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REMARKS REGARDING HEISENBERG'S PAPER ON THE LEE MODEL

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The case of two real discrete states of the V particle in the Lee model is considered. It is shown that Heisenberg's method can be extended to this case.

1. Heisenberg[1] showed, with the Lee model as an example, that in the special case when an unphysical discrete state with negative norm is combined with a discrete state with positive norm one can exclude transitions to unphysical states in a consistent way, at the same time guaranteeing the unitarity of the S matrix for scattering between physical states. We shall show that the requirement of combining the discrete states is unnecessary, at least in the Lee model. We make use of the results and notations of Heisenberg.

The Lee model introduces three real particles that interact according to the scheme $V \rightleftarrows N + \theta$. All states are divided into the sectors $(N + z\theta, V + (z-1)\theta)$ which do not go over into each other. Each sector can therefore be considered separately.

The sector $(N+\theta, V)$ contains the eigenstates of the continuous spectrum (describing the scattering of θ particles by N particles) and two discrete states. The state with the highest energy has a positive norm and may be interpreted as a physical state of the V particle. The other state has a negative norm. It is called a "ghost state" of the V particle.

Any state of the sector $(N+z\theta, V+(z-1)\theta)$ can be considered as a superposition of states formed by a state of the sector $(N+\theta, V)$ and $(z-1)$ free θ particles. These states are not eigenstates of the system. This means, in particular, that transitions are possible from states with positive norm, which can be interpreted physically, to states with negative norm, which have no physical meaning. This led Pauli and Källén[2] to the conclusion that the Lee model is inadequate. We shall show, however, that it is indeed possible to derive meaningful physical results from a theory involving states with negative norm, if the initial states are subjected to certain conditions.

2. We first discuss the scattering of a θ particle from a V particle (sector $(N+2\theta, V+\theta)$). This is meaningful, since one obtains, in contrast to the case considered by Heisenberg, a discrete V particle state with positive norm for the two discrete states. We seek a solution of the Schrödinger equation

$$H\Phi\rangle = E\Phi\rangle \tag{1}$$

in the form

$$\Phi\rangle = (\psi_V^* \int \varphi(\mathbf{k})\, a^*(\mathbf{k})\, d\mathbf{k} + \psi_N^* \int \varphi(\mathbf{k}_1, \mathbf{k}_2)\, a^*(\mathbf{k}_1)\, a^*(\mathbf{k}_2)\, d\mathbf{k}_1 d\mathbf{k}_2) \,|\, 0\rangle \tag{2}$$

or shorter,

$$\Phi = \begin{cases} \varphi(\mathbf{k}), \\ \varphi(\mathbf{k}_1, \mathbf{k}_2). \end{cases} \tag{3}$$

Repeating the calculations of Heisenberg (reference 1, sec. 3.1), we find that the function $\varphi(\mathbf{k})$, describing the scattering of the θ particle in the momentum representation, satisfies the equation [formula (68) of reference 1]

$$h^+(E-\omega)\,\varphi(\omega) = \int_{m_\theta}^{\infty} K(\omega\omega')\,\varphi(\omega')\, d\omega', \tag{4}$$

where

$$K(\omega\omega') = -\frac{1}{2}\frac{k'}{\omega+\omega'-E-i\gamma}\sqrt{\omega'/\omega},$$

the other notations are, as always, taken from reference 1. Formula (4) differs from formula (68) of Heisenberg's paper[1] only by the fact that the function $h^+(E-\omega)$ has two separate roots and that the inhomogeneous term $\varphi_{0i}(\mathbf{k})$ is absent, since in our case the state $N+2\theta$ is not present in the initial state. The general solution of (4) is of the form

$$\varphi(\omega) = a\delta(\omega-\omega_p) + b\delta(\omega-\omega_g) + \chi(\omega)/h^+(E-\omega), \tag{5}$$

where $\omega_{p,g} = E - E_{p,g}$; E_p and E_g are the energies of the physical state and the unphysical "ghost state" of the V particle, i.e., the zeroes of the function $h^+(\omega)$; $\chi(\omega)$ is a function which is regular at the points $\omega_{p,g}$. We rewrite the last term in (5):

$$\frac{\chi(\omega)}{h^+(E-\omega)} = \left[\frac{(\omega-\omega_p)(\omega-\omega_g)\chi(\omega)}{(E_p-E_g)h^+(E-\omega)}\right]\frac{1}{\omega-\omega_g-i\gamma} - \left[\frac{(\omega-\omega_p)(\omega-\omega_g)\chi(\omega)}{(E_p-E_g)h^+(E-\omega)}\right]\frac{1}{\omega-\omega_p-i\gamma}. \quad (6)$$

The function inside the square brackets is regular at the points $\omega_{p,g}$.

The first two terms in (5) thus describe stationary S waves, while the last term describes outgoing waves of the system $V+\theta$ in the physical and "ghost" states. A physical interpretation of the solution (5) is possible only if there are no transitions of the system from the physical to the unphysical states. This requirement can be met by setting $\chi(\omega_g) = 0$. This corresponds to the absence of divergent scattered waves of the unphysical state at infinity. It is not necessary to assume that the amplitude of the stationary wave of the unphysical state is zero, since the stationary wave corresponds to an unchanged state of an unphysical system at $t = \pm\infty$, which may readily be left out of the physical interpretation (see also reference 3). We therefore choose this amplitude such that

$$\chi(\omega_g) = 0 = K(\omega_g, \omega_p)\,a + K(\omega_g, \omega_g)\,b + \int_{m_\theta}^{\infty} K(\omega_g, \omega')\,(\chi(\omega')/h^+(E-\omega'))\,d\omega'. \quad (7)$$

$\chi(\omega)$ is then uniquely determined by the equation

$$\chi(\omega) = G(\omega, \omega_p)\,a + \int_{m_\theta}^{\infty} G(\omega, \omega')\frac{\chi(\omega')\,d\omega'}{h^+(E-\omega')}, \quad (8)$$

where

$$G(\omega, \omega') = K(\omega, \omega') - \frac{K(\omega, \omega_g)K(\omega_g, \omega')}{K(\omega_g, \omega_g)} = -\frac{k'}{2}\sqrt{\frac{\omega'}{\omega}}\frac{(\omega-\omega_g)(\omega'-\omega_g)}{(\omega-E_g)(\omega'-E_g)(\omega+\omega'-E)}. \quad (9)$$

For $E > 2m_\theta$ inelastic processes $(V+\theta \rightarrow N+2\theta)$ are possible. They are described by an outgoing wave [formula (66) of reference 1]:

$$\varphi(\mathbf{k}_1, \mathbf{k}_2) = \frac{g_0}{2\sqrt{4\pi}(\omega_1+\omega_2-E-i\gamma)}\left[\frac{\varphi(\omega_2)}{\sqrt{2\omega_1}} + \frac{\varphi(\omega_1)}{\sqrt{2\omega_2}}\right]. \quad (10)$$

Then the wave of the unphysical state will not be scattered, if its amplitude is related to the amplitude a of the physical wave by the equation [cf. formula (7)]

$$b = -a\left\{\frac{K(\omega_g, \omega_p)}{K(\omega_g, \omega_g)} + \int_{m_\theta}^{\infty}\frac{K(\omega_g, \omega')}{K(\omega_g, \omega_g)}\frac{[\chi(\omega')/a]\,d\omega'}{h^+(E-\omega')}\right\}; \quad (11)$$

the function $\chi(\omega')/a$ is independent of a [cf. formula (8)].

3. In exactly the same way it is easily shown that the scattered wave of "ghost states" can be made to vanish for the system $(V+(z-1)\theta, N+z\theta)$, if a definite unphysical state amplitude is added to the initial state of the system. This obviously guarantees the conservation of the probability for finding the system in the physical region, i.e., the unitarity of the physical S matrix.

Repeating Heisenberg's calculations, we arrive at an interesting result: the scattered wave of "ghost states" vanishes for a definite value, not of the amplitude b of the incoming wave of the "ghost states" itself, but of the following integral of b:

$$B = \int b(\mathbf{k}_1 \ldots \mathbf{k}_{z-1})\,\delta\left(E - E_g - \sum_{i=1}^{z-1}\omega_i\right)d\mathbf{k}_{z-1}. \quad (12)$$

The adding of an arbitrary "ghost" wave giving a prescribed value to the integral (12), to the incoming physical wave does not alter any physical results.

4. We now discuss in more detail the question of the choice of basis states of the system. To be definite, we shall talk about the sector $(N+2\theta, V+\theta)$. We seek eigenstates of the Hamiltonian in the form of expansions in terms of states with a definite number of bare particles [cf. formula (2)]. The state $\psi_V^* a^* |0\rangle$ has a negative norm. Nevertheless, we look for a physical interpretation for the amplitude $\varphi(\mathbf{k})$ of this state, and show that it can be found. In order to resolve this seeming contradiction, we look at the solutions of the previous sector $(N+\theta, V)$. The discrete states of this sector are described by the function [formula (7R) of reference 1]

$$\Phi_{p,g}\rangle = c\left[-\psi_V^* + \frac{g_0}{\sqrt{4\pi}}\psi_N^*\int\frac{1}{\sqrt{2\omega}(\omega-E_{p,g})}a^*(\mathbf{k})\,d\mathbf{k}\right]|0\rangle. \quad (13)$$

Its norm is [formula (34) of reference 1]

$$\langle\Phi|\Phi\rangle_{p,g} = \left[-1 + |g_0|^2\int\frac{k^2dk}{2\omega(\omega-E_{p,g})^2}\right]|c|^2. \quad (14)$$

The second term is greater than unity for the physical state of the V particle, and the system is mainly in the state $N+\theta$. We have the opposite situation for the "ghost state."

In the sector $(N+2\theta, V+\theta)$ it would have more physical meaning, if we expanded in terms of eigenstates of the system $(N+\theta, V)$ plus a θ particle in a free state, as was done by Heisenberg in the discussion of the higher sectors (reference 1, Sec. 5.1). However, it is more convenient for the calculations to use an expansion in terms of bare particle states. $\varphi(\mathbf{k})$ can then be regarded as a superposition of the amplitudes of the two states of the V particle, which is what is effectively done in reference 1 as well as in this paper [if $E > 2m_\theta$, $\varphi(\mathbf{k})$ also contains the amplitude of the virtual state of the V particle coming from the continuous spectrum of the $(N+\theta, V)$ system].

The use of $\varphi(\mathbf{k})$ is justified by the fact that it possible, as a consequence of the formal conservation laws concerning the asymptotic behavior of the function $\varphi(\mathbf{k})$ at great distances (the δ-function in the Hamiltonian (2R) in reference 1), to tell which one of the states of the system $(N+\theta, V)$ belongs to the scattered θ particle. In view of what has been said earlier, we shall regard the basis states of the following section as states of the type $(N+\theta, V) + z\theta$.

5. In conclusion we investigate the conditions which have to be imposed on the unphysical states in the more general case.

Let the initial state of the system be described by the function $\begin{pmatrix}\Phi\\ G\end{pmatrix}$, where Φ is a state with positive norm, and G is a state with negative norm. Writing this function in the form of a column vector underscores the orthogonality of Φ and G with respect to each other. Φ and G may themselves be many-component functions.

The scattering matrix mixes states with positive and negative norm belonging to the same energy eigenvalue (other integrals of the motion may also be identical). Owing to the pseudo-unitarity of the scattering matrix $(S^+S = 1)$,[1] the norm of the system remains unchanged:

$$|\Phi|^2 - |G|^2. \tag{15}$$

If the initial state contains only states with either positive or negative norm, the final state will be a mixture of both:

$$S\binom{1}{0} = \binom{A}{b}, \quad |A|^2 - |b|^2 = +1, \tag{16}$$

$$S\binom{0}{1} = \binom{a}{B}, \quad |a|^2 - |B|^2 = -1. \tag{17}$$

If the initial state is a mixture, then

$$S\left[\binom{1}{0} + \xi\binom{0}{1}\right] = \binom{A+\xi a}{b+\xi B}, \tag{18}$$

$$|A+\xi a|^2 - |b+\xi B|^2 = 1 - |\xi|^2.$$

Choosing ξ such that

$$|\xi|^2 = |b+\xi B|^2, \tag{19}$$

we obtain

$$|A+\xi a|^2 = 1 \tag{20}$$

The S matrix then takes the form

$$\begin{pmatrix}S_1 & 0\\ 0 & S_2\end{pmatrix}, \tag{21}$$

with both $S_1^+S_1 = 1$ and $S_2^+S_2 = 1$, where the hermitian conjugation is to be understood in the usual sense.

With the unitarity of the S matrix describing transitions between states with positive norm guaranteed, one may now construct physical quantities by postulating that they are completely defined by states with positive norm. The probability $\langle\Phi|\Phi\rangle$, the energy $\langle\Phi|H|\Phi\rangle$, and the other integrals of the motion will be conserved, as should be required of these physical quantities.

With regard to the foregoing section we emphasize again that the states Φ must have a physical meaning, whereas the states with negative norm G have the character of boundary conditions and arise solely from unitarity considerations for the scattering matrix for physical transitions.

We add a few remarks on the uniqueness of the determination of the states G. The only condition on the amplitude of the admixture G is the requirement that the scattering matrix for transitions between unphysical states be unitary [cf. formula (19)]. Then we can apply physical language to the "ghost states." In particular, if there is only one unphysical state, condition (19) can be said to be the requirement of elastic scattering of the unphysical admixture.

It is clear that the requirement of elastic scattering of the "ghost state" makes the problem nonunique, since it depends on the choice of the scattering phase of this state. In this respect the case of an indefinite metric is different from the case considered by Heisenberg. There the exclusion of the unphysical states is unique. The simplest procedure to obtain unique physical results is given by the more incisive requirement (which we used in the present paper) that the unphysical admixture does not scatter at all.[3] In this case condition (19)

$$|G_{t=+\infty}|^2 = |G_{t=-\infty}|^2 \tag{19a}$$

is replaced by the condition

$$G_{t=+\infty} = G_{t=-\infty}. \tag{19b}$$

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[1] W. Heisenberg, Nucl. Phys. **4**, 532 (1957).

[2] G. Källén and W. Pauli, Dan. Mat. Fys. Medd. **30**, No. 7 (1955). Russ. Transl. in Usp. Fiz. Nauk **60**, No. 3 (1956).

[3] Bogolyubov, Medvedev, and Polivanov, Научн. докл. Высш. школы серия физ.-мат. (Scient. Reports of the Higher Schools Phys.-Math. Series) **1**, (1958).

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