ANGULAR MOMENTUM AND PARITY CONSERVATION LAWS IN THE STATISTICAL THEORY OF MULTIPLE PRODUCTION

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The general statistical method of the microcanonical distribution is employed to calculate the statistical weights of a many-particle system obeying an arbitrary statistics. The conservation laws for angular momentum and parity are taken into account. A general computational formula is obtained under the assumption that all particles obey Boltzmann statistics.

We consider a system of N identical particles obeying arbitrary statistics. The particles have an intrinsic parity λ. The total angular momentum L and the total parity I are given. Furthermore, it is assumed that the orbital angular momentum of each particle is bounded from above by some maximal value $\bar{l}$, since the particles are produced in a limited region and possess, by the conservation laws, a finite momentum. Hence $0 \leq l \leq \bar{l}$.

For the determination of the statistical weight (the number of states) of the system under consideration, for given L and I, it is sufficient to calculate the number of states $F_{N,\bar{l}}(M, I)$ with a given projection M of the total angular momentum (and with a given I). The number of states with given L and I, $G_{N,\bar{l}}(L, I)$, is then determined with the help of well-known Slater condition (see, e.g., reference 1):

$$G_{N,\bar{l}}(L, I) = F_{N,\bar{l}}(L, I) - F_{N,\bar{l}}(L+1, I). \tag{1}$$

For the calculation of $F_{N,\bar{l}}(M, I)$ one can effectively use the microcanonical distribution, since M is an additive integral of the system.

Each microstate of our system is completely determined by the set of occupation numbers n_s, where s denotes the set of magnetic (m) and azimuthal (l) quantum numbers defining the state of the particle.

The number of states with given M and I is equal to

$$F_{N,\bar{l}}(M, I) = \sum_{(n)} \delta\left(M - \sum_s m n_s\right) \delta\left(N - \sum_s n_s\right) \frac{1}{2} \prod_s \tag{2}$$
$$\times \left\{1 + I \prod_s [\lambda(-1)^l]^{n_s}\right\} \Omega(n) \equiv \frac{1}{2} F_{N,\bar{l}}(M) + \frac{I}{2} F'_{N,\bar{l}}(M),$$

where $\Omega(n)$ is the degeneracy of a given microstate:

$$\Omega(n) = \begin{cases} = 1 \text{ for bosons; } n_s = 0, 1, 2\ldots \\ = 1 \text{ for fermions; } n_s = 0, 1. \\ = N!\left(\prod_s n_s!\right)^{-1} \text{ in the case of Boltzmann} \end{cases}$$

statistics; $n_s = 0, 1, 2\ldots$

The factor $\frac{1}{2}\{1 + I \prod_s [\lambda(-1)^l]^{n_s}\}$ guarantees parity conservation in each microstate; $\delta(m-n) \equiv \delta_{m,n}$, the Kronecker symbol.

The summation goes over all microstates of the system, and also over all quantum numbers within each microstate:

$$\sum_s f(s) = \sum_{l=0}^{\bar{l}} \sum_{m=-l}^{l} f(l, m). \tag{3}$$

In the following calculations we make use of the integral representation of the δ symbol:

$$\delta(a-b) = (2\pi i)^{-1} \oint z^{b-a-1} dz, \tag{4}$$

Each term in the expression under the integral for $F_{N,\bar{l}}(M)$ and $F'_{N,\bar{l}}(M)$ is rewritten according to (4).

We obtain in the usual fashion (see, e.g., reference 2)

$$F'_{N,\bar{l}}(M) = (2\pi i)^{-2} \oint\oint x^{-M-1} y^{-N-1} dx\, dy$$
$$\times \sum_{(n)} x^{\Sigma m n_s} y^{\Sigma n_s} \prod_s [\lambda(-1)^l]^{n_s} \Omega(n)$$
$$= (2\pi i)^{-2} \oint\oint x^{-M-1} y^{-N-1} dx\, dy \prod_s \sum_{n_s} [\lambda y x^m (-1)^l]^{n_s} \Omega(n)$$
$$= (2\pi i)^{-2} \oint\oint x^{-M-1} y^{-N-1} dx\, dy \exp \Phi(x, y), \tag{5}$$

where

$$\Phi(x, y) = \alpha \sum_s \ln[1 - \alpha y x^m (-1)^l]; \tag{6}$$

$\alpha = +1$ for bosons, and $\alpha = -1$ for fermions. Expanding the logarithm into a series and summing each term over s, we obtain

$$\Phi(x, y) = \sum_{k=1}^{\infty} \frac{\alpha^{k+1}\lambda^k}{k} \Phi_1(x^k) y^k, \qquad (7)$$

where

$$\Phi_1(z) = \sum_{l=0}^{\bar{l}} (-1)^l \sum_{m=-l}^{l} z^m$$

$$= z^{-\bar{l}}(-1)^{\bar{l}}(1 - z^{2(\bar{l}+1)})/(1 - z^2). \qquad (8)$$

We integrate (5) over y, and obtain, by the law of residues,

$$F'_{N,\bar{l}}(M) = (2\pi i)^{-1} \oint x^{-M-1} dx \frac{1}{N!} \left[\frac{d^N}{dy^N} \exp \Phi(x, y)\right]_{y=0}, \qquad (9)$$

or, since $F'_{N,\bar{l}}(M) = F'_{N,\bar{l}}(-M)$,

$$F'_{N,\bar{l}}(M) = (2\pi i)^{-1} \oint x^{M-1} dx \frac{1}{N!} \left[\frac{d^N}{dy^N} \exp \Phi(x, y)\right]_{y=0}. \qquad (9')$$

Using the expression for the N-th derivative of a complex function (see, e.g., reference 3), we may also write this result in the explicit form

$$F'_{N,\bar{l}}(M) = \lambda^N (-1)^{N\bar{l}} \sum \frac{\alpha^{N+l}}{1^i 2^j \ldots l^k\, i!\, j! \ldots k!}$$
$$\times (2\pi i)^{-1} \oint x^{M-N\bar{l}-1} \varphi^i(x) \varphi^j(x^2) \ldots \varphi^k(x^l)\, dx, \qquad (10)$$

where the summation goes over all positive integral roots of the equation $i + 2j + \ldots + lk = N$, and

$$\varphi(z) = (1 - z^{2(\bar{l}+1)})/(1 - z^2). \qquad (11)$$

In exactly the same way we obtain

$$F_{N,\bar{l}}(M) = \sum \frac{\alpha^{N+l}}{1^i 2^j \ldots l^k i!\, j! \ldots k!} (2\pi i)^{-1}$$
$$\times \oint x^{M-N\bar{l}-1} \psi^{2i}(x) \psi^{2j}(x^2) \ldots \psi^{2k}(x^l)\, dx, \qquad (12)$$

where

$$\psi(x) = (1 - x^{\bar{l}+1})/(1 - x)$$

Finally,

$$F_{N,\bar{l}}(M, I) = \frac{1}{2} F_{N,\bar{l}}(M) + \frac{I}{2} F'_{N,\bar{l}}(M). \qquad (13)$$

Formulas (10) to (13) give the solution of our problem in the general form, for an arbitrary statistics.

This result can be easily generalized to the case of a mixture of particles obeying different statistics, as was done in reference 2.

The formulas are considerably simplified in the case of Boltzmann statistics. The solution can then be given in explicit form. It is easily seen[2] that it is sufficient to sum the expressions (10) and (12) only over the terms with $i = N$, $j = \ldots = k = 0$ and to multiply the result by N!. We then have

$$F_{N,\bar{l}}(M) = (2\pi i)^{-1} \oint x^{M-N\bar{l}-1} \psi^{2N}(x)\, dx$$
$$= (2\pi i)^{-1} \oint \frac{dx}{x^{N\bar{l}-M+1}} \frac{(1 - x^{\bar{l}+1})^{2N}}{(1 - x)^{2N}}, \qquad (12')$$

$$F'_{N,\bar{l}}(M) = \frac{\lambda^N (-1)^{N\bar{l}}}{2\pi i} \oint x^{M-N\bar{l}-1} \varphi^N(x)\, dx$$
$$= \frac{\lambda^N (-1)^{N\bar{l}}}{2\pi i} \oint \frac{dx}{x^{N\bar{l}-M+1}} \frac{(1 - x^{2(\bar{l}+1)})^N}{(1 - x^2)^N}. \qquad (10')$$

The integrals (10′) and (12′) are equal to the residue of the argument function at the point x = 0.

We apply the expansion

$$\left(\frac{1 - z^m}{1 - z}\right)^n = \sum_{i,j} (-1)^i \binom{n}{i} \binom{n + j - 1}{n - 1} z^{mi+j}, \qquad (14)$$

where $\binom{n}{i} = C_n^i = n!/i!\,(n-i)!$, to the functions under the integral sign in (10′) and (12′). By keeping the terms containing $x^{N\bar{l}-M}$, we find

$$F_{N,\bar{l}}(M) = \sum_{k=0}^{\left[\frac{N\bar{l}-M}{\bar{l}+1}\right]} (-1)^k \binom{2N}{k} \binom{N\bar{l} - M + 2N - 1 - (\bar{l} + 1)k}{2N - 1}, \qquad (15)$$

$$F'_{N,\bar{l}}(M) = \lambda^N (-1)^{N\bar{l}}$$
$$\times \sum_{k=0}^{\left[\frac{N\bar{l}-M}{2(\bar{l}+1)}\right]} (-1)^k \binom{N}{k} \binom{(N\bar{l} - M)/2 + N - 1 - (\bar{l} + 1)k}{N - 1}, \qquad (16)$$

where [x] is the integral part of the number x.

We note that $F'_{N,\bar{l}}(M) = 0$ if $N\bar{l} - M$ is odd. The number of states with given M and I is given by formula (13). Here

$$F_{N,\bar{l}}(M) = \sum_{I=\pm 1} F_{N,\bar{l}}(M, I). \qquad (17)$$

Using formula (1), we now find the number of states with given L and I:

$$G_{N,\bar{l}}(L, I) = F_{N,\bar{l}}(L, I) - F_{N,\bar{l}}(L + 1, I)$$
$$= \frac{1}{2}[F_{N,\bar{l}}(L) - F_{N,\bar{l}}(L + 1)]$$
$$+ \frac{I}{2} \begin{cases} F'_{N,\bar{l}}(L), & \text{if } N\bar{l} - L \text{ is even} \\ -F'_{N,\bar{l}}(L + 1), & \text{if } N\bar{l} - L \text{ is odd,} \end{cases} \qquad (18)$$

or, with obvious notations,

$$G_{N,\bar{l}}(L, I) = \frac{1}{2} G_{N,\bar{l}}(L)$$
$$+ \frac{1}{2} I (-1)^L G'_{N,\bar{l}}\left[L + \frac{1}{2}(1 - (-1)^{N\bar{l}-L})\right]. \qquad (19)$$

By termwise substitution of expansions (15) and (16) for M = L and M = L + 1, we find, with

$$\binom{n}{m} - \binom{n-1}{m} = \binom{n-1}{m-1},$$

that

$$G_{N,\bar{l}}(L) = \sum_{k=0}^{\left[\frac{N\bar{l}-L}{\bar{l}+1}\right]} (-1)^k \binom{2N}{k}\binom{N\bar{l}-L+2N-2-(\bar{l}+1)k}{2N-2}, \quad (20)$$

$$G'_{N,\bar{l}}(p) = \lambda^N \sum_{k=0}^{\left[\frac{N\bar{l}-p}{2(\bar{l}+1)}\right]} \binom{N}{k}\binom{\frac{1}{2}(N\bar{l}-p)+N-1-(\bar{l}+1)k}{N-1}. \quad (21)$$

It follows from (19) that

$$G_{N,\bar{l}}(L) = \sum_{I=\pm 1} G_{N,\bar{l}}(L, I) \quad (22)$$

is the number of states with a given L (and arbitrary I).

Corrections taking into account the type of statistics can, in any specific case, be calculated with the help of the general formulas (10) and (12).

This method allows the solution of a whole series of similar problems, as, for example, the determination of the statistical weight of a system of particles with arbitrary spin.

If the system consists of N particles with spin s obeying an arbitrary statistics, the statistical weight $g_{N,s} = \sum_S g_{N,s}(S)$ is given by

$$g_{N,s}(S) = f_{N,s}(S) - f_{N,s}(S+1), \quad (1')$$

$$g_{N,s} = \begin{cases} f_{N,s}(0) & \text{for integer } Ns, \\ f_{N,s}(1/2) & \text{for half odd-integer } Ns, \end{cases} \quad (23)$$

where $g_{N,s}(S)$ is the number of states with total spin S, and

$$f_{N,s}(M) = \sum_{(n)} \delta\left(M - \sum_{m=-s}^{s} m n_m\right) \delta\left(N - \sum_{m=-s}^{s} n_m\right) \Omega(n) \quad (24)$$

is the number of states with a given projection M of the total spin.

Omitting intermediate calculations analogous to those done above, we give the final result.

In our notations, we obtain for an arbitrary statistics,

$$f_{N,s}(M) = \sum \frac{\alpha^{N+l}}{1^i 2^j \ldots l^k i!\, j! \ldots k!} (2\pi i)^{-1}$$

$$\times \oint x^{M-Ns-1} \Phi^i(x)\, \Phi^j(x^2) \ldots \Phi^k(x^l)\, dx, \quad (25)$$

$$\Phi(z) = (1 - z^{2s+1}) / (1 - z).$$

For the special case of Boltzmann statistics we obtain the closed expressions

$$g_{N,s}(S) = \sum_{k=0}^{\left[\frac{sN-S}{2s+1}\right]} (-1)^k \binom{N}{k}\binom{Ns - S + N - 2 - (2s+1)k}{N-2}, \quad (26)$$

$$g_{N,s} = \begin{cases} \displaystyle\sum_{k=0}^{\left[\frac{sN}{2s+1}\right]} (-1)^k \binom{N}{k}\binom{sN + N - 1 - (2s+1)k}{N-1}, & \text{if } sN \text{ is integral,} \\ \displaystyle\sum_{k=0}^{\left[\frac{sN-1/2}{2s+1}\right]} (-1)^k \binom{N}{k}\binom{sN - 1/2 + N - 1 - (2s+1)k}{N-1}, & \text{if } sN \text{ is half odd-integral.} \end{cases} \quad (27)$$

For the special case s = ½ formulas (26) and (27) give a well-known result (see, e.g., reference 1). For s = 1 we get the result[4] obtained by the combinatorial method.

A problem similar to the one just considered was solved by Barashenkov and Barbashev[5] with the use of recurrence relations. We further remark that the results of references 1, 4, and 5 were obtained for the case of Boltzmann statistics.

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