INTERACTION BETWEEN A MEDIUM AND A RING CURRENT INCIDENT ON IT

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An analysis is given of the spectral distribution of the current in a turn that impinges with a constant nonrelativistic velocity on a medium possessing arbitrary $\epsilon(\omega)$ and μ. Derivations are given of the criterion for reflection at nonrelativistic velocities, and of the expression for the acting forces.

1. FORMULATION OF THE PROBLEM

IN an earlier paper[1] the author has calculated the interaction between a semi-infinite medium and a constant rectilinear current incident on it with a constant velocity $\beta = v/c$. If the current is incident on plasma (or on metal), the force of repulsion does not depend on the electron density or on other characteristics of the plasma, and as $\beta \rightarrow 1$ the force increases as $1/(1-\beta^2)^{1/2}$, provided the velocity exceeds a certain critical value $\beta \gg \beta_{cr} = c\gamma_0/\omega_0^2 r$ (cf. reference 1). When $\beta \ll \beta_{cr}$ the force decreases, practically linearly with decreasing β (if the logarithmic term is not taken into account). These results are not applicable to a ring current even if the distances to the plasma are sufficiently small so that the curvature of the current is negligible, since the change in the magnetic flux through the current circuit as it approaches the plasma will change significantly the magntidue of the current, and therefore also the magnitude of the forces, unless we assume that such a change in current is compensated by some additional electromotive forces. Consequently the restuls of the earlier paper[1] are applicable to a ring current only under the assumption that the magnitude of the incident current is at its limit.

In the present paper we consider the field of a ring current of radius a, which is incident with a constant velocity v on a medium occupying the semi-infinite space $z \leq 0$. The medium is described by arbitrary μ and $\epsilon = \epsilon(\omega)$. It is assumed that the plane of the ring current is parallel to the plane surface of the medium. Further, we assume that the ring is characterized by infinite conductivity, so that the total flux of the magnetic field linked with the current circuit is a constant of the motion. When the distance from the circuit to the medium is infinite, $z \rightarrow \infty$ (cf. Fig. 1), the

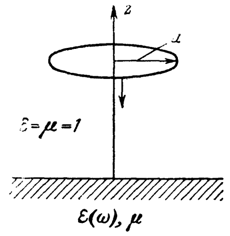

FIG. 1

magnitude of the current I_∞ is considered to be given, and its subsequent variation is determined by the motion of the circuit. It is assumed that the radius of the ring does not change in the course of the motion. Thus, the current density can be written

$$j_z = j_\rho = 0, \quad j_\varphi = I(t)\,\delta(z+vt)\,D(\rho, a), \qquad (1)$$

where a is the radius of the ring, $I(-\infty) = I_\infty$.

A change in the current in the course of the motion of the ring means that of definite frequencies are excited in the ring. Without solving the problem it is possible to make a number of qualitative statements with respect to the frequency distribution of the excited currents from the uncertainty relation $\Delta\omega\Delta t \sim 1$ in the same way as is done in the case of radiation in accelerators. The order of magnitude of the maximum excited frequency can be estimated as $\omega_{max} \sim 1/\tau_{col}$, where τ_{col} is the characteristic time for the collision of the ring with the wall. At speeds $v \sim c$, as a result of relativistic contraction, this time is of order of magnitude $a(1-\beta^2)^{1/2}/c$, i.e., $\omega_{max} \sim c/a(1-\beta^2)^{1/2}$. Thus, as $\beta \rightarrow 1$ successively higher harmonics will be excited in the ring. The object of the present work is a quantitative investigation of this effect.

To analyze the frequency distribution we expand all the quantities in Fourier integrals. In particular,

$$I(t) = \int I_\omega e^{-i\omega t} d\omega, \qquad (2)$$

$$I(t)\,\delta(z + vt) = \frac{1}{2\pi}\int I_{\omega+\varkappa v} e^{-i\omega t + i\varkappa z} d\omega\, d\varkappa. \qquad (3)$$

In view of the cylindrical symmetry of the problem it is convenient to expand the radial function $D(\rho, a)$ in terms of Bessel functions:

$$D(\rho, a) = a\int_0^\infty g(k)\, kJ_1(k\rho)\, J_1(ka)\, dk. \qquad (4)$$

A linear ring current corresponds to $g(k) = 1$ and $D(\rho, a) = \delta(\rho - a)$. The quantity $g(k)$ has the meaning of a form factor that characterizes the "smearing" of the current and which must cut off values of k greater than $1/\delta$, where δ is a dimension that characterizes the current cross section. Since the specific form of $g(k)$ is not important in the case of thin rings, which will be discussed from now on, it is convenient in specific calculations to take $g(k) = \exp\{-k\delta\}$. In accordance with Eqs. (1) to (4), the potential of the field of the ring current can be written in the form*

$$A^0 = \frac{2a}{c}\int \frac{I_{\omega+\varkappa v}}{k^2 + \varkappa^2 - \omega^2/c^2} e^{i(\varkappa z - \omega t)} kJ_1(k\rho) J_1(ka)\, g(k)\, dk\, d\varkappa\, d\omega. \qquad (5)$$

Generally speaking, it is necessary to add to the expression $1/(k^2 + \kappa^2 - \omega^2/c^2)$ a δ-function term that leads to retarded potentials, and to interpret the integral containing $1/(k^2 + \kappa^2 - \omega^2/c^2)$ in the sense of its principal value (cf., for example, Bolotovskii[2]). This operation is equivalent to specifying a definite method of going around the poles by means of infinitesimally small imaginary (or, in general, complex) additions to the denominator. We assume that the rules of going around the poles are specified by prescribing a complex ϵ which differs from unity by an arbitrarily small amount, while the sign of the imaginary term in the complex expression for ϵ must correspond to dissipative processes for a given choice of the sign of the frequency (cf. Bolotovskii[2]). This requirement leads in an unambiguous way to the displacement of the poles into the upper half-plane of complex ω, which is equivalent to adding to the principal value of $P(1/(k^2 + \kappa^2 - \omega^2/c^2))$ the δ-function term $(\pi i\omega/|\omega|)\,\delta(k^2 + \kappa^2 - \omega^2/c^2)$. Expressions containing such denominators will be interpreted henceforth in this sense.

2. THE METHOD OF IMAGES

Let the medium be described by certain $\epsilon(\omega)$ and μ. We assume that the field for $z \geq 0$ consists of the field A^0 in the absence of the medium and the field of the "reflected" current, the potential $A^{(1)}$ due to the latter being obtainable from A^0 by letting $z \to -z$ and by multiplying the Fourier components of the current by $\Phi_1(\omega, \kappa)$:

$$A|_{z\geq 0} = A^0 + A^{(1)}, \qquad (6)$$

$$A^{(1)} = \frac{2a}{c}\int \Phi_1(\omega, \varkappa) \times \frac{I_{\omega+\varkappa v}}{k^2 + \varkappa^2 - \omega^2/c^2} e^{-i(\varkappa z + \omega t)}\, dk\, d\varkappa\, d\omega\, kJ_1(k\rho)\, J_1(ka)\, g(k). \qquad (7)$$

The field for $z \leq 0$ is the field whose potential $A^{(2)}$ can be obtained from A^0 by letting $z \to \zeta z$, where $\zeta = \zeta(\omega, \kappa)$, by multiplying the Fourier component of the current by $\Phi_2(\omega, \kappa)$, and by replacing c by $c/\sqrt{\epsilon\mu}$:

$$A|_{z\leq 0} = A^{(2)} = \frac{2a\mu}{c}\int \Phi_2(\omega, \varkappa)\, \frac{I_{\omega+\varkappa v}\exp\{i[\varkappa\zeta(\omega, \varkappa).z - \omega t]\}}{k^2 - \varepsilon\mu\omega^2/c^2 + \varkappa^2\zeta^2(\omega, \varkappa)} \times kJ_1(k\rho)\, J_1(ka)\, g(k)\, dk\, d\varkappa\, d\omega. \qquad (8)$$

In the special case when the current does not vary, and the medium is not a dispersive one, the above replacement means that the field due to the medium at $z \geq 0$ is the field of the reflected current whose magnitude is reduced by the factor Φ_1.

From the boundary conditions we obtain:

$$\zeta = \{1 + (\omega/c\varkappa)^2[\mu\varepsilon(\omega) - 1]\}^{1/2}, \quad \mathrm{Re}\,\zeta > 0; \qquad (9)$$

$$\Phi_1(\omega, \varkappa) = (\mu - \zeta(\omega, \varkappa))/(\mu + \zeta(\omega, \varkappa)); \quad \Phi_2(\omega, \varkappa) = 2\mu/(\mu + \zeta(\omega, \varkappa)). \qquad (10)$$

In order that Eqs. (6) to (8) be fields that satisfy the foregoing equations, it is also necessary to show that if ζ, Φ_1 and Φ_2 are defined by relations (9) and (10), expressions (7) and (8) satisfy the homogeneous equations in the corresponding regions $z \geq 0$ and $z \leq 0$. To verify that the currents obtained from (7) and (8) vanish in these regions it is sufficient to let $\omega \to \omega - \kappa v$ in the integrands of the expressions for the currents and to deform the path of integration into the lower half-plane of complex κ. In virtue of $\mathrm{Re}\,\zeta > 0$ and of the form of $\kappa\zeta$, Φ_1 and Φ_2, the only singularities of the integrands will be the singular points ϵ and $1/\epsilon$ (cf., for example, references 3 and 4). We have taken the time-dependent factor in the present case to be the same as in reference 4 $(\exp\{-i\omega t\})$, but the expressions in which we are interested contain $\omega - \kappa v$, so that the poles with respect to κ lie in the upper half-plane of complex κ. Integration over the large semi-circle in the lower half-plane gives zero for certain ranges of values $z \geq 0$ and $z \leq 0$, since $\zeta \to 1$ as $|\kappa| \to \infty$, because $\epsilon\mu \to 1$ as $|\omega| \to \infty$.

*The only nonvanishing component A_φ of the four-potential will be denoted throughout simply by A.

3. INTEGRAL EQUATION FOR THE FOURIER COMPONENTS OF THE CURRENT

The change in the current is determined by the flux Φ, which can be obtained from (5) and (7):

$$\Phi = 2\pi a A \Big|_{\substack{\rho=a\\ z=-vt}} = \frac{4\pi a^2}{c}\int \frac{I_{\omega+\varkappa v}}{k^2+\varkappa^2-\omega^2/c^2} g(k)\, kJ_1^2(ka)\, dk\, d\varkappa\, d\omega \times \left\{e^{-i(\varkappa v+\omega)t} + \frac{\mu-\zeta(\omega,\varkappa)}{\mu+\zeta(\omega,\varkappa)} e^{-i(\omega-\varkappa v)t}\right\}. \quad (11)$$

For a superconducting ring we can write

$$\Phi(t) = \Phi(-\infty) = LI_\infty / c(1-\beta^2)^{1/2}. \quad (12)$$

On multiplying this equation by $\exp\{i\lambda t\}$ and integrating with respect to t we obtain:

$$\frac{4\pi a^2}{c}\int g(k)\, kJ_1^2(ka)\, dk\, d\varkappa \left\{\frac{I_\lambda}{k^2+\varkappa^2-(\lambda-\varkappa v)^2/c^2} + \frac{\mu-\zeta(\lambda+\varkappa v,\varkappa)}{\mu+\zeta(\lambda+\varkappa v,\varkappa)} \frac{I_{\lambda+2\varkappa v}}{k^2+\varkappa^2-(\lambda+\varkappa v)^2/c^2}\right\} = \frac{LI_\infty}{c(1-\beta^2)^{1/2}}\delta(\lambda). \quad (13)$$

Here L is the self-inductance of the cirucit

$$L = 4\pi^2 a^2 \int_0^\infty g(k) J_1^2(ka)\, dk. \quad (14)$$

If we set $g(k) = \exp\{-k\delta\}$, we obtain from (14) for $\delta \ll a$,

$$L = 4\pi a\{\ln(8a/\delta) - 2\}.$$

We transform Eq. (13) by introducing in the second term of the left hand side the new variables $\kappa v = (\omega-\lambda)/2$, $\lambda + \kappa v = (\omega+\lambda)/2$. We obtain:

$$I_\lambda K(\lambda) + \int K(\lambda,\omega) I_\omega d\omega = \frac{LI_\infty}{4\pi a^2(1-\beta^2)^{1/2}}\delta(\lambda), \quad (15)$$

where

$$K(\lambda) = \int \frac{kJ_1^2(ka)\, g(k)\, dk\, d\varkappa}{k^2+\varkappa^2-(\lambda-\varkappa v)^2/c^2}, \quad (16)$$

$$K(\lambda,\omega) = \frac{1-\zeta_\lambda}{1+\zeta_\lambda}\frac{1}{2v}\int_0^\infty \frac{kJ_1^2(ka)\, g(k)\, dk}{k^2+(\lambda-\omega)^2/4v^2-(\lambda+\omega^2)/4c^2}, \quad (17)$$

with

$$\zeta_\lambda = \left\{1+\beta^2\left(\frac{\lambda+\omega}{\lambda-\omega}\right)^2\left[\mu\varepsilon\left(\frac{\lambda+\omega}{2}\right)-1\right]\right\}^{1/2}. \quad (18)$$

In virtue of the remark made above with respect to the rules for going around the singularities of the integrands, the kernels $K(\lambda)$ and $K(\lambda,\omega)$ are complex; it follows from $I_\lambda = I_\lambda^*$ that

$$K^*(-\lambda) = K(\lambda); \qquad K^*(-\lambda,-\omega) = K(\lambda,\omega). \quad (19)$$

We now evaluate the integrals (16) and (17). The integral (16) with respect to κ can be easily found by taking into account the rules stated above for going around the singularities

$$K(\lambda) = \frac{\pi i}{a(1-\beta^2)^{1/2}}\int_0^\infty \frac{kg(k/a)\, dkJ_1^2(k)}{(\psi^2-k^2)^{1/2}}, \qquad \lambda > 0, \quad (20)$$

where $(\psi^2-k^2)^{1/2} = 1(k^2-\psi^2)^{1/2}$ if $k > \psi$. Here $\psi = \lambda a/c(1-\beta^2)^{1/2}$, while, in the case of negative λ, $K(\lambda)$ can be obtained in accordance with (19). The function $g(k/a)$, which appears in (20), cuts off $k > 1/\gamma \equiv a/\delta$. For $\psi \gg 1/\gamma$ we can set in the integrand of (20) $(\psi^2-k^2)^{1/2} = \psi$. If, in addition, we set $g(k/a) = \exp\{-k\gamma\}$, we obtain for $\gamma \ll 1$

$$K(\lambda) = i/a(1-\beta^2)^{1/2}\psi\gamma. \quad (21)$$

In the other limiting case, $\psi\gamma \ll 1$, we simplify the calculation by setting $g(k/a) = \exp\{-\gamma(k^2-\psi^2)^{1/2}\}$, which corresponds to cutting off $k^2 > \psi^2+\gamma^{-2} \approx \gamma^{-2}$, and obtain:

$$K(\lambda) = K(0) - \frac{\pi}{2a(1-\beta^2)^{1/2}}\left\{\frac{\lambda}{|\lambda|} i\int_0^{2\psi} J_2(x)\, dx - \int_0^{2\psi}\Omega_2(x)dx\right\}, \quad (22)$$

$$K(0) = L/4\pi a^2(1-\beta^2)^{1/2},$$

J_2 and Ω_2 are the Bessel and the Lommel-Weber functions of the second order, L is the coefficient of self-inductance.

Further, the integral (17) can be written

$$K(\lambda,\omega) = \frac{1}{2v}\frac{1-\zeta_\lambda}{1+\zeta_\lambda}\int_0^\infty \frac{kg(k/a)J_1^2(k)\, dk}{k^2+\chi^2}, \quad (23)$$

where

$$\chi = (a/2v)\{(\lambda-\omega)^2-\beta^2(\lambda+\omega)^2\}^{1/2}. \quad (24)$$

If $\chi \gg 1/\gamma$, we can neglect k^2 compared to χ^2 in the denominator of (23); we then obtain

$$K(\lambda,\omega) = \frac{1}{2v\pi\chi^2\gamma}\frac{1-\zeta_\lambda}{1+\zeta_\lambda}. \quad (25)$$

If $\chi \ll 1/\gamma$ we can set $g(k/a) \approx 1$. By writing one of the $J_1(k)$ in the form $J_1(k) = (H_1^{(1)}(k) + H_2^{(2)}(k))/2$, we can write (23) in the form of an integral from $-\infty$ to $+\infty$. The resulting integral can be evaluated by the method of residues, taking into account the rule given above for going around the singularities on the real axis; we then have

$$K(\lambda,\omega) = \frac{1}{2v}\frac{1-\zeta_\lambda}{1+\zeta_\lambda} I_1(\chi)K_1(\chi), \qquad I_1(z) = \frac{1}{i}J_1(iz); \qquad K_1(z) = -\frac{\pi}{2}H_1^{(1)}(iz). \quad (26)$$

4. INVESTIGATION OF THE FIRST APPROXIMATION

If the current is far from the medium or if the medium in general has a small effect on the cur-

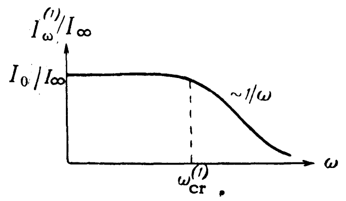

FIG. 2

rent (as is the case, for example, for a dielectric), we can restrict ourselves in solving Eq. (15) to the first approximation obtained by the method of iterations, in which we take for the zero-order approximation the expression for the Fourier components of the time-invariant current, which is equal to the current at infinity $I_\lambda^0 = I_\infty \delta(\lambda)$. The first approximation has the form

$$I_\lambda^{(1)} = -I_\infty K(\lambda, 0)/K(\lambda). \tag{27}$$

On substituting expressions (22) and (26) into (27) we obtain

$$I_\lambda^{(1)} = I_\infty \frac{\zeta_\lambda - 1}{\zeta_\lambda + 1} \frac{a(1-\beta^2)^{1/2}}{2v} I_1(\chi_0) K_1(\chi_0) \Big\{ l - \frac{\pi}{2} \int\limits_0^{2\psi} \Omega_2(x)\, dx + i \frac{\pi}{2} \frac{\lambda}{|\lambda|} \int\limits_0^{2\psi} J_2(x) dx \Big\}^{-1}, \tag{28}$$

$$\chi_0 = (a\lambda/2v)(1-\beta^2)^{1/2}, \quad \psi = \lambda a / c(1-\beta^2)^{1/2},$$

where l is one-half the linear inductance.

Thus we can introduce two characteristic critical frequencies $\omega_{cr}^{(1)}$ and $\omega_{cr}^{(2)}$ which determine the shape of the frequency distribution of the additional currents induced in the ring. For these frequencies we have respectively $\chi_0 = 1$ and $\psi_0 = 1$:

$$\omega_{cr}^{(1)} = 2v/a(1-\beta^2)^{1/2}; \qquad \omega_{cr}^{(2)} = (c/a)(1-\beta^2)^{1/2}. \tag{29}$$

In the nonrelativistic limit $\beta \ll 1$ we always have $\omega_{cr}^{(1)} \ll \omega_{cr}^{(2)}$, therefore the denominator in (28) is equal to l; if, moreover, $\omega_{cr}^{(1)}$ is appreciably lower than the characteristic frequencies of the medium, then $\epsilon \approx \epsilon(0)$ and, consequently:

$$I_\omega^{(1)} = I_\infty \beta(\varepsilon(0)\mu - 1) \frac{a}{2cl} I_1\Big(\frac{a|\omega|}{2v}\Big) K_1\Big(\frac{a|\omega|}{2v}\Big). \tag{30}$$

From the asymptotic behavior of the Bessel functions I_1 and K_1 it follows that frequencies lower than $\omega_{cr}^{(1)}$ are excited in the ring with approximately the same amplitude, and $I_0/I_\infty = (\beta a\ 4cl) \cdot (\epsilon(0)\mu - 1)$ (cf. Fig. 2); the amplitudes of the frequencies $\omega > \omega_{cr}^{(1)}$ fall off as $1/\omega$.

A frequency distribution of the same form, but with a different numerical factor

$$I_\omega^{(1)} = I_\infty \frac{a}{2vl} I_1\Big(\frac{a|\omega|}{2v}\Big) K_1\Big(\frac{a|\omega|}{2v}\Big) \tag{31}$$

can occur under certain conditions also in the case of metals and plasma. If one can assume that in metals the number ν of collisions between electrons and ions considerably exceeds $\omega_{cr}^{(1)}$, then one can set $\epsilon - 1 = -\omega_0^2/i\nu\omega$; if, moreover, $\omega_{cr}^{(1)} \ll \omega_0^2\beta^2/\gamma$, and, consequently, $\beta \gg 2\gamma\lambda\!\!\!^-_0/\omega_0 a$, then in (28) $(\zeta_\lambda - 1)/(\zeta_\lambda + 1) \approx 1$. Consequently, under these assumptions the amplitude of the frequency distribution increases with decreasing velocity and does not decrease as in (30). The condition $\beta \gg 2\gamma\lambda\!\!\!^-_0/\omega_0 a$ in fact means, as can be easily seen, that the skin depth in the medium at a frequency $\omega_{cr}^{(1)}$ is considerably smaller than the radius of the ring a. If we can assume that the number ν of collisions between electrons and ions in the plasma is considerably smaller than the critical frequency $\omega_{cr}^{(1)}$, the distribution (31) will hold for most of the excited frequencies provided only the plasma oscillation frequency ω_0 is small in comparison with the critical frequency $\omega_{cr}^{(1)}$. However, if $\nu \gg \omega_{cr}^{(1)}$ and $\omega_{cr}^{(1)} \gg \beta^2 4\pi\sigma$, then for most frequencies the distribution has the form:

$$I_\omega^{(1)} = I_\infty \frac{a\pi}{cl} \frac{i\beta\sigma}{2\omega} I_1\Big(\frac{a|\omega|}{2v}\Big) K_1\Big(\frac{a|\omega|}{2v}\Big); \quad \sigma = \frac{\omega_0^2}{4\pi\nu}. \tag{32}$$

If the conditions $\nu \ll \omega_{cr}^{(1)}$ and $\omega_{cr}^{(1)} \gg \omega_0$ are satisfied, we obtain for most frequencies

$$I_\omega^{(1)} = I_\infty \frac{a\omega_0^2}{2vl} \frac{1}{\omega^2} I_1\Big(\frac{a|\omega|}{2v}\Big) K_1\Big(\frac{a|\omega|}{2v}\Big). \tag{33}$$

Thus, for $\nu \gg \omega_{cr}^{(1)}$ and $\omega_{cr}^{(1)} \gg \beta^2 4\pi\sigma$ only the frequencies $\omega \ll \beta^2 4\pi\sigma$ are excited with a constant amplitude, while for $\nu \ll \omega_{cr}^{(1)}$, $\omega_{cr}^{(1)} \gg \omega_0$ only the frequencies $\omega \ll \omega_0$ are so excited.

With the aid of the foregoing frequency distributions we can find the dependence of the current on the time, or, which is the same thing, on the instantaneous distance $r = -vt$ to the medium. The current and its time derivative are expressed in terms of the integral

$$\int\limits_{-\infty}^{\infty} e^{-i\omega t} d\omega K_1(a|\omega|/2v) I_1(a|\omega|/2v) = (v/2\pi a^2) M, \tag{34}$$

where M is the coefficient of mutual inductance between the current and its image in the case of a perfectly conducting wall. The limiting expressions for the current defined by (30) and (31) may be obtained by utilizing the limiting values for the coefficient of mutual inductance:

$$\frac{I^{(1)}(t)}{I_\infty} = \begin{cases} \frac{\pi}{16 l} \frac{a^3}{r^3} \text{ for } r \gg a \text{ and } \frac{v}{r} \ll \nu,\ r \gg \frac{c^2}{4\pi\sigma v} \text{ or } \frac{v}{r} \gg \nu,\ r \gg \frac{c}{\omega_0} \\ \frac{1}{l}\Big\{\ln\frac{4a}{r} - 2\Big\} \text{ for } r \ll a \text{ and } \frac{v}{r} \ll \nu,\ r \gg \frac{c^2}{4\pi\sigma v} \text{ or } \frac{v}{r} \gg \nu,\ r \gg \frac{c}{\omega_0}. \end{cases} \tag{35}$$

For $r \ll c^2/4\pi\sigma v$, $v/r \ll \nu$ the current for the distribution (32) can be obtained by integrating (34)

$$\frac{I^{(1)}(t)}{I_\infty} = \frac{\pi\beta\sigma a}{cl}\begin{cases} \frac{\pi}{16}\left(\frac{a}{r}\right)^2 & \text{for } r \gg a \\ 1 - \frac{r}{a}\left(\ln\frac{4a}{r} + 1\right) & \text{for } r \ll a. \end{cases} \tag{36}$$

For $v/r \gg \nu$, $r \ll c/\omega_0$ we obtain

$$\frac{I^{(1)}(t)}{I_\infty} = \frac{\omega_0^2 a^2}{4c^2 l}\begin{cases} \pi a/8r & \text{for } r \gg a \\ 1/3 & \text{for } r \ll a. \end{cases} \tag{37}$$

Let us now examine the question as to the manner in which the frequency distribution of the currents in the ring will change as its velocity increases right up to ultrarelativistic values. As $\beta \to 1$ the critical frequency $\omega_{cr}^{(1)}$ increases, while $\omega_{cr}^{(2)}$ decreases. There exists a critical velocity at which these two frequencies become equal, $\beta_{cr} = \sqrt{2} - 1$; since $1/\sqrt{1-\beta^2}$ is not large at this point, the speed β_{cr} is not ultrarelativistic. In the extreme ultrarelativistic case, when not only $\omega_{cr}^{(1)}$ $\omega_{cr}^{(2)}$ but also $\omega_{cr}^{(1)} \gg \gamma\omega_{cr}^{(2)}$, $\gamma = \delta/a$, we can substitute into (27) the value of $K(\lambda)$ from (21):

$$I_\omega = I_\infty \frac{\zeta - 1}{\zeta + 1}\frac{a\delta}{2ic^2}\,\omega\, I_1\left(\frac{a\,|\,\omega\,|\,(1-\beta^2)^{1/2}}{2c}\right) \times K_1\left(\frac{a\,|\,\omega\,|\,(1-\beta^2)^{1/2}}{2c}\right) \quad \text{for } \omega \ll \frac{\omega_{\text{кр}}^{(1)}}{\gamma}, \tag{38}$$

$$I_\omega = I_\infty \frac{\zeta - 1}{\zeta + 1}\frac{2}{i(1-\beta^2)\,\omega} \quad \text{for } \omega \gg \frac{\omega_{\text{кр}}^{(1)}}{\gamma}. \tag{39}$$

If $\omega_{cr}^{(1)}/\gamma \ll \omega^{(0)}$, on the order of the characteristic resonance frequencies of the medium, the frequency distribution has the approximate form shown in Fig. 3, since the factor $(\zeta - 1)/(\zeta + 1)$ can be taken to be independent of the frequency, and

$$\frac{I_{\max}}{I_\infty} = \frac{\zeta - 1}{\zeta + 1}\frac{\delta}{c(1-\beta^2)^{1/2}}.$$

As β increases $\omega_{cr}^{(1)}$ also increases, i.e., successively higher harmonics are excited in the ring. Finally, at sufficiently large β the quantity $\omega_{cr}^{(1)}/\gamma$ becomes of order $\omega^{(0)}$; the frequency dependence of the factor $(\zeta - 1)/(\zeta + 1)$ becomes important, for it produces peaks in the frequency

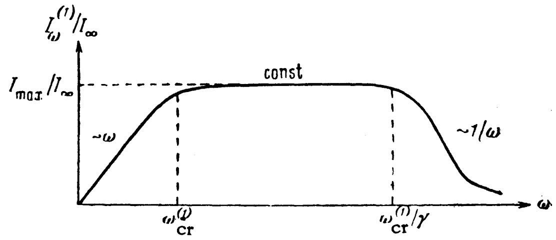

FIG. 3

distribution at frequencies close to the characteristic frequencies of the medium.

In the case of plasma or of a metal the characteristic resonance frequency must be set equal to zero, since the induced electric field is a purely rotational one, and the frequency dependence of $(\zeta - 1)/(\zeta + 1)$ is therefore always important. In the following special cases the factor $(\zeta - 1)/(\zeta + 1) = S$ has the following form:

$S = 1$ for $\omega_{cr}^{(1)}/\gamma \ll \nu$ and $\omega_{cr}^{(2)}/\gamma \ll 4\pi\sigma$;

$S = \pi\sigma i/\omega$ for $\omega_{cr}^{(1)}/\gamma \ll \nu$ and $\omega_{cr}^{(1)}/\gamma \gg 4\pi\sigma$;

$S = 1$ for $\omega_{cr}^{(1)}/\gamma \gg \nu$ and $\omega_{cr}^{(1)}/\gamma \ll \omega_0$;

$S = -\omega_0^2/4\omega^2$ for $\omega_{cr}^{(1)} \gg \nu\gamma$ and $\omega_{cr}^{(1)} \gg \omega_0\gamma$.

The form of the frequency distribution in these special cases may be easily found from (38) and (39).

From the frequency distributions we can obtain the expression for the current. For a number of limiting cases the analytic expressions for the dependence $I^{(1)}(t)/I_\infty$ of the current on the time are collected in the last column of the table. It is assumed here that $1/(1-\beta^2)^{1/2} \gg 1$, while ϵ is taken to be of the form: $\epsilon = 1 - \omega_0^2/(\omega^2 + i\nu\omega)$, $\sigma = \omega_0^2/4\pi\nu$.

5. THE CASE OF LOW VELOCITIES

In the case of low velocities the problem can be solved in much greater detail. Here we investigate only the case of incidence on a plasma characterized by low losses, i.e., $\epsilon = 1 - \omega_0^2/\omega^2$; $\beta \ll 1$. When these two conditions are satisfied the integral equation for the Fourier components of the current can be solved exactly, since

$$\zeta_\lambda = \{1 - [2\beta\omega_0/(\lambda - \omega)]^2\}^{1/2}; \quad \chi = (a/2v)\,|\,\lambda - \omega\,|$$

and therefore the kernel of the integral equation depends only on the difference $\lambda - \omega$; $K(\lambda, \omega) = K(|\lambda - \omega|)$; moreover, in the case under consideration we may also set $K(\lambda) = K(0) = L/4\pi a^2$. On multiplying (15) by $\exp\{-i\omega t\}$ and integrating with respect to t, we obtain

$$I(t) = I(r) = I_\infty/(1 - M(r)/L); \quad r = -vt, \tag{40}$$

where after integrating with respect to ω (by deforming the path of integration into the upper half-plane of complex ω) we can write for $M(r)$

Dependence of the current on the time

$c/r \ll \nu$	$r \gg c/4\pi\sigma$	$r \gg a_1$*	$(3\pi\delta/16a)(1-\beta^2)(a/r)^4$
,,	,,	$r \ll a_1$	$\delta/r\,(1-\beta^2)^{1/2}$
,,	$r \ll c/4\pi\sigma$	$r \gg a_1$	$(\delta\pi^2\sigma/8c)(1-\beta^2)(a/r)^3$
,,	,,	$r \ll a_1$	$\dfrac{\delta 2\pi\sigma}{c(1-\beta^2)^{1/2}}\left\{\ln\dfrac{4a(1-\beta^2)^{1/2}}{r}-2\right\}$
$c/r \gg \nu$	$r \gg c/\omega_0$	$r \gg a_1$	$(3\pi\sigma/16a)(1-\beta^2)(a/r)^4$
,,	,,	$r \ll a_1$	$\delta/r\,(1-\beta^2)^{1/2}$
,,	$r \ll c/\omega_0$	$r \gg a_1$	$(\delta\pi a\omega^2{}_0/32c^2)(1-\beta^2)(a/r)^2$
,,	,,	$r \ll a_1$	$\delta a\pi\omega_0{}^2/4c^2$

* $a_1 = a(1-\beta^2)^{1/2}$

$$M(r) = 4\pi^2a^2\int_0^\infty \frac{\{1+(\omega_0/kc)^2\}^{1/2}-1}{\{1+(\omega_0/kc)^2\}^{1/2}+1} J_1^2(ka)\,e^{-2kr}\,dk. \quad (41)$$

Thus, the magnetic flux linking the current circuit and produced by the medium has the form

$$\Phi_{cr} = IM/c.$$

Thus, in spite of the fact that the image current in the plasma has a different magnitude for each frequency, the magnetic flux produced by the image currents in the plasma is such as if there exists in the non superconducting plasma an image current I which is numerically equal and oppositely directed to the incident current at a given distance from the plasma (just as in the case of a superconducting wall), but the coefficient of mutual inductance is changed and depends on the frequency of plasma oscillations ω_0.

When $r \gg c/\omega_0$ the quantity M is simply the coefficient of mutual inductance of two circular currents. This coefficient can be expressed in terms of elliptic integrals whose asymptotic values are well known. When $r \ll c/\omega_0$ the coefficient M approaches the constant value

$$M = \frac{\pi a\omega_0^2a^2}{c^2}\begin{cases}\pi a/8r, & r \gg a,\\ 4/3, & r \ll a.\end{cases} \quad (42)$$

Thus, the current in the ring does not increase indefinitely, but reaches a certain maximum. Therefore the magnetic energy which can be stored in the ring cannot exceed

$$U_{\max} = \Phi^2/2(L-M_{\max}) - \Phi^2/2L \quad (43)$$

or, since $\Phi^2 = \text{const} = L^2I_\infty^2/c^2$,

$$U_{\max} = LI_\infty^2M_{\max}/2c^2(L-M_{\max}). \quad (44)$$

The last equation also means that the ring will not be reflected from the plasma for all energies of translational motion of the ring. This energy must be less than $U_{\max}$. In the two limiting cases $a \gg c/\omega_0$ and $a \ll c/\omega_0$ the conditions under which reflection occurs can be written in the form

$$mv^2 \leqslant \frac{|I_\infty|^2}{c^2}\,\frac{4\pi a\{\ln(4a\omega_0/c)-2\}\,l}{\ln(2c/\delta\omega_0)}, \quad a \gg \frac{c}{\omega_0}, \quad (45)$$

$$mv^2c^2/I_\infty^2 n \leqslant (16\pi/3)\,a^3r_0; \quad a \ll c/\omega_0; \quad r_0 = e^2/\mu c^2; \quad (46)$$

where m is the mass of the ring, $n = \mu\omega_0^2/4\pi e^2$, μ is the electron mass. If the density of the plasma is so great that $\delta \gg c/\omega_0$, then for any arbitrary values of r the plasma can be considered to be superconducting.

The point at which reflection occurs can be found by equating $U_{\max} = mv^2/2$. The force of repulsion between the ring and the plasma can be obtained both directly from the expression for the fields, and by differentiating V with respect to $r = -vt$

$$F_z = -\frac{I^2}{2c^2}\frac{\partial M}{\partial r} \approx \frac{I_\infty^2}{c^2}\,\frac{\pi^2a^2\omega_0^2}{16c^2}\left(\frac{a}{r}\right)^2 \text{ for } r \ll \frac{c}{\omega_0};\ r \gg a, \quad (47)$$

$$F_z = \frac{I_\infty^2}{2c^2}\,\frac{\pi^2a^2\omega_0^2}{c^2} \text{ for } r \ll \frac{c}{\omega_0};\ r \ll a. \quad (48)$$

The force that stretches the ring can also be found in a similar manner:

$$F_\rho = -\frac{I^2}{2c^2}\frac{\partial M}{\partial a} = \frac{I_\infty^2}{c^2}\,\frac{\pi^2}{4}\,\frac{\omega_0^2a^2}{c^2}\,\frac{a}{r} \text{ for } r \ll \frac{c}{\omega_0};\ r \gg a. \quad (49)$$

$$F_\rho = \frac{I_\infty^2}{c^2}\,2\pi a^2\frac{\omega_0^2}{c^2} \text{ for } r \ll \frac{c}{\omega_0};\ r \ll a. \quad (50)$$

6. FORCES AT HIGH VELOCITIES FOR CONSTANT CURRENT

We consider here only the case of plasma with low damping, on the assumption that the current is constant.

The force repelling the ring can be obtained from (7). After integration with respect to κ we obtain

$$F_z = \frac{I_\infty}{c} 2\pi a \frac{\partial A^{(1)}}{\partial z}\Big|_{\substack{z=-vt\\ \rho=a}} = \frac{(2\pi a)^2 I_\infty^2}{c^2(1-\beta^2)} \int_0^\infty \frac{\{1+\beta^2[\varepsilon(ikv/(1-\beta^2)^{1/2})-1]\}^{1/2}-1}{\{1+\beta^2[\varepsilon(ikv/(1-\beta^2)^{1/2})-1]\}^{1/2}+1} k \exp\left\{-\frac{2kr}{(1-\beta^2)^{1/2}}\right\} J_1^2(ka)\, dk. \quad (51)$$

When $r = -vt \gg a(1-\beta^2)^{1/2}$, if we restrict ourselves to the first term in the series expansion of a Bessel function, we obtain

$$F_z = \frac{\pi^2 a^4 I_\infty^2}{c^2}(1-\beta^2) \int_0^\infty \frac{\{1+\beta^2[\varepsilon(ikv)-1]\}^{1/2}-1}{\{1+\beta^2(\varepsilon(ikv)-1)\}^{1/2}+1} k^3 e^{-2kr} dk. \quad (52)$$

In the limiting cases we have

$$F_z = (3\pi^2 I_\infty^2/4c^2)(1-\beta^2)a^4/r^4 \quad \text{for } r \gg c/\omega_0, \quad (53)$$

$$F_z = (\pi^2 I_\infty^2/2c^2)\,\omega_0^2 a^2/c^2 \quad \text{for } r \ll c/\omega_0. \quad (54)$$

When $r \ll a\sqrt{1-\beta^2}$ the use of the asymptotic expansion for J_1 leads to the results given in reference 1.

From (7) we obtain the force that stretches the ring:

$$F_\rho = -\frac{2\pi a}{\rho}\frac{I_\infty}{c}\frac{\partial}{\partial\rho}(\rho A^{(1)})$$

$$= \frac{(2\pi a)^2 I_\infty^2}{c^2}(1-\beta^2)^{1/2} \int_0^\infty \frac{\{1+\beta^2(\varepsilon(ikv)-1)\}^{1/2}-1}{\{1+\beta^2(\varepsilon(ikv)-1)\}^{1/2}+1} k e^{-2kr} dk \quad (55)$$

$$\times J_0(ka(1-\beta^2)^{1/2})\, J_1(ka(1-\beta^2)^{1/2}).$$

In limiting cases we obtain the following expressions

$$F_\rho = \pi^2 a^3 I_\infty^2 (1-\beta^2)/4c^2 r^2 \text{ for } r \gg a\sqrt{1-\beta^2} \text{ and } r \gg c/\omega_0, \quad (56)$$

$$F_\rho = 4\pi I_\infty^2/c^2\sqrt{1-\beta^2} \text{ for } r \ll a\sqrt{1-\beta^2} \text{ and } r \gg c/\omega_0. \quad (57)$$

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