THE STABILITY OF SHOCK WAVES IN MAGNETOHYDRODYNAMICS

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The interaction between shock waves in a magnetic field and magnetohydrodynamic waves of small amplitude is considered. The condition for stability with respect to spontaneous emission of weak magnetohydrodynamic waves by a shock wave has been obtained. The conditions are found under which the linear equations for a small perturbation do not have a solution. This case is interpreted as the decay of the shock wave.

THE stability of shock waves in magnetic hydrodynamics, so far as we know, has not yet been considered. The usual scheme of investigation of the stability consists in finding the time behavior of arbitrarily small perturbations of the initial flow. Investigations of this type were completed in references 1, 2 for a tangential discontinuity and for a magnetohydrodynamic wave in an incompressible fluid. A similar investigation for shock waves in a compressible medium requires the furnishing of equations of state of the medium and in general is very difficult. However, we can obtain some evidence on the stability of shock waves without having recourse to the general scheme mentioned above. For this purpose we must consider the possibility of spontaneous emission from a shock wave of waves of infinitesimally small amplitudes,[3] and also the interaction of a shock wave with such waves.

In magnetohydrodynamics an arbitrarily small perturbation can be represented in the form of the superposition of waves of four types:[4] fast and slow magneto-acoustical waves, a magnetohydrodynamic wave, and an entropy wave. We shall limit ourselves throughout to a consideration of the interaction of the shock wave with magnetohydrodynamic waves of small amplitude. In such a case, important use is made of the fact that in magnetohydrodynamics a shock wave is two dimensional, that is, in a suitable coordinate system, the vectors of a magnetic field and velocity on both sides of the shock lie in the same plane. Furthermore, there is a system of coordinates in which the discontinuity is at rest, and on each side of which the velocity is directed along the magnetic field intensity. In this case we eliminate from consideration the perpendicular shock wave which is propagated at right angles to the field and also a tangential discontinuity and a rotational discontinuity[5] (the magnetohydrodynamical field in a compressible medium), the investigation of the stability of which is comparatively simple by the usual method. Thus we limit ourselves to the consideration of the so-called oblique shock waves which include as a special case the ordinary shock waves which propagates along the magnetic field.

We assume that the velocity of the medium and the magnetic field intensity are parallel to the plane (x, y) while the shock front coincides with the plane $x = 0$. In the case of an oblique shock, the boundary equations for a perturbation on the surface of the discontinuity $x = 0$ fall into two groups,[2] one of which contains the z components of the perturbations $\mathbf{w}$ and $\mathbf{h}$ of the velocity and magnetic field intensity, while the second does not contain these components. The first group of boundary equations at $x = 0$ has the form

$$\{v_x h_z - H_x w_z\} = 0; \tag{1}$$

$$\{\rho v_x (w_z + v_x \partial\xi/\partial z) - H_x h_z/4\pi\} = 0, \tag{2}$$

where the v_x and H_x are the normal components of the velocity and magnetic field intensity in the unperturbed shock wave, ρ is the density of the medium and $\xi = \xi(x, y, z, t)$ is the displacement of the surface of discontinuity from its position $x = 0$ as a result of the perturbation; the curly brackets denote differences in the values of the quantities contained in them on the two sides of the discontinuity. Equation (1) expresses the continuity of the y component of the electric field, while (2) exhibits the continuity of the flux of the z component of the momentum. The second group of equations contain perturbations of the density, pressure and x and y components of the field and the velocity. Its coupling with the first group is maintained only by the displacement ξ of the surface of discontinuity. However, if the perturbation is itself a plane wave, the wave vector of which lies in the xy plane, then $\partial\xi/\partial z = 0$, and both groups of boundary equations are completely independent. In what follows we shall limit our attention to perturbations of the form

$$e^{i(kx-\omega t)}, \tag{3}$$

which are propagated along the normal to the shock front. Therefore, if only z components of the field and velocity appear on one side of the discontinuity, then, by virtue of the boundary equations, only these components will arise on the other side.

A wave in which the perturbation of the velocity and the intensity of the field are perpendicular to the direction of propagation (x axis) and the unperturbed magnetic field $\mathbf{H}(H_x, H_y, 0)$, is a magnetohydrodynamic wave (of small amplitude)[4,5] and has the properties

$$\omega_0 = \pm kV_A = \pm kH_x/\sqrt{4\pi\rho}, \quad w_z = \mp h_z/\sqrt{4\pi\rho}, \tag{4}$$

where ω_0 is the frequency of the wave in the system of coordinates in which the medium is at rest, $V_A = H/\sqrt{4\pi\rho}$ is the velocity of the wave relative to the liquid, and where it is taken into account that $\mathbf{k} = (k, 0, 0)$. No perturbations of the density, pressure and entropy in a magnetic field exist in a magnetohydrodynamic wave of small amplitude. Thus, magnetohydrodynamic waves of small amplitude which propagate in the plane of the flow and which are polarized perpendicular to this plane can lead to the appearance of waves only of the same type when they interact with the shock wave.

If the medium moves with velocity $\mathbf{v}$, then the coupling of the frequency with the wave vector is evidently defined by the expression

$$\omega = kv_x + \omega_0 = k(v_x \pm V_A). \tag{5}$$

Therefore, on each side of the discontinuity, in general, two waves are possible, one of which is propagated in the direction of the flow,

$$\omega^+ = k^+(v_x + V_A), \quad w_z^+ = -h_z^+/\sqrt{4\pi\rho}, \tag{6}$$

while the other is propagated against the flow, and for it,

$$\omega^- = k^-(v_x - V_A), \quad w_z^- = h_z^-/\sqrt{4\pi\rho}. \tag{7}$$

By virtue of the boundary equations for the perturbation on the surface of discontinuity $x = 0$, the frequencies ω of all arriving and departing waves must be identical, that is,

$$\begin{aligned} k_1^+(v_{1x} + V_{A1}) &= k_1^-(v_{1x} - V_{A1}) \\ &= k_2^+(v_{2x} + V_{A2}) = k_2^-(v_{2x} - V_{A2}). \end{aligned} \tag{8}$$

Here the indices 1 and 2 denote the different sides of the surface of discontinuity, $x < 0$ and $x > 0$, respectively. Without loss of generality, we can assume that $\omega > 0$, while the motion of the medium and the magnetic field intensity are directed toward positive values of x, that is, $v_{1x} > 0$, $v_{2x} > 0$, $H_{1x} = H_{2x} > 0$. Therefore, waves propagating in the positive direction of the x axis correspond to positive k, while waves in the opposite direction correspond to negative k. The choice of the boundary conditions at infinity is determined by the particular arrangement of the problem.

Let us first consider the possibility of spontaneous emission of weak magnetohydrodynamic waves of small amplitude from the shock wave. Since in this case all the waves should be departing from the surface of discontinuity, that is, they should be propagated in the direction of negative x from the side 1 of the discontinuity, and to the direction of positive x with regard to the side 2, then it follows from Eq. (8) that in the region $x < 0$, which corresponds to the incident flow, one or zero departing waves is possible, depending on whether the condition

$$v_{1x} < V_{A1} = H_x/\sqrt{4\pi\rho_1}, \tag{9}$$

is satisfied or not. To the right of the shock front in the region 2, two diverging waves are possible for the condition

$$v_{2x} > V_{A2} = H_x/\sqrt{4\pi\rho_2}, \tag{10}$$

or only one if this equation is not satisfied.

In the case of three emitted waves, two are expressed by the amplitude of the third through relations which follow from the boundary equations (1) and (2)

$$\begin{aligned} (v_{1x} - V_{A1})h_{1z}^- &= (v_{2x} + V_{A2})h_{2z}^+ + (v_{2x} - V_{A2})h_{2z}^-, \\ \sqrt{\rho_1/\rho_2}(v_{1x} - V_{A1})h_{1z}^- &= -(v_{2x} + V_{A2})h_{2z}^+ + (v_{2x} - V_{A2})h_{2z}^-. \end{aligned} \tag{11}$$

and the problem of the spontaneous emission from the shock wave of weak magnetohydrodynamic waves thus has a solution that contains one free parameter if conditions (9) and (10) are satisfied. If either of conditions (9) and (10) is not satisfied, the number of diverging waves is less than three. In this case, as can easily be seen from Eq. (11), setting h_{1z}^- and h_{2z}^- separately or simultaneously equal to zero, the boundary equations have only trivial solutions. Thus Eqs. (9) and (10) are necessary and sufficient for the instability of the shock wave relative to emission of magnetohydrodynamic waves of small amplitude.

The equation of continuity of flow $\rho_1 v_{1x} = \rho_2 v_{2x}$, together with the conditions (9) and (10), show that spontaneous emission is possible only for v_{1x}/v_{2x}

$= \rho_2/\rho_1 < \sqrt{\rho_2/\rho_1}$, or $\rho_2/\rho_1 < 1$, that is, only for rarefaction waves. These conditions cannot be simultaneously satisfied in a compression wave, which is consequently stable relative to spontaneous emission of magnetohydrodynamic waves. This result is similar to the well-known result from ordinary hydrodynamics, and testifies to the fact that stationary shock waves are possible in magnetohydrodynamics only as compression waves. True, the proof given above of the instability of rarefaction waves applies only to the case in which the magnetic field intensity satisfies conditions (9) and (10). This is connected with the fact that we have considered as emitted waves only magnetohydrodynamic waves. A general result can be obtained for consideration of all possible waves of small amplitude in magnetic hydrodynamics. However there is no fundamental need of this, since a derivation of the fact that shock waves in magnetic hydrodynamics can only be compressional waves follows from a consideration of the change of entropy in the shock wave.[6]

We now consider the collision between a shock wave and a weak magnetohydrodynamic wave propagated toward it. In addition to the incident wave, there are in general three possible diverging waves, as we have seen: one reflected and two transmitted. However, the presence of three diverging waves is equivalent to satisfying the conditions (9) and (10) and consequently to the instability of the initial shock wave, which in this case is a rarefaction wave. Therefore, of the three departing waves in the stationary discontinuity, only two can be realized: either one is reflected and one is transmitted under the conditions

$$v_{1x} < V_{A1}, \quad v_{2x} \leqslant V_{A2}, \tag{12}$$

or else two are transmitted under the conditions

$$v_{1x} \geqslant V_{A1}, \quad v_{2x} > V_{A2}, \tag{13}$$

while the reflected wave is impossible, since the velocity of the current flow is greater than the velocity of this wave relative to the medium. In both these cases the wave incident on the discontinuity leads to the production of two departing waves whose amplitudes are expressed in terms of the amplitude of the incident wave by means of the boundary conditions.

However, in addition to the cases considered, another possibility is

$$v_{1x} \geqslant V_{A1}, \quad v_{2x} \leqslant V_{A2}. \tag{14}$$

in this case the reflected wave and one of the transmitted ones are impossible, that is, besides the incident wave there is only one departing wave (k_2^+), which is clearly insufficient to satisfy the boundary equations. Actually, in this case, the boundary equations reduce to

$$\begin{aligned} (v_{1x} + V_{A1})\, h_{1z}^+ &= (v_{2x} + V_{A2})\, h_{2z}^+, \\ \sqrt{\rho_1/\rho_2}\,(v_{1x} + V_{A1})\, h_{1z}^+ &= (v_{2x} + V_{A2})\, h_{2z}^+. \end{aligned} \tag{15}$$

These equations are consistent only if $\rho_1 = \rho_2$, that is, either in the absence of a discontinuity or in a magnetohydrodynamic wave of finite amplitude (a rotational discontinuity), for which the equations of magnetic hydrodynamics reduce to linear equations and, consequently, two such waves do not interact. In the opposite case, the equations are not consistent and have no solution.

Equations (14) do not violate the equation of discontinuity and the equation for the increase of the density at the shock, and actually serve only as conditions for the magnetic field intensity. In particular, it is easy to establish that these conditions can be realized in a parallel shock wave — an ordinary shock wave which is propagated along the field — but are also always satisfied in the fundamental oblique shock wave in which the velocity on one side of the discontinuity and the magnetic field are parallel to the normal, while on the other side they are connected by the relation $\mathbf{v} = \mathbf{H}/\sqrt{4\pi\rho}$ [See reference 4, Eqs. (3.39) and (3.40)].

In our case, the absence of a solution of the linearized system of equations for an infinitesimally small distrubance can only mean that a magnetohydrodynamic wave of infinitesimally small amplitude incident on the shock wave produces a finite change in the original flow, which leads either to disintegration of the shock wave into several new shock waves or to a more general non-stationary flow of the medium.

It is not difficult to demonstrate the possibility of disintegration of a parallel shock wave satisfying conditions (14) into two fundamental oblique shocks. In this case both resultant shock waves move at a constant distance apart, while the direction the magnetic field in the region between them changes from its original direction, re-establishing it after passage of both waves. We can show that the parallel shock under conditions (14) represents two merging fundamental oblique waves, and separates into these waves under the action of a small perturbation. However, such a system of disintegrated shock waves is again unstable relative to a magnetohydrodynamic perturbation oriented perpendicular to the plane of the motion, as a consequence of the instability of the fundamental oblique waves. This means that the disintegration of the initial shock wave into a stationary sequence of shock waves cannot eliminate the instability of the type under consideration. We therefore reach the conclusion that a parallel shock wave satisfying conditions (14) is converted into a certain nonstationary flow under the action of infinitely small perturbations. This nonstationary flow can evidently contain one or several diverging shock waves.

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