A MODEL FOR THE PROCESS OF MULTIPLE PRODUCTION

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We discuss the idea of constructing a "table of random stars" which could reproduce, in a form suitable for comparison with experiment, theoretical concepts concerning the multiple production of elementary particles. It is shown possible to construct such a table for energies up to 10 Bev.

MOST papers on the theory of multiple particle production are devoted to the solution of two problems. The first is the determination of the statistical weights of the various different reactions, and the second is the determination of the momentum distributions. Ordinarily one does not discuss the question of angular correlations or of other correlations between the directions and velocities of the secondary particles. Yet the solution of this question would allow one to reach more definite conclusions on the applicability of the theory to the process of multiple production. Indeed, correlations between the directions of the particles must depend on the nature of the interaction between the particles at the time they are produced. Thus in Fermi's original statistical model of independent secondary particles, the correlations should depend only on restrictions due to the conservation laws, while according to the presently accepted ideas of a resonance interaction between the nucleon and meson, the correlation should be stronger.

A qualitative estimate of the "forward-backward" correlation was already given by Fermi[1] from considerations of angular-momentum conservation. The difficulty in obtaining quantitative results lies in the very complicated calculations necessary. But even if these calculations could be carried out with very simple assumptions, the complication in the form of the interaction matrix (the natural way to proceed with the theory) would present new difficulties. In view of these conditions, an obvious way out would be to establish a model for multiple particle production. But since our ideas as to the mechanism for multiple production are still too cloudy for us to find among large-scale processes one with the appropriate regularities, we may make use of a numerical model. Such a model should be capable of reproducing a large class of physical assumptions on the form of the interaction.

M. I. Podgoretskii and M. Ia. Danysh (in a private communication) have suggested the idea for such a model in the form of a "table of random stars" for the isobaric model. The essence of their proposal is to construct a table of random variables which satisfy the same laws as the momenta of the secondary particles as predicted by the isobaric model. The table of random stars should contain a series of entries each of which represents one case of particle production, i.e., contains the magnitudes and directions of the individual secondary-particle momenta. The statistical analysis of such a table will give the same kind of information on the multiple-production process (statistical weights, momentum distributions, angles, charges, etc.) as will the analysis of actual stars in photographic emulsions or chambers.

The distributions or correlations which one is able to obtain from stars, say, in a hydrogen chamber, can be duplicated on the corresponding table. A shortcoming of the method is the low accuracy obtained from a small table, the large amount of calculation necessary to construct and analyze the table, and the impossibility of obtaining analytic expressions for the results.

It is shown in the present work how this idea can be realized using an electronic computer. It is possible, however, to construct a table containing 100 or 200 lines with some simplifying assumptions for five or six secondary particles by our method without resort to machines.

Our method takes into account energy and momentum conservation in the reaction. It can be generalized to a relatively wide class of concepts concerning the interaction of particles in multiple production. For this reason we are able to answer affirmatively the question of whether it is possible to construct a model for multiple production more accurately than it is possible to test numerically

any particular physical model for multiple production.

1. TABLE OF THREE-PARTICLE REACTIONS

Consider the reaction of two particles with total energy E and zero momentum, resulting in the production of three particles with masses m_i, momenta $\mathbf{p}_i$, and total energy e_i (with i = 1, 2, 3). According to Fermi's theory, the probability for this reaction is proportional to

$$W(E) = \int d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 \delta\Big(\sum_1^3 e_i - E\Big)\delta\Big(\sum_1^3 \mathbf{p}_i\Big). \quad (1.1)$$

The product of δ-functions represents the momentum distribution density in nine-dimensional momentum space, and the entire problem of constructing a model lies just in achieving such a distribution. But the narrow bands along the intersections of the $\Sigma e_i - E = 0$ and $\Sigma \mathbf{p}_i = 0$ surfaces of the nine-dimensional rectangle have too small a volume for the randomly chosen points with coordinates $\mathbf{p}_i$ to fall sufficiently often into these bands. On the other hand the integral

$$W(E) = 8\pi^2 \int_{D_3} p_1 p_2 (E - e_1 - e_2)\, dp_1 dp_2, \quad (1.2)$$

obtained from (1.1) by simple operations, can be used to achieve the necessary momentum distribution. In this integral the region of integration D_3

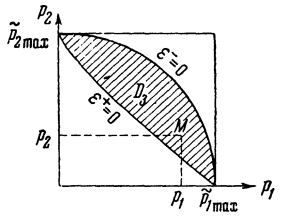

FIG. 1

(Fig. 1) is bounded by the curves $\cos\theta = \pm 1$ (where θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$). The equations of these curves are

$$\varepsilon^+ = 0, \quad \varepsilon^- = 0, \quad (1.3)$$

where

$$\varepsilon^\pm = e_1 + e_2 + [(p_1 \pm p_2)^2 + m_3^2]^{1/2} - E. \quad (1.4)$$

It follows from (1.2) that the joint probability that the momentum $\mathbf{p}_1$ lies in the interval $d\mathbf{p}_1$ and $\mathbf{p}_2$ lies in the interval $d\mathbf{p}_2$ is distributed with a density (which is thus also the density of points $M(\mathbf{p}_1, \mathbf{p}_2)$ in D_3) proportional to the function

$$w(p_1, p_2) = p_1 p_2 (E - e_1 - e_2). \quad (1.5)$$

According to (1.5) the points M are not uniformly distributed in D_3. There are two ways of obtaining random variables α distributed in (a, b) with density $f(\alpha)$. These are (a) the rejection method and (b) the "direct" method.[2] The two-dimensional distribution of (1.5) is conveniently obtained by the "rejection method". To do this one must know the maximum of $W(p_1, p_2)$ on D_3. It is easily shown that the values of $p_{1,2}$, which we shall call $\bar{p}_{1,2}$, for which this maximum is attained are the roots of the set of equations

$$p_1^2/e_1 = p_2^2/e_2 = E - e_1 - e_2, \quad (1.6)$$

if they lie within D_3. If they do not, then $\bar{p}_{1,2}$ lie on the boundary of D_3 (the curve $\epsilon^- = 0$) and satisfy the set of equations

$$\frac{p_2[E - e_1 - e_2 - (p_1^2/e_1)]}{p_1[E - e_1 - e_2 - (p_2^2/e_2)]} = \frac{p_2 - (E - e_2)(p_1/e_1)}{p_1 - (E - e_1)(p_2/e_2)}, \quad \varepsilon^- = 0. \quad (1.7)$$

For high values of the energy E, it is (1.6) which must be solved, while for low ones, it is (1.7).

A pair of numbers p_1, p_2 determines the momenta of all three particles in magnitude and direction. Therefore the procedure for obtaining a single entry in the table of three-particle stars corresponding to Fermi's model is the following.

1. The random variables p_1 and p_2 are picked uniformly in the intervals $(0, \tilde{p}_{1\max})$ and $(0, \tilde{p}_{2\max})$.* Here

$$\tilde{p}_{1\max} = \{[E^2 - (m_1 + m_2 + m_3)^2] \times [E^2 - (m_1 - m_2 - m_3)^2]\}^{1/2}/2E, \quad (1.8)$$

with a similar expression for $\tilde{p}_{2\max}$.

2. One must check that (p_1, p_2) is a point in D_3. The easiest way to do this is to verify that

$$\varepsilon^- < 0, \quad \varepsilon^+ > 0. \quad (1.9)$$

If these inequalities are fulfilled, one can always find an angle θ between $\mathbf{p}_1$ and $\mathbf{p}_2$ such that

$$e_1 + e_2 + [(\mathbf{p}_1 + \mathbf{p}_2)^2 + m_3^2]^{1/2} = E. \quad (1.10)$$

If (1.10) is not fulfilled, however, (p_1, p_2) is rejected and a new pair is picked.

3. The random variable β is picked uniformly in the interval $(0, w_{\max} \equiv w(\bar{p}_1, \bar{p}_2))$ and the inequality

$$\beta \leqslant w(p_1, p_2). \quad (1.11)$$

is checked. If this inequality is not fulfilled, the (p_1, p_2) pair is rejected.

4. If (1.11) is fulfilled, (1.10) is used to calculate

$$\cos\vartheta = (p_3^2 - p_1^2 - p_2^2)/2p_1p_2. \quad (1.12)$$

The momenta of the three particles can now be

*The terminology is explained elsewhere.[2,3]

taken as

$$\mathbf{p}_1 = \{p_1 \cos\psi,\ -p_1 \sin\psi,\ 0\},$$
$$p_2 = \{p_2 \cos(\vartheta - \psi),\ p_2 \sin(\vartheta - \psi),\ 0\}, \qquad (1.13)$$
$$\mathbf{p}_3 = -\mathbf{p}_1 - \mathbf{p}_2,$$

where ψ is a random angle uniformly distributed in $(0, 2\pi)$.

A table in which each line is calculated from (1.13) will be a table of plane stars. This means that in analyzing an experiment one must reduce all the three-particle stars to a single plane before comparing the experimental and the tabulated statistics. To obtain a table of three-particle stars not restricted to a single plane, what is necessary, obviously, is to allow the end point of the unit normal to the $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ plane to be uniformly distributed over the unit sphere.

In the next section we give a method for transforming plane stars to stars uniformly distributed in three-space. We also evaluate in this section the efficiency with which our method yields stars.

2. TABLE OF REACTIONS WITH n SECONDARY PARTICLES

General case. Consider the case in which n secondary particles are formed in the interaction, the square of the interaction matrix element $F(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ being a function which remains bounded everywhere. This function F can also depend on the energy E and on the parameters of the initial particles, but in our discussion such dependence is of no import. The momentum part of the expression for the statistical weight is of the form

$$W(E, \mathbf{P}) = \int d^3\mathbf{p}_1 \ldots d^3\mathbf{p}_n F(\mathbf{p}_1, \ldots, \mathbf{p}_n)\,\delta$$
$$\times \Big(\sum_1^n e_i - E\Big)\,\delta\Big(\sum_1^n \mathbf{p}_i - \mathbf{P}\Big), \qquad (2.1)$$

where $\mathbf{P}$ is the total momentum of the system. The region of integration over the first $(k-1)$ momenta will be denoted by D_k, and the other momenta $\mathbf{p}_k, \ldots, \mathbf{p}_n$ are not restricted in any way except in so far as they are involved in the conservation laws. Let us denote by d_k the region of integration over $\mathbf{p}_k$ for fixed $\mathbf{p}_1, \ldots, \mathbf{p}_{k-1}$ and arbitrary (except for the conservation laws) $\mathbf{p}_{k+1}, \ldots, \mathbf{p}_n$. We shall introduce the following special notation for the energies and momenta of particles $k, \ldots, n$ when the momenta of particles $1, \ldots, k-1$ are fixed:

$$E_k = E - \sum_1^{k-1} e_i, \qquad \mathbf{P}_k = \sum_1^{k-1} \mathbf{p}_i - \mathbf{P}; \qquad (2.2)$$

we also denote the characteristic ("maximum") energy and momentum of particle k in the center-of-mass system of particles $k, \ldots, n$ for fixed values of $\mathbf{p}_1, \ldots, \mathbf{p}_{k-1}$ by

$$E_k^* = (M_k^2 + m_k^2 - \mu_k^2)/2\,M_k (k = 1, \ldots, n-1),$$
$$E_n^* = (M_{n-1}^2 - m_{n-1}^2 + m_n^2)/2M_{n-1},$$
$$p_k^* = (E_k^{*2} - m_k^2)^{1/2} \qquad (2.3)$$
$$= \{(M_k^2 - m_k^2 - \mu_k^2) - (2m_k\mu_k)^2\}^{1/2}/2M_k,$$

where M_k is the effective mass of the system of particles $k, \ldots, n$, and μ_k is the mass of the compound particle composed of particles $k+1, \ldots, n$, i.e.,

$$M_k^2 = E_k^2 - P_k^2, \qquad \mu_k = \sum_{k+1}^{n} m_i. \qquad (2.4)$$

We shall specify $\mathbf{p}_k$ by its spherical coordinates $\mathbf{p}_k = \{p_k, \theta_k, \varphi_k\}$ in a system fixed with respect to the sum of the $\mathbf{p}_1, \ldots, \mathbf{p}_{k-1}$, momenta which have already been picked. We take the polar axis along the $\mathbf{P}_k$ direction, θ_k to be the angle between $\mathbf{P}_k$ and $\mathbf{p}_k$, and calculate φ_k from the vertical plane containing $\mathbf{P}_k$ (Fig. 2). This coordinate system is convenient in that the magnitude of $\mathbf{P}_k + \mathbf{p}_k$ is independent of φ_k; indeed,

$$P_{k+1}^2 = P_k^2 + p_k^2 + 2P_k p_k \cos\vartheta_k. \qquad (2.5)$$

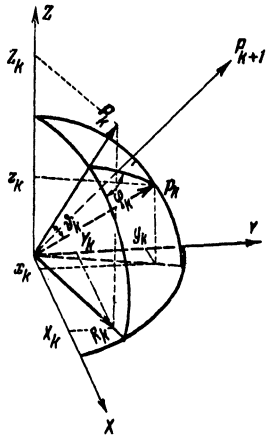

FIG. 2

The Cartesian coordinates of the momentum (see Fig. 2) are given by

$$\mathbf{P}_k = \{X_k, Y_k, Z_k\}, \qquad R_k = (X_k^2 + Y_k^2)^{1/2}; \qquad (2.6)$$

$$x_k = p_k\left[\frac{X_k}{P_k}\cos\vartheta_k + \left(\frac{Z_k}{P_k}\frac{X_k}{R_k}\cos\varphi_k - \frac{Y_k}{R_k}\sin\varphi_k\right)\sin\vartheta_k\right],$$

$$y_k = p_k\left[\frac{Y_k}{P_k}\cos\vartheta_k + \left(\frac{Z_k}{P_k}\frac{Y_k}{R_k}\cos\varphi_k + \frac{X_k}{R_k}\sin\varphi_k\right)\sin\vartheta_k\right],$$

$$z_k = p_k\left[\frac{Z_k}{P_k}\cos\vartheta_k - \frac{R_k}{P_k}\sin\vartheta_k\cos\varphi_k\right]. \qquad (2.7)$$

Let us now rewrite (2.1). Integrating over $\mathbf{p}_n$

and θ_{n-1}, we eliminate the δ-functions in the integrand, obtaining

$$W(E, \mathbf{P}) = \int d^3\mathbf{p}_1 \ldots d^3\mathbf{p}_{n-2} dp_{n-1} d\varphi_{n-1} p_{n-1} \frac{E_n}{P_{n-1}} \times F(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, -\mathbf{P}_{n-1} - \mathbf{p}_{n-1}), \quad (2.8)$$

where $\mathbf{p}_{n-1}$ is the vector whose components are

$$\mathbf{p}_{n-1} = \left\{ p_{n-1}, \cos^{-1} \frac{E_n^2 - m_n^2 - P_{n-1}^2 - p_{n-1}^2}{2P_{n-1}p_{n-1}}, \varphi_{n-1} \right\}, \quad (2.9)$$

and the region D_n of integration is bounded by the surfaces $\cos\theta_{n-1} = \pm 1$, i.e.,

$$\{(P_{n-1} \pm p_{n-1})^2 + m_n^2\}^{1/2} = E_{n-1} - e_{n-1}. \quad (2.10)$$

Now (2.8) can in principle be used to solve the problem of achieving the distribution associated with the model whose interaction is given by F. There are no dependent variables of integration in (2.8), so that the sets of values of $\mathbf{p}_1, \ldots, \mathbf{p}_{n-2}$, p_{n-1}, φ_{n-1} must satisfy certain inequalities rather than equations (as is the case for (2.1)), and this increases greatly the efficiency of drawing. However the amount of calculation necessary to obtain a distribution over $\mathbf{p}_1, \ldots, \mathbf{p}_{n-2}$, p_{n-1}, φ_{n-1} with density $p_{n-1}E_nF/P_{n-1}$ is still extremely large. All further operations should be directed toward decreasing the amount of calculation. An obvious way to do this is to perform as many integrations as possible in (2.8), since this reduces the dimensionality of the region of integration and thus increases the ratio between the volume of this region and the volume of the rectangle under consideration. This goal is not, however, possible to achieve in general with arbitrary forms of F. Since we wish to develop a method suitable for all F, we shall proceed to reduce the amount of calculations in a different way. We write d^3p_k in the form $p_k^2 dp_k \times d\cos\theta_k d\varphi_k$, making use of the spherical coordinate system described above. Then the density of distribution over p_1, $\cos\theta_1$, φ_1; $p_2, \ldots$; $\ldots, \varphi_{n-2}$; p_{n-1}, φ_{n-1} will be of the form

$$\Phi(p_1, \ldots, \varphi_{n-1}) = p_{n-1} \frac{E_n}{P_{n-1}} F \prod_{k=1}^{n-2} p_k^2, \quad (2.11)$$

where the arguments of F are the same as in (2.8). The essential result is that the regions $D_2, \ldots, D_k, \ldots, D_n$ for the solved values $\mathbf{p}_1$; $\mathbf{p}_1, \mathbf{p}_2$; $\ldots$; $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{k-1}$; $\ldots$; $\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}$ are independent of $\varphi_1, \ldots, \varphi_{n-1}$. We proceed to prove this assertion.

Let the momenta $\mathbf{p}_1, \ldots, \mathbf{p}_{k-1}$ (where $k \leq n-2$) be fixed. Let us find the region of variation d_k of the momentum $\mathbf{p}_k$. The remaining k, k+1, $\ldots$, n particles fulfill the relations

$$\sum_k^n e_i = E_k, \qquad \sum_k^n \mathbf{p}_i = -\mathbf{P}_k. \quad (2.12)$$

It is known[4] that if in multiple production one particle has its maximum possible momentum, then all the other secondary particles move as a whole, i.e., as a particle whose mass is equal to the sum of the masses. For particle k this mass is μ_k [see (2.4)]. The conservation laws for two particles with masses m_k and μ_k, total energy E_k and momentum $-\mathbf{P}_k$ can be written in the form

$$(\mathbf{p}_k^2 + m_k^2)^{1/2} + \{(-\mathbf{P}_k - \mathbf{p}_k)^2 + \mu_k^2\}^{1/2} = E_k.$$

This is the equation of the surface which bounds the region d_k of possible positions of the end points of $\mathbf{p}_k$. The location and shape of d_k is most clearly found by a graphical representation of the conservation laws.[5] On going from the center-of-mass system to the laboratory system, the $p_k^* = \text{const}$ sphere is deformed into an ellipsoid of revolution prolate along the direction of relative motion of the two coordinate systems. In the case we are considering the center of the ellipsoid lies at O (whose position vector is $-E_k^*\mathbf{P}_k/M_k$), the major axis (the axis of revolution) is of length $2p_k^*E_k/M_k$ and is directed along $\mathbf{P}_k$, and the minor semiaxis is of length p_k^* (Fig. 3). Here E_k^* and p_k^* are given by (2.3).

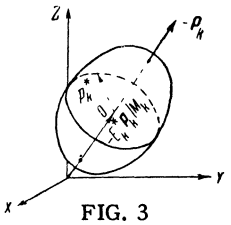

FIG. 3

We can now find the region D_k. Since the energy E_k^* of particle k in the center-of-mass system of particles k, k+1, $\ldots$, n is no less than its mass m_k, it follows that [see (2.3)]

$$M_k \geqslant \mu_{k-1} \quad (k \leqslant n-1). \quad (2.13)$$

This is not only a necessary condition on $\mathbf{p}_1, \ldots, \mathbf{p}_{k-1}$, but also a sufficient one. That is, when it is fulfilled there always exist $\mathbf{p}_k, \ldots, \mathbf{p}_n$ such that the conservation laws will be fulfilled; for instance, particles k+1, $\ldots$, n may all be moving in the same direction (in the coordinate system in which $\mathbf{P}_k = 0$), while particle k moves in the opposite direction.

But M_k depends only on the magnitude of $\mathbf{P}_k$, as indicated by (2.4), and this in turn is independent of φ_k [see (2.5)]. Therefore the shape of D_k as given by (2.3) is independent of φ_k (for $k \leq$

$n-1$). For D_n this follows from (2.10).

The fact that the shape of D_k is independent of φ_k for all k from 1 to n makes it possible to decrease the dimensionality of each D_k by a factor of $\frac{2}{3}$; thus, for instance, the dimensionality of D_n will be $2(n-1)$ instead of $3(n-1)$. This, in turn, greatly increases the ratio between the volume of D_n and the volume of our rectangle of the same number of dimensions.

Thus we can use the following procedure for obtaining n-particle reactions for an interaction the square of whose matrix element is F. First p_1 and $\cos\theta_1$ are picked from a uniform distribution in $(\tilde{p}_{1\min}, \tilde{p}_{1\max})$ and $(-1, 1)$; then p_2 and $\cos\theta_2$ are picked from $(\tilde{p}_{2\min}, \tilde{p}_{2\max})$ and $(-1, 1)$, and (2.13) is checked with $k = 3; \ldots;$ then p_k and $\cos\theta_k$ are picked from* $(\tilde{p}_{k\min}, \tilde{p}_{k\max})$ and $(-1, 1)$, E_{k+1} is calculated from (2.2), and P_{k+1} from (2.5), and (2.13) is verified. If it is not fulfilled, we pick again from p_1, while if it is, we proceed to $k+1$, etc., up to $k = n-1$. For $k = n-1$ only p_{n-1} is picked, within the limits $|E^*_{n-1}P^*_{n-1} \mp p^*_{n-1}E_{n-1}|/M_{n-1}$ (but not between $\tilde{p}_{n-1\min}$ and $\tilde{p}_{n-1\max}$!), and (2.13) need not be verified. Only now are $\varphi_1, \ldots, \varphi_{n-1}$ picked from $(0, 2\pi)$, value of $\cos\theta_{n-1}$ calculated by (2.9), and the value of $\Phi(p_1, \ldots, \varphi_{n-1})$ calculated by (2.11).

The use of the rejection method to obtain the distribution with density Φ has its peculiarities in the present case. It cannot be used by rote, since if F is everywhere finite, $\Phi_{\max} = \infty$ when $P_{n-1} = 0$. But it is just in the limit $P_{n-1} \to 0$ that the interval of variation of p_{n-1} contracts to a point, so that p_{n-1} is uniquely specified and need not be picked.

After obtaining the distribution with density Φ, the construction of the table is completed by calculating $\mathbf{p}_n = \mathbf{p}_{n-1} - \mathbf{P}_{n-1}$ and writing all the momenta in the cartesian XYZ system using Eqs. (2.6) and (2.7).

Fermi's model ($F \equiv 1$) allows some simplification of the method of computation. In this case it is possible to carry out the integration over φ_{n-1} and p_{n-1} in (2.8), or more accurately over e_{n-1} between the limits

$$e_{\substack{\min\\\max}} = (E^*_{n-1}E_{n-1} \mp p^*_{n-1}P_{n-1})/M_{n-1}, \qquad (2.14)$$

and to obtain the distribution density†

$$\Phi'(\mathbf{p}_1, \ldots, \mathbf{p}_{n-2}) = \frac{4\pi}{3}p^*_{n-1}\frac{3E^*_{n-1}E^*_nE^2_{n-1} - p^{*2}_{n-1}P^2_{n-1}}{M^3_{n-1}}\prod_{i=1}^{n-2}p_i^2. \qquad (2.15)$$

The sequence of picking is here the same as in the general case, but ends with the picking of p_{n-2} and $\cos\theta_{n-2}$. Having obtained uniformly distributed points in D_{n-1}, one obtains a nonuniform distribution with density Φ' by the rejection method. In addition, $\Phi'_{\max}$ can be calculated beforehand; it can be shown, namely, that the maximum value of Φ' is obtained when $P_{n-1} = 0$ and the values of $\mathbf{p}_1, \ldots, \mathbf{p}_{n-2}$ satisfy the set of equations

$$\frac{p_1^2}{e_1} = \frac{p_2^2}{e_2} = \ldots = \frac{p^2_{n-2}}{e_{n-2}} = \frac{2p^{*2}_{n-1}E^*_{n-1}E^*_nE_{n-1}}{(E^*_{n-1}E^*_n)^2 + p^{*2}_{n-1}(E^{*2}_{n-1} - E^*_{n-1}E^*_n + E^{*2}_n)}. \qquad (2.16)$$

These equations are easily solved by an iteration method.

We must still pick the momentum p_{n-1}. Instead, it is better to pick the energy e_{n-1} by the "direct" method, i.e., to solve the equation

$$2e^3_{n-1} - 3E_{n-1}e^2_{n-1} = (2e^3_{\min} - 3E_{n-1}e^2_{\min})(1-\alpha) + (2e^3_{\max} - 3E_{n-1}e^2_{\max})\alpha, \qquad (2.17)$$

where α is uniformly distributed over $(0, 1)$. Only after this has been done does it make sense to pick $\varphi_1, \ldots, \varphi_{n-1}$.

A characteristic feature of the methods described in Secs. 1 and 2 is the occurrence of unsuccessful pickings, which means that the uniform distributions are obtained by rejection. It is obvious that if there are too many unsuccessful pickings, this method for constructing the table becomes practically useless. It is impossible to evaluate the efficiency of this method in general. A numerical experiment has shown, however, that for meson production by 10-Bev protons, the method is fifty percent efficient (for production of one π meson) and ten percent efficient (for two π mesons). For more mesons, the efficiency is too low for the method to be usable by hand calculation. Nevertheless, it is possible to construct a table for production of 3 or 4 mesons by hand if one integrates Φ' several more times. The use of an electronic computer makes it possible to construct large tables for 5 or 6-particle reactions by our method.

3. THE TABLE OF RANDOM STARS

It is difficult to compare an n-particle table with experiment, since experiment does not observe neutral particles. Such a table must enter

*For $n = 3$, Eq. (2.15) becomes the same as Block's[6] Eq. (4).

†The values of $\tilde{p}_{k\min}$ and $\tilde{p}_{k\max}$ are calculated by formulas similar to (1.8).

as a component part into a table of random stars formed by the collision of two given particles for a given energy E. To form such a table one must consider all forms of reactions which have observable statistical weight. The statistical weight of a reaction determines the average number of entries corresponding to it.

Although in Fermi's model the statistical weights can be calculated and are already known for many reactions, this is not true for a reaction with an arbitrary value of $F(p_1, \ldots, p_n)$; it may turn out to be no less difficult to calculate the statistical weight for such a model than it is to construct the table of random stars. The above method for constructing the table of n-particle reactions, however, can in principle be used to determine the volume W in phase space, the fundamental part of the expression for the statistical weight. Indeed, it is clear that W is equal to the product of the volume of D_n by the average value of Φ as given by (2.11) over this region. But the volume of D_n is equal to the fraction of successful drawings of $p_1, \ldots, p_{n-1}$ in this volume multiplied by the volume of the rectangle of known dimensions circumscribed about D_n. In actuality, the use of this method to calculate the volume in phase space requires a very large number of drawings, because the Φ values are so widely scattered; the smaller the volume of phase space, the greater the number of drawings required.

But if all the statistical weights of the most important reactions which take place at a given energy are known, there is no great difficulty involved in constructing the table of random stars. We arrange all the reactions $N_1, N_2, \ldots, N_j, \ldots$ in an arbitrary order, and with each we associate an interval δ_j on the segment (0, 1), such that the length of δ_j is proportional to the weight of N_j. Then in calculating an entry in the table, one chooses a random point on (0, 1). The index of the δ_j interval in which this point lies gives the reaction N_j which is represented by this entry. One finds (if necessary), again by drawing, which of the momenta in this entry are to belong to charged particles. The momenta of the other particles, since they are not observed, need not be included in the entry.

In a table constructed in this way, (a) the reactions will be randomly shuffled, and (b) the number of entries referring to each reaction will fluctuate in accordance with their statistical weights. These statistical fluctuations will obey the same laws as in the observation of actual stars. On the other hand, one may wish to increase the accuracy of small tables by forbidding such fluctuations.

Using a computer giving stars with $n = 6$, this method can take into account most reactions involving protons with kinetic energies up to 10 Bev. This will give rise to stars with varying numbers of prongs.

The accuracy of the results obtained from a table of random stars depends on essentially the same factors as in the case of real stars. Errors in the determination of the energy and angles of emission of the particles do not occur; one may drop the fluctuations of the frequency of occurrence of the various reactions about the statistical weights; the results will, however, be affected by the error in calculating the statistical weights themselves. If the errors in the weights are of the order of ten percent, it is hardly worth having tables of more than one thousand entries. The allowed margin of error is necessary to obtain nominal distributions and to compare distributions for different energies. The errors in the statistical weights for different E cancel, since they are more or less systematic, and thus comparison of the distributions for different E can be very accurate.

An interesting possibility is that of increasing the accuracy by constructing the table of random stars in the laboratory coordinate system. It is then no longer necessary to transform the actual stars into the center-of-mass system, an operation which can be handled only very roughly for high energies. This simplifies the analysis of the experimental data and increases the accuracy with which the table of random stars can be compared with experiment.

CONCLUSION

The purpose of this paper was to clarify the possibility of constructing a table of random stars. A method for such a construction has been given. It is first of all possible, clearly, to represent Fermi's model and the isobaric model in the form of tables. One may hope that modern computers are entirely able to reproduce reactions with 6 or 7 secondary particles, which means that it should be possible to construct tables of the stars obtained in proton-proton collisions in a 10-Bev accelerator.

The method for constructing the table for these models will give an isotropic angular distribution for the reaction products. In order to explain the actually observed nonisotropic distributions, it is necessary to give the appropriate momentum and angular dependence of the square of the matrix element F.

Our method for constructing the table of random stars is applicable to different forms of F.

The fact that F may be a complicated functions plays no role, since most of the operating time of the computer is spent, in our method, in rejecting the unsuccessfully picked momentum components, rather than performing operations with the successfully picked momenta.

It thus becomes possible to find a phenomenologically satisfactory description of the interaction by testing various hypotheses and comparing the statistical weights so obtained, the distributions, and the correlations with the experimental data. The same kind of an approach can be used to study the decay of unstable particles.

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