_2 = \frac{\Theta}{c_s}\frac{1}{N}\sum_{\mathbf{k}\neq 0}\frac{\partial n_{\mathbf{k}}}{\partial\Theta}, \quad (16')$$

and the quantities $B_{\gamma\gamma}$, $B_{\gamma\eta}$, $B_{\Theta\Theta}$, and $B_{\eta\eta}$ are related to the collision operators by

$$\gamma B_{\gamma\gamma} = \frac{c_s + c_p}{c_s c_p}\frac{\Theta}{\Gamma_2}\frac{1}{N}\sum_{\mathbf{k}\neq 0} L_w\{n_{\mathbf{k}}\},$$

$$\eta B_{\gamma\eta} = -\frac{c_s + c_p}{c_s c_p}\frac{\Theta}{\Gamma_2}\frac{1}{N} L_a\{n_{\mathbf{k}}\}_{\mathbf{k}=0},$$

$$\Delta\Theta B_{\Theta\Theta} = \frac{c_s + c_p}{c_s c_p}\frac{1}{N}\sum_{\mathbf{k}\neq 0} \varepsilon_{\mathbf{k}} L_p\{n_{\mathbf{k}}, N_{fs}\},$$

$$\eta B_{\eta\eta} = \frac{\varepsilon_0}{N} L_a\{n_{\mathbf{k}}\}_{\mathbf{k}=0}$$

($L_a\{n_{\mathbf{k}}\}_{\mathbf{k}=0}$ is the value of the collision integral L_a when $\gamma = 0$ and a value n_0 is specified).

Using formulas (8) and (10) we obtain for the coefficients B

$$B_{\gamma\gamma} = -\frac{4}{5\pi}\frac{\mu^3 M_0}{\hbar a^3}\left(\frac{\Theta}{\Theta_c}\right)^2 \frac{c_s + c_p}{c_s c_p}\frac{K(\xi)}{\Theta_c \Gamma_2};$$

$$B_{\gamma\eta} = \frac{3}{(2\pi)^3}\beta^2\frac{\mu^4}{\hbar a^6}\left(\frac{\Theta}{\Theta_c}\right)^2 \frac{c_s + c_p}{c_s c_p}\frac{G(\xi)}{\Theta_c \Gamma_2};$$

$$B_{\eta\eta} = -\frac{3}{(2\pi)^3}\beta^2\frac{\mu^4 \varepsilon_0 \Theta}{\hbar a^6 \Theta_c^3} G(\xi); \tag{17}$$

$$B_{\Theta\Theta} = -\frac{1}{(2\pi)^3}\frac{\hbar}{\rho a^5}\left(\frac{\Theta}{\Theta_l}\right)^8 \frac{c_s + c_p}{c_s c_p} J_l(\alpha_l)$$

$$-\frac{1}{(2\pi)^3}\frac{\hbar}{\rho a^5}\left(\frac{\Theta_t}{\Theta_c}\right)^3\left(\frac{\Theta}{\Theta_t}\right)^5 J_t(\alpha_t)\frac{c_s + c_p}{c_s c_p},$$

where

$$\xi = \varepsilon_0/\Theta,\quad \alpha_l = \Theta_l^2/\Theta\Theta_c,\quad \alpha_t = \Theta_t^2/\Theta\Theta_c;$$

$$K(\xi) = \xi^2\int_0^\infty dx \int_{1/4x}^\infty \frac{1/3 + 1/4xy}{(e^{\xi(1+x)} - 1)(e^{\xi(1+y)} - 1)(1 - e^{-\xi(x+y+2)})} dy;$$

$$G(\xi) = \int_\xi^\infty \frac{dx}{(e^x - 1)(1 - e^{\xi - x})}\ln\frac{1 - e^{-x}}{1 - e^{-\xi}};$$

$$J_t(\alpha_t) = \beta_1^2\int_0^\infty \frac{y^4 dy}{e^y - 1}\int_{\frac{(y+\alpha_t)^2}{4\alpha_t}}^\infty \frac{x - (y + \alpha_t)^2/4\alpha_t}{(1 - e^{-x})(e^{x-y} - 1)} dx;$$

$$J_l(\alpha_l) = \int_0^\infty \frac{y^4 dy}{e^y - 1}$$

$$\times \int_{(y+\alpha_l)^2/4\alpha_l}^\infty \frac{\{\tfrac{1}{2}\beta_1(\alpha_l^2 - y^2) + \beta_2[\alpha_l x - \tfrac{1}{2}y(y + \alpha_l)]\}^2}{(1 - e^{-x})(e^{x-y} - 1)} dx.$$

Assuming in (16) that the quantities γ, η, and $\Delta\Theta$ vary as $\exp\{-\lambda t\}$, we obtain the following value for the relaxation constants:

$$\lambda_1 = \frac{\Gamma_1 B_{\Theta\Theta} + B_{\gamma\gamma} + \{(\Gamma_1 B_{\Theta\Theta} - B_{\gamma\gamma})^2 + 4\Gamma_2 B_{\Theta\Theta} B_{\gamma\gamma}\}^{1/2}}{2(\Gamma_2 - \Gamma_1)};$$

$$\lambda_2 = \frac{\Gamma_1 B_{\Theta\Theta} + B_{\gamma\gamma} - \{(\Gamma_1 B_{\Theta\Theta} - B_{\gamma\gamma})^2 + 4\Gamma_2 B_{\Theta\Theta} B_{\gamma\gamma}\}^{1/2}}{2(\Gamma_2 - \Gamma_1)}; \tag{18}$$

$$\lambda_3 = -B_{\eta\eta}.$$

In the temperature range $\Theta_c \gg \Theta \gg \epsilon_0$ ($\xi \ll 1$), the expressions for λ_2 and λ_3 are greatly simplified and become

$$\lambda_2 \approx \frac{32\chi}{5}\frac{\mu M_0}{\hbar}\frac{\mu M_0}{\sqrt{\varepsilon_0 \Theta_c}}\frac{\Theta}{\Theta_c},\quad \lambda_3 \approx \frac{\beta^2}{16\pi}\frac{\mu^4}{\hbar a^6 \Theta_c}\left(\frac{\Theta}{\Theta_c}\right)^2, \tag{19}$$

where

$$\chi = \frac{1}{12}\int_0^\infty dx \int_{1/4x}^\infty \frac{4 + 3/xy}{(1 + x)(1 + y)(2 + x + y)} dy \sim 10^{-1}.$$

We present expressions for λ_1 in the limiting cases of "low" and "high" temperatures:[3,4]

$$\lambda_1 = \begin{cases} \dfrac{155\zeta(5)}{2\pi^5}\beta_1^2\left(2 + \dfrac{\Theta_t^3}{\Theta_l^3}\right)\dfrac{\hbar}{\rho a^5}\dfrac{\Theta^8}{\Theta_c^3\Theta_t^5}\exp\left\{-\dfrac{\Theta_t^2}{4\Theta\Theta_c}\right\}, & \Theta \ll \dfrac{\Theta_t^2}{\Theta_c}, \\ \dfrac{2^6\pi^{5/2}}{765\zeta(5/2)}\dfrac{\hbar}{\rho a^5}[2\beta_1^2 + (2\beta_1 + \beta_2)^2]\left(\dfrac{\Theta}{\Theta_c}\right)^{5/2}, & \Theta \gg \dfrac{\Theta_l^2}{\Theta_c}. \end{cases} \tag{20}$$

Using (14) to (16) we can show that the time variation of $\Delta\Theta$, $\mathfrak{M}_z$, and $\mathfrak{M}^2$ is determined by the following formulas

$$\Delta\Theta = \frac{n_0}{N}\frac{\lambda_3}{\lambda_1}\frac{\varepsilon_0}{c_s}e^{-\lambda_3 t} + \Delta\Theta_1 e^{-\lambda_1 t} + \Delta\Theta_2 e^{-\lambda_2 t};$$

$$\mathfrak{M}_z = \overline{\mathfrak{M}}_z - 2\mu n_0(1 - \zeta)e^{-\lambda_3 t} - 2\mu N\frac{c_p c_s}{c_p + c_s}\frac{\Gamma_2 B_{\gamma\gamma}}{\lambda_1\Gamma_1 + B_{\gamma\gamma}}\frac{\Delta\Theta_1}{\Theta}e^{-\lambda_1 t}$$

$$-2\mu N\frac{c_p c_s}{c_p + c_s}\frac{\Gamma_2 B_{\gamma\gamma}}{\lambda_2\Gamma_1 + B_{\gamma\gamma}}\frac{\Delta\Theta_2}{\Theta}e^{-\lambda_2 t};$$

$$\mathfrak{M}^2 = \overline{\mathfrak{M}^2} + 4\mu M_0 V\zeta n_0 e^{-\lambda_3 t} \tag{21}$$

$$-4\mu M_0 NV\frac{c_p c_s}{c_p + c_s}\frac{\Gamma_2 B_{\gamma\gamma}}{\Gamma_1\lambda_1 + B_{\gamma\gamma}}\frac{\Delta\Theta_1}{\Theta}e^{-\lambda_1 t}$$

$$-4\mu M_0 NV\frac{c_p c_s}{c_p + c_s}\frac{\Gamma_2 B_{\gamma\gamma}}{\Gamma_1\lambda_2 + B_{\gamma\gamma}}\frac{\Delta\Theta_2}{\Theta}e^{-\lambda_2 t},$$

where $\overline{\mathfrak{M}_z}$ and $\overline{\mathfrak{M}^2}$ are the equilibrium values of $\mathfrak{M}_z$ and $\mathfrak{M}^2$ for the given temperature,

$$\zeta = -\frac{\lambda_3}{\lambda_1} + \frac{\varepsilon_0}{\Theta}\frac{\Gamma_2}{c_s + c_p}\left(c_s - \frac{\lambda_3}{\lambda_1}c_p\right) \ll 1$$

and the integration constants n_0, $\Delta\Theta_1$, and $\Delta\Theta_2$ are determined by the initial values of $\Delta\Theta$, $\mathfrak{M}_z$, and $\mathfrak{M}^2$.

We note that of the three quantities λ_1, λ_2, and λ_3 the smallest is λ_3. Therefore, if $t \gg 1/\lambda_1$ or $1/\lambda_2$, it is necessary to retain in (21) only one term, proportional to $\exp\{-\lambda_3 t\}$. The values of $\mathfrak{M}_z$, $\mathfrak{M}^2$, and $\Delta\Theta$ now become

$$\mathfrak{M}_z = \overline{\mathfrak{M}}_z - 2\mu n_0(1 - \zeta)e^{-\lambda_3 t},\quad \mathfrak{M}^2 = \overline{\mathfrak{M}^2} + 4\mu M_0 V n_0 \zeta e^{-\lambda_3 t},$$

$$\Delta\Theta = \frac{n_0}{N}\frac{\lambda_3}{\lambda_1}\frac{\varepsilon_0}{c_s}e^{-\lambda_3 t}. \tag{22}$$

Since $\zeta \ll 1$, $(\mathfrak{M}_z - \overline{\mathfrak{M}_z})/\overline{\mathfrak{M}_z}$ is considerably greater than $(\mathfrak{M}^2 - \overline{\mathfrak{M}^2})/\mathfrak{M}^2$. In other words, first to be established is the equilibrium of the square of the moment at $\mathfrak{M}^2$, this is followed by a slow rotation of the magnetic moment towards the axis of least magnetization, which in turn establishes the equilibrium value of $\mathfrak{M}_z$. Such a relaxation process can be described phenomenologically by the well-known Landau-Lifshitz equation.[1] The time of relaxation of the magnetic moment towards the axis of least magnetization is

$$\tau = 1/\lambda_3 = (16\pi/\beta^2)(\hbar a^6\Theta_c/\mu^4)(\Theta_c/\Theta)^2.$$

Putting $\beta \sim 10$, $\Theta_c \sim 10^{-13}$ erg, $a \sim 2 \times 10^{-8}$ cm, and $\Theta/\Theta_c \sim 10^{-1}$, we get $\tau \sim 10^{-5}$ sec.

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