GREEN'S FUNCTION FOR THE RESONANCE RADIATION DIFFUSION EQUATION

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An analytic expression for the Green's function of the resonance radiation diffusion equation has been derived for the case of a homogeneous infinite space. The properties of the Green's function have been investigated for dispersion and Doppler spectral lines. An analytic expression has been derived for the mean time required for photon to move as a result of diffusion over a distance greater than some prescribed value. In conclusion, the Green's function has been determined for the stationary equation and its asymptotic expression is given in explicit form for a dispersion spectral line.

1. INTRODUCTION

THE theory of resonance radiation leads to a Fredholm integral equation of the second order relative to the concentration of the excited atoms. Biberman[1] formulated this equation for stationary problems under the following assumptions: (1) all the atoms have one resonant level, (2) the diffusion of the atoms can be neglected compared with the diffusion of the photons; (3) one can neglect the reduction in the number of normal atoms due to excitation or ionization of some of the normal atoms; (4) the role of negative absorption is unimportant; (5) the mean-free-path time of the photon is small compared with the duration of the excited state of the atom; (6) the frequency of the photon emitted by the atom is independent of the frequency of the absorbed quantum within the limits of the given spectral line.

The integral equation of diffusion of resonance radiation is first encountered for non-stationary problems in the papers of Holstein.[2,3] However, since he investigated the rate of emission of the gases by determining the cessation of their excitation, Holstein was interested only in the first eigenvalue of the equation, which he calculated by the Ritz method.

We shall show here that by retaining Biberman's assumptions and by considering the diffusion of radiation in an infinite homogeneous medium it is possible to obtain an analytic expression for the Green's function $f(\mathbf{r}, t)$ for this problem.

In this case the function $f(\mathbf{r}, t)\,dV$ is the probability of the excited atom staying at the instant t in the vicinity of the point $\mathbf{r}$, if at the initial instant of time $(t = 0)$ the space contains only one excited atom, located at $\mathbf{r} = 0$. The unknown Green's functions should satisfy the following equation, considered in the papers of Biberman and Holstein,

$$\frac{\partial f(\mathbf{r}, t)}{\partial t} = \frac{1}{4\pi\tau}\int\limits_V\int\limits_0^\infty \varepsilon_\nu k_\nu f(\mathbf{r}_1, t)\,\frac{\exp(-k_\nu|\mathbf{r}-\mathbf{r}_1|)}{|\mathbf{r}-\mathbf{r}_1|^2}\,d\nu\,dV_1 - \left(\frac{1}{\tau}+\sigma\right)f(\mathbf{r}, t) \qquad (1.1)$$

with the initial condition $f(\mathbf{r}, t) = \delta(\mathbf{r})$ when $t = 0$.

The following symbols are used in Eq. (1.1): τ is the average lifetime of the excited state of the atom, σ is the probability of an extinction collision per single excited atom, and ν is the frequency of the photon. The functions ϵ_ν and k_ν characterize the shapes of the emission and absorption lights respectively.

2. SOLUTION OF THE RADIATION DIFFUSION EQUATION

By virtue of the symmetry of the problem, we have for a homogeneous unbounded medium, $f(\mathbf{r}, t) = f(r, t)$, where r is the modulus of the vector $\mathbf{r}$.

We use the Ambartsamyan transformation.[4] We write Eq. (1.1) in a rectangular system of coordinates (x, y, z) and integrate it over the two variables y and z from $-\infty$ to $+\infty$.

Let us consider the transformation term by term. We have

$$\iint_{-\infty}^{\infty} f(\sqrt{x^2+y^2+z^2}, t)\, dy\, dz = 2\pi \int_{|x|}^{\infty} f(r, t)\, r\, dr,$$

where $r^2 = x^2 + y^2 + z^2$.

We introduce the notation

$$A(x, t) = 2\pi \int_{|x|}^{\infty} f(r, t)\, r\, dr. \tag{2.1}$$

Differentiating Eq. (2.1) with respect to x we find

$$f(x, t) = -(1/2\pi x)\, \partial A(x, t)/\partial x. \tag{2.2}$$

Next we have

$$\iint_{-\infty}^{\infty} \frac{\exp[-k_\nu |\mathbf{r} - \mathbf{r}'|]}{|\mathbf{r} - \mathbf{r}'|^2}\, dy\, dz = 2\pi\, \mathrm{Ei}\,(k_\nu |x - x'|).$$

Here Ei is the exponential integral.

We thus obtain for the function $A(x, t)$ a one-dimensional integro-differential equation.

$$\frac{\partial A(x, t)}{\partial t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} \varepsilon_\nu k_\nu\, \mathrm{Ei}\,(k_\nu |x - x'|)\, A(x', t)\, d\nu\, dx'$$

$$-\left(\frac{1}{\tau} + \sigma\right) A(x, t) \tag{2.3}$$

with an initial condition $A(x, t) = \delta(x)$ when $t = 0$. We apply the Fourier transformation to Eq. (2.3), denoting the Fourier transforms with the same symbols as the originals, merely changing the arguments from x to p. We find

$$\frac{\partial A(p, t)}{\partial t} = \frac{\sqrt{2\pi}}{2\pi} A(p, t) \int_{0}^{\infty} \varepsilon_\nu k_\nu\, \mathrm{Ei}\,(k_\nu p)\, d\nu - \left(\frac{1}{\tau} + \sigma\right) A(p, t). \tag{2.4}$$

The initial conditions becomes $A(p, t) = 1/\sqrt{2\pi}$ when $t = 0$.

By definition

$$\mathrm{Ei}\,(k_\nu p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{Ei}\,(k_\nu |x|)\, e^{ipx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{0}^{1} e^{ipx}\, \frac{\exp(-k_\nu |x| / y)}{y}\, dy\, dx.$$

After integrating first with respect to x and then with respect to y we get

$$\mathrm{Ei}\,(k_\nu p) = \frac{1}{p} \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{p}{k_\nu}. \tag{2.5}$$

Solving Eq. (2.4) and using Eq. (2.5) we get

$$A(p, t) = \frac{1}{\sqrt{2\pi}} \exp\left[t \left(\frac{1}{\tau p} \int_{0}^{\infty} \varepsilon_\nu k_\nu \tan^{-1} \frac{p}{k_\nu}\, d\nu - \frac{1}{\tau} - \sigma\right)\right].$$

Using the inverse Fourier and Ambartsamyan transformations, we obtain finally

$$f(r, t) = -\frac{e^{-t(1/\tau + \sigma)}}{(2\pi)^2 r}$$

$$\times \frac{\partial}{\partial r} \left\{ \int_{-\infty}^{\infty} e^{-ipr} \left[\exp\left\{\frac{t}{\tau} J(p)\right\} - 1 \right] dp + 2\pi\delta(r) \right\}, \tag{2.6}$$

where

$$J(p) = \frac{1}{p} \int_{0}^{\infty} \varepsilon_\nu k_\nu \tan^{-1} \frac{p}{k_\nu}\, d\nu. \tag{2.7}$$

We shall henceforth replace the argument x, which denotes the distance between two points in space, by r.

Equation (2.6) gives a complete solution of the problem for an arbitrary form of the spectral line. The first term in (2.6) takes into account the diffusion portion of the solution. The second corresponds to the probability of the excited atom staying at the point $r = 0$ during the entire time t, without experiencing a single act of radiation or extinction.

We turn now to the limiting case. Let the spectral line be monochromatic. Then

$$J(p) = \frac{k}{p} \tan^{-1} \frac{p}{k}. \tag{2.8}$$

The Green's function can be calculated further by approximate methods. Its asymptotic expression for larger values of r can be obtained by expanding $J(p)$ in powers of small p and retaining the first two terms. We then obtain

$$f(r, t) \approx \frac{e^{-\sigma t}}{(2\pi)^2 r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} \exp\left\{-ipr - \frac{tp^2}{3\tau k^2}\right\} dp$$

$$= (4\pi D t)^{-3/2} \exp\left(-\sigma t - \frac{r^2}{4Dt}\right), \tag{2.9}$$

where

$$D = 1/3\tau k^2.$$

The diffusion of monochromatic radiation is thus analogous in the first approximation to the diffusion of particles. This analogy was noted by Compton.[6]

3. DIFFUSION OF PHOTONS FOR A DISPERSION SPECTRAL LINE

A dispersion spectral line is defined by the functions

$$k_\nu = \frac{k_0}{1 + [2(\nu - \nu_0)/\Delta\nu_c]^2}; \quad \varepsilon_\nu = \frac{2}{\pi \Delta\nu_c} \frac{1}{1 + [2(\nu - \nu_0)/\Delta\nu_c]^2}. \tag{3.1}$$

Here ν_0 is the frequency of the center of the spectral line, $\Delta\nu_c$ is the width of the line, and k_0 is the coefficient of absorption at the frequency ν_0.

If the spectral line is of this form, the integral (2.7) can be calculated explicitly.[5] For an asymptotic expression of the Green's function it is enough to know the behavior of the function J (p) for small arguments only. We determine this expression:

$$J(p) = \int_0^\infty \int_0^1 \frac{\varepsilon_\nu k_\nu^2}{k_\nu^2 + p^2\mu^2}\, d\mu d\nu$$

$$= \frac{k_0^2}{\pi} \int_{-2\nu_0/\Delta\nu_c}^{\infty} \int_0^1 \frac{1}{(1+\omega^2)^3} \frac{d\mu d\omega}{\left(\frac{k_0}{1+\omega^2}\right)^2 + p^2\mu^2},$$

where $\omega = 2(\nu - \nu_0)/\Delta\nu_c$. Since $\Delta\nu_c \lesssim \nu_0$ for the spectral lines of the atoms, the lower limit of integration can be extended to $-\infty$. Using the normalization condition of the function ϵ_ν and making the following change in the integration variable

$$(k_0/p\mu)/(1+\omega^2) = x; \qquad (p \geqslant 0),$$

we obtain

$$J(p) = 1 + \frac{1}{\pi} \int_0^1 \sqrt{\frac{p\mu}{k_0}} \int_{k_0/p\mu}^{0} \frac{dx d\mu}{(1+x^2)\sqrt{(1 - xp\mu/k_0)\,x}}. \tag{3.2}$$

At small values of p the lower limit of integration can be extended to infinity and the second term under the radical can be neglected. Then

$$J(p) \approx 1 - {}^1/_3 \sqrt{\frac{2p}{k_0}}; \qquad (p \geqslant 0). \tag{3.3}$$

Inserting (3.3) into (3.6) we get

$$f(r,t) \approx -\frac{e^{-\sigma t}}{(2\pi)^2 r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} e^{-ipr} \exp\left(-\frac{t}{3\tau}\sqrt{\frac{2|p|}{k_0}}\right) dp.$$

After making the substitution $rp = q^2$ for the integration variable and integrating by parts we obtain as the final result

$$f(r,t) = -\frac{1}{\pi^2 r} \frac{\partial}{\partial r}\left(\frac{e^{-\sigma t}}{r}\varphi(y)\right), \tag{3.4}$$

where

$$\varphi(y) = y\int_0^\infty e^{-2yq}\sin q^2 dq; \quad y = t/3\tau\sqrt{2k_0 r}. \tag{3.5}$$

For very large distances it follows from (3.4) and (3.5) that

$$f(r,t) \approx \frac{1}{(4\pi)^{3/2}} \frac{t}{\tau} \frac{e^{-\sigma t}}{\sqrt{k_0 r}} \frac{1}{r^3}. \tag{3.6}$$

Compared with (3.9), the distribution (3.6) diminishes slowly at infinity. This is explained by the slow character of decrease in the kernel of Eq. (1.1), as pointed out by Biberman.[1]

The function $\varphi(y)$ can be expressed in terms of the Fresnel integrals:[7]

$$\varphi(y) = \sqrt{\frac{\pi}{2}}\, y\left\{\sin y^2\left[\frac{1}{2} - S(y)\right] + \cos y^2\left[\frac{1}{2} - C(y)\right]\right\}, \tag{3.7}$$

where

$$S(y) = \frac{2}{\sqrt{2\pi}}\int_0^y \sin t^2 dt; \qquad C(y) = \frac{2}{\sqrt{2\pi}}\int_0^y \cos t^2 dt.$$

It is easy to verify that the distribution (3.4), like (3.9), satisfies the following normalization condition

$$e^{\sigma t}\int_0^\infty f(r,t)\, 4\pi r^2 dr = 1. \tag{3.8}$$

For the subsequent calculations we shall find it convenient to use certain integral relations. We have

$$\int_0^r A(r_1,t)\, dr_1 = \frac{1}{2} - S(y)[1 - S(y)] - C(y)[1 - C(y)]. \tag{3.9}$$

Using the properties of the Fresnel integrals, we obtain the following asymptotic formulas

$$\begin{aligned} \int_0^r A(r_1,t)\, dr_1 &\approx \frac{1}{2\pi y^2} && \text{for } y \to \infty, \\ \int_0^r A(r_1,t)\, dr_1 &\approx \frac{1}{2} - \frac{2y}{\sqrt{2\pi}} && \text{for } y \to 0. \end{aligned} \tag{3.10}$$

Furthermore

$$\begin{aligned} \int_0^t A(r,t')\, dt' &= \frac{1}{\sqrt{2\pi}} \frac{t}{yr}\left(\frac{1}{2} - \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2yq}\cos q^2 dq\right), \\ \int_0^t\int_0^r A(r_1,t')\, dr_1 dt' &= t\int_0^r A(r_1,t)\, dr_1 + 2r\int_0^t A(r,t')\, dt'. \end{aligned} \tag{3.11}$$

4. DIFFUSION OF PHOTONS FOR A DOPPLER SPECTRAL LINE

The Doppler spectral line, resulting from the thermal motion of atoms with Maxwellian velocity distribution, is determined by the functions

$$\begin{aligned} k_\nu &= k_0 \exp\left\{-\left[\frac{2(\nu-\nu_0)}{\Delta\nu_D}\sqrt{\ln 2}\right]^2\right\}; \\ \varepsilon_\nu &= \frac{2}{\Delta\nu_D}\sqrt{\frac{\ln 2}{\pi}}\exp\left\{-\left[\frac{2(\nu-\nu_0)}{\Delta\nu_D}\sqrt{\ln 2}\right]^2\right\}, \end{aligned} \tag{4.1}$$

where $\Delta\nu_D$ is the Doppler width.

The integral (2.7) cannot be calculated in explicit form for a Doppler spectral line. For small values of the argument p, it is possible to obtain by the method indicated above the asymptotic value

$$J(p) \approx 1 - \sqrt{\pi p/4k_0}\,\sqrt{\ln k_0 - \ln p}. \tag{4.2}$$

Equations (2.6) and (4.2) lead to the following asymptotic expression for the Green's function

$$f(r,t) = -\frac{e^{-\sigma t}}{2\pi^2 r}\frac{\partial}{\partial r}\mathrm{Re}\int_0^{k_0} e^{-ipr} \times \exp\left(-\frac{t}{\tau}\frac{\sqrt{\pi}}{4}\frac{p}{k_0}\frac{1}{\sqrt{\ln k_0 - \ln p}}\right) dp. \tag{4.3}$$

For large optical distances $k_0 r$, formula (4.3) can be further simplified to

$$f(r,t) \approx \frac{2}{(4\pi)^{3/2}} \frac{t}{\tau} \frac{e^{-\sigma t}}{k_0 r^4 \sqrt{\ln k_0 r}}. \qquad (4.4)$$

It follows from (4.4), like from (3.6), that the photon distribution function $f(r, t)$ diminishes slowly with increasing distance.

5. NUMERICAL CHARACTERISTICS OF THE DIFFUSION OF PHOTONS

The principal numerical characteristics of photon diffusion, like those of any distribution, are the mathematical expectation value and the dispersion. The mathematical expectation value vanishes because the function $f(r, t)$ is even. The dispersion $\overline{r^2}$ provides an estimate of the mean distance covered by the photon during the time t.

Let the photons diffuse without a change of frequency and in the absence of extinction ($\sigma = 0$). Calculating r^2 with the aid of (2.9), we arrive at the Einstein equation $r^2 = 6Dt$.

If the spectral lines have Doppler or dispersion shapes, the slow decrease in the function $f(r, t)$ at infinity causes the integral that determines r^2 to diverge, and the foregoing arguments lose their meaning. We must forego the definition of the mean distance covered by the photon during the time t. We can speak, however of the mean time t required for the photon to shift more than a specified distance r.

Assume that there is no extinction. Then the expression

$$\int_0^r 4\pi r_1^2 f(r_1, t)\, dr_1$$

represents the probability of the photon staying within a sphere of radius r. Therefore

$$-\frac{\partial}{\partial t}\int_0^r 4\pi r_1^2 f(r_1, t)\, dr_1 dt \qquad (5.1)$$

is the probability of the photon leaving this sphere within a time dt. Consequently, the average time required for the photon to shift a distance greater than r is defined as the mathematical expectation value of the function (5.1)

$$\bar{t} = -\int_0^\infty t \frac{\partial}{\partial t}\int_0^r 4\pi r_1^2 f(r_1, t)\, dr_1 dt. \qquad (5.2)$$

Let us turn now to the dispersion spectral line. Integration of expression (5.2) by parts and relations (3.10) lead to

$$\bar{t} = \int_0^\infty \int_0^r 4\pi r_1^2 f(r_1, t)\, dr_1 dt. \qquad (5.3)$$

Integrating by part once more

$$\bar{t} = -2r\int_0^\infty A(r, t)\, dt + 2\int_0^\infty \int_0^r A(r_1, t)\, dr_1 dt.$$

These integrals have been given above. We therefore obtain as the final result

$$\bar{t} = 3\tau\sqrt{k_0 r/\pi} = 1.7\tau\sqrt{k_0 r}. \qquad (5.4)$$

Formula (5.4), obtained with the aid of an asymptotic expression for the function $f(r, t)$, does not hold for small $k_0 r$. In practice it gives correct results even if $k_0 r$ is on the order of several units.

Since we consider the photon to have an infinite velocity, the time t owes its existence to the finite duration τ of the excited state of the atom. Consequently $\bar{t} = n\tau$, where n is the average number of photon reradiation events over an optical distance greater than r. From this we get

$$n = 3\sqrt{k_0 r/\pi}.$$

Holstein[2,3] investigated the de-excitation of finite volumes of gas after cessation of the excitation of the atoms. He considered a planar layer of gas of thickness L and a cylindrical volume of radius R. For the mean de-excitation times of these volumes he obtained (using our symbols) the following expressions, respectively:

$$\bar{t} = 2.2\tau\sqrt{k_0 L/2}; \quad \bar{t} = 1.6\tau\sqrt{k_0 R}. \qquad (5.5)$$

Comparison of (5.5) with (5.4) shows that Holstein's results are close to ours. The difference lies only in the numerical factor, which is determined by the configuration of the scattering space.

Let us turn now to the Doppler spectral line. From (5.2) and (4.3) we find

$$\bar{t} = \frac{2}{\pi}\int_0^\infty t \frac{\partial}{\partial t}\int_0^r r_1 \frac{\partial}{\partial r_1}\int_0^{k_0} \cos pr_1$$

$$\times \exp\left(-\frac{t}{\tau}\frac{\sqrt{\pi}}{4}\frac{p}{k_0}\frac{1}{\sqrt{\ln k_0 - \ln p}}\right) dp\, dr_1 dt.$$

Interchanging the order of integration and performing operations connected with t and r_1 we arrive at the expression

$$\bar{t} = \frac{8}{\pi^{3/2}}\tau k_0 r \int_0^{k_0 r}\left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right)\sqrt{\ln k_0 r - \ln x}\, dx,$$

where $x = pr$. If the optical distance $k_0 r$ is sufficiently small, then

$$\bar{t} = 8\pi^{-3/2}\tau k_0 r\sqrt{\ln k_0 r}\int_0^\infty\left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right) dx.$$

After integrating, we obtain the final result

$$\bar{t} = 8\pi^{-3/2}\tau k_0 r\sqrt{\ln k_0 r} = 1.4\tau k_0 r\sqrt{\ln k_0 r}. \qquad (5.6)$$

Compare these with Holstein's calculated results

$$\bar{t} = 1.9\tau k_0 \frac{L}{2} \sqrt{\ln k_0 \frac{L}{2}}; \quad \bar{t} = 1.1\tau k_0 R \sqrt{\ln k_0 R}. \quad (5.7)$$

His first formula is for a planar layer, the second for a volume with cylindrical configuration. Holstein's relations (5.7) are in good agreement with experiment.[8]

Comparison of (5.6) and (5.7) leads to the same results as obtained for a dispersion spectral line.

6. CONCERNING STATIONARY SOLUTIONS

Let a source of excited atoms act for an infinitely long time at a certain point in space ($\mathbf{r} = 0$). The distribution function $f(\mathbf{r})$ satisfies in this case the equation

$$\frac{1}{4\pi\tau} \int_V \int_0^\infty \varepsilon_\nu k_\nu f(\mathbf{r}_1) \frac{\exp(-k_\nu |\mathbf{r} - \mathbf{r}_1|)}{|\mathbf{r} - \mathbf{r}_1|^2} d\nu dV_1$$

$$-f(\mathbf{r})\left(\frac{1}{\tau} + \sigma\right) + \delta(\mathbf{r}) = 0. \quad (6.1)$$

Using, as previously, the Ambartsamyan and Fourier transformations, we find

$$f(r) = -\frac{1}{(2\pi)^2 r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} \frac{e^{-ipr}\,dp}{\sigma + \tau^{-1}[1 - J(p)]}, \quad (6.2)$$

where the function $J(p)$ is defined by Eq. (2.7).

Assume that there is no extinction and that the spectral line has a dispersion shape. Using (3.3) we arrive at the asymptotic expression

$$f(r) \approx 3\tau \sqrt{\pi k_0 r}/2\,(2\pi)^2 r^3.$$

The same result can be obtained by using the solution of Eq. (1.1) and the formula

$$f(r) = \int_0^\infty f(r, t)\,dt.$$

In conclusion, I express deep gratitude to L. M. Biberman, who was in charge of this research.

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