THE OSCILLATIONS OF A DEGENERATE ELECTRON FLUID

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The oscillations of an electron fluid are treated on the basis of the Landau theory of the Fermi fluid.[1] The oscillations reduce to longitudinal plasma waves, transverse electromagnetic waves, zeroth sound, and spin waves.

1. In treating the oscillations of a degenerate electron fluid on the basis of the usual theory of such a fluid one can deal with longitudinal plasma waves, and also with transverse electromagnetic waves.[2-4] The treatment includes the interaction between the electrons caused by the electromagnetic field. Such a treatment, however, is valid only under the condition that the interaction between the electrons is small in comparison with their kinetic energy. In the opposite case the correlation of the particles, and in particular the exchange correlation of the electrons, leads to important changes in the kinetic equations.[5]

The general theory of the degenerate Fermi fluid has been examined by Landau,[1] and the use of such a theory for the electron fluid has been discussed in an earlier paper.[6] We examine below the oscillations of a degenerate electron fluid on the basis of the Landau theory. The analogous problem for the uncharged degenerate Fermi fluid has been discussed in references 7 to 9.

2. For our study of the oscillations of a degenerate electron fluid we use the kinetic equation for the quasi-particles, which has the form[6]

$$\frac{\partial n}{\partial t} - \frac{i}{\hbar}[\varepsilon, n] + \frac{1}{2}\left(\frac{\partial \varepsilon}{\partial \mathbf{p}}\frac{\partial n}{\partial \mathbf{r}} + \frac{\partial n}{\partial \mathbf{r}}\frac{\partial \varepsilon}{\partial \mathbf{p}}\right) - \frac{1}{2}\left(\frac{\partial \varepsilon}{\partial \mathbf{r}}\frac{\partial n}{\partial \mathbf{p}} + \frac{\partial n}{\partial \mathbf{p}}\frac{\partial \varepsilon}{\partial \mathbf{r}}\right) + e\mathbf{E}\frac{\partial n}{\partial \mathbf{p}} + \frac{1}{2}\frac{e}{c}\left(\left[\frac{\partial \varepsilon}{\partial \mathbf{p}} \times \mathbf{H}\right]\frac{\partial n}{\partial \mathbf{p}} + \frac{\partial n}{\partial \mathbf{p}}\left[\frac{\partial \varepsilon}{\partial \mathbf{p}} \times \mathbf{H}\right]\right) = \hat{J}, \tag{1}$$

where n is the density matrix, ϵ is the Hamiltonian function of the quasiparticles, and $[\epsilon, n]$ is the commutator of these quantities; these are all matrices in the spin space. $\hat{J}$ is the collision operator. The field strengths obey the Maxwell equations:

$$\operatorname{div}\mathbf{E} = 4\pi e\,\mathrm{Sp}_\sigma\int d\mathbf{p}'\delta n,$$

$$\operatorname{curl}\mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi e}{c}\mathrm{Sp}_\sigma\int d\mathbf{p}\,\frac{\partial \varepsilon}{\partial \mathbf{p}}n + 4\pi\beta\,\mathrm{rot}\,\mathrm{Sp}_\sigma\int d\mathbf{p}\hat{\sigma}n,$$

$$\operatorname{div}\mathbf{H} = 0, \operatorname{curl}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = 0, \tag{2}$$

where σ are the Pauli spin matrices ($\sigma_i\sigma_k + \sigma_k\sigma_i = 2\delta_{ik}$), and δn is the nonequilibrium contribution to the density matrix. For small deviations from the equilibrium state

$$\delta\varepsilon(\mathbf{p}, \mathbf{r}) = -\beta(\hat{\sigma}\cdot\mathbf{H}) + \mathrm{Sp}_{\sigma'}\int\{\varphi(\mathbf{p}, \mathbf{p}') + (\hat{\sigma}\cdot\hat{\sigma}')\psi(\mathbf{p}, \mathbf{p}')\}\,\delta n(\mathbf{p}', \mathbf{r})\,d\mathbf{p}'. \tag{3}$$

Here the spin-orbit interactions have been neglected.

It is convenient to use instead of n the distribution function $f = \mathrm{Sp}\, n$ of the particles in the phase space of the coordinates and momenta, and the vector function $\boldsymbol{\sigma} = \mathrm{Sp}\,\hat{\sigma}n$ of the spin density in the phase space.

Then (cf. reference 7), recalling that $\epsilon_{mn} = \delta_{mn}\epsilon_1 + \hat{\sigma}_{mn}\epsilon_2$, we have:

$$\frac{\partial f}{\partial t} + \frac{\partial \varepsilon_1}{\partial \mathbf{p}}\frac{\partial f}{\partial \mathbf{r}} - \frac{\partial \varepsilon_1}{\partial \mathbf{r}}\frac{\partial f}{\partial \mathbf{p}} + \frac{\partial \varepsilon_2}{\partial p_j}\frac{\partial \boldsymbol{\sigma}}{\partial r_j} - \frac{\partial \varepsilon_2}{\partial r_j}\frac{\partial \boldsymbol{\sigma}}{\partial p_j} + e\mathbf{E}\frac{\partial f}{\partial \mathbf{p}} + \frac{e}{c}\left[\frac{\partial \varepsilon_1}{\partial \mathbf{p}} \times \mathbf{H}\right]\frac{\partial f}{\partial \mathbf{p}} + \frac{e}{c}\left[\frac{\partial \varepsilon_{2j}}{\partial p} \times \mathbf{H}\right]\frac{\partial \sigma_j}{\partial \mathbf{p}} = J, \tag{4}$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} + \left(\frac{\partial \varepsilon_1}{\partial \mathbf{p}}\cdot\frac{\partial}{\partial \mathbf{r}}\right)\boldsymbol{\sigma} - \left(\frac{\partial \varepsilon_1}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{p}}\right)\boldsymbol{\sigma} + \frac{2}{\hbar}[\boldsymbol{\varepsilon}_2\times\boldsymbol{\sigma}] + \left(\frac{\partial f}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{p}}\right)\boldsymbol{\varepsilon}_3 - \left(\frac{\partial f}{\partial \mathbf{p}}\cdot\frac{\partial}{\partial \mathbf{r}}\right)\boldsymbol{\varepsilon}_2 + e\left(\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{p}}\right)\boldsymbol{\sigma} + \frac{e}{c}\left(\left[\frac{\partial \varepsilon_1}{\partial \mathbf{p}} \times \mathbf{H}\right]\frac{\partial}{\partial \mathbf{p}}\right)\boldsymbol{\sigma} - \frac{e}{c}\left(\left[\frac{\partial f}{\partial \mathbf{p}} \times \mathbf{H}\right]\frac{\partial}{\partial \mathbf{p}}\right)\boldsymbol{\varepsilon}_2 = \mathbf{J}. \tag{5}$$

In the equilibrium state, on the assumption that we can expand in powers of the constant field H_0 and keep only the first-order terms, we have:

$$\boldsymbol{\varepsilon}_{20} = -\beta\mathbf{H}_0 + \int d\mathbf{p}'\psi(\mathbf{p}, \mathbf{p}')\,\boldsymbol{\sigma}_0(\mathbf{p}') \tag{6}$$

In virtue of the fact that

$$\boldsymbol{\sigma}_0 = \frac{\partial f_0}{\partial \varepsilon}\boldsymbol{\varepsilon}_{20}, \tag{7}$$

Equation (6) is an integral equation for ϵ_{20}. It is easy to see that the solution of this equation has the form*

*It is assumed throughout that the Fermi surface has the shape of a sphere.

$$\varepsilon_{20} = -\gamma(p)\,\mathbf{H}_0,$$

where

$$\gamma(p) = \beta\left\{1 + \frac{2}{(2\pi\hbar)^3}\frac{p_0^2}{v_0}\int do'\,[\psi(\mathbf{p}_0, \mathbf{p}_0') - \psi(\mathbf{p}, \mathbf{p}_0')]\right\} \times \left\{1 + \frac{2}{(2\pi\hbar)^3}\frac{p_0^2}{v_0}\int do'\psi(\mathbf{p}_0, \mathbf{p}_0')\right\}^{-1}. \tag{8}$$

For a momentum equal to the limiting momentum p_0 of the Fermi distribution Eq. (8) gives an expression agreeing with that obtained by Landau.[1]

The oscillations of the degenerate electron fluid, regarded as small deviations from the equilibrium state, can be described by kinetic equations obtained from Eqs. (4) and (5) by neglect of terms of the order of H_0^2:

$$\frac{\partial\delta f}{\partial t} + \left(\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\right)\left(\delta f - \frac{\partial f_0}{\partial\varepsilon}\delta\varepsilon_1\right) + e\mathbf{E}\frac{\partial f_0}{\partial\mathbf{p}} + \frac{e}{c}[\mathbf{v}\times\mathbf{H}_0]\frac{\partial}{\partial\mathbf{p}}\left(\delta f - \frac{\partial f_0}{\partial\varepsilon}\delta\varepsilon_1\right) = J; \tag{4'}$$

$$\frac{\partial\delta\sigma}{\partial t} + \left(\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\right)\left(\delta\sigma - \frac{\partial f_0}{\partial\varepsilon}\delta\varepsilon_2\right) - \frac{2\gamma(p)}{\hbar}\left[\mathbf{H}_0, \delta\sigma - \frac{\partial f_0}{\partial\varepsilon}\delta\boldsymbol{\varepsilon}_2\right] + \frac{e}{c}\left([\mathbf{v}\times\mathbf{H}_0]\frac{\partial}{\partial\mathbf{p}}\right)\left(\delta\sigma - \frac{\partial f_0}{\partial\varepsilon}\delta\varepsilon_2\right) = \mathbf{J}. \tag{5'}$$

Here*

$$\mathbf{v} = \partial\varepsilon_1/\partial\mathbf{p},$$

$$\delta\varepsilon_1(\mathbf{p}, \mathbf{r}) = \int\varphi(\mathbf{p}, \mathbf{p}')\,\delta f(\mathbf{p}', \mathbf{r})\,d\mathbf{p}',$$

$$\delta\varepsilon_2(\mathbf{p}, \mathbf{r}) = -\beta\mathbf{H} + \int\psi(\mathbf{p}, \mathbf{p}')\,\delta\sigma(\mathbf{p}', \mathbf{r})\,d\mathbf{p}'.$$

3. Of the various types, we first single out the oscillations that are not accompanied by changes of the spin distribution function. We then get the following equation for the nonequilibrium contribution δf to the distribution function, which describes such oscillations:

$$-i\omega\delta f + \left(i\,\mathbf{k}\cdot\mathbf{v} + \frac{e}{c}[\mathbf{v}\times\mathbf{H}_0]\frac{\partial}{\partial\mathbf{p}}\right)\left(\delta f - \frac{\partial f_0}{\partial\varepsilon}\delta\varepsilon_1\right) + \frac{\partial f_0}{\partial\varepsilon}\frac{4\pi i e^2c^2}{\omega^2 - c^2k^2}\left\{\mathbf{k}\cdot\mathbf{v}\int d\mathbf{p}'\delta f' - \frac{\omega}{c^2}\int d\mathbf{p}'\mathbf{v}\cdot\mathbf{v}'\left(\delta f' - \frac{\partial f_0}{\partial\varepsilon'}\delta\varepsilon_1'\right)\right\}$$
$$= -\frac{1}{\tau}\left\{\delta f - \frac{\partial f_0}{\partial\varepsilon}\delta\varepsilon_1 - \int\frac{do'}{4\pi}\left(\delta f' - \delta\varepsilon_1'\frac{\partial f_0}{\partial\varepsilon}\right)\right\}. \tag{9}$$

Here it is assumed that the dependence of δf on the time and the coordinates has the form $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$, and in addition it is assumed that there is no constant electric field.

When we represent δf in the form of a series of spherical harmonics (with the polar axis directed along H_0),

$$\delta f = \frac{\partial f_0}{\partial\varepsilon}\sum_{n, m}F_{n, m}Y_n^m(\theta, \varphi),$$

$$Y_n^m(\theta, \varphi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\,e^{im\varphi}P_n^m(\cos\theta) \tag{10}$$

and set

$$\frac{2}{(2\pi\hbar)^3}\frac{4\pi p_0^2}{v_0}\varphi = \Phi = \sum_l A_l P_l(\cos\chi), \tag{11}$$

where χ is the angle between the vectors $\mathbf{p}$ and $\mathbf{p}'$, we get the following dispersion equation for the determination of the frequencies of the characteristic oscillations of the electron fluid:

$$\Bigg|\delta_{nn'}\delta_{mm'} - \frac{A_{n'}}{2n'+1}\Bigg\{\left(-im'\Omega + \frac{1}{\tau}\right)N_{nn'}^{mm'} - ikv_0\Bigg[\left(\sqrt{\frac{(n'+1)^2 - m'^2}{4(n'+1)^2 - 1}}N_{n,n'+1}^{mm'} + \sqrt{\frac{n'^2 - m'^2}{4n'^2 - 1}}N_{n,n'-1}^{m,m'}\right)\cos\theta_k$$
$$+ \frac{1}{2}\sin\theta_k e^{-i\varphi_k}\left(\sqrt{\frac{(n'-m'-1)(n'-m')}{4n'^2 - 1}}N_{n,n'-1}^{m,m'+1} - \sqrt{\frac{(n'+m'+1)(n'+m'+2)}{4(n'+1)^2 - 1}}N_{n,n'+1}^{m,m'+1}\right)$$
$$+ \frac{1}{2}\sin\theta_k e^{i\varphi_k}\left(\sqrt{\frac{(n'-m'+1)(n'-m'+2)}{4(n'+1)^2 - 1}}N_{n,n'+1}^{m,m'-1} - \sqrt{\frac{(n'+m')(n'+m'-1)}{4n'^2 - 1}}N_{n,n'-1}^{m,m'-1}\right)\Bigg]\Bigg\} - \frac{1+A_0}{\tau}\delta_{n'0}\delta_{m'0}N_{nn'}^{mm'}$$
$$+ \frac{4\pi ie^2c^2}{\omega^2 - c^2k^2}\frac{8\pi p_0^2}{(2\pi\hbar)^3}\Bigg\{\frac{k}{\sqrt{3}}\delta_{n'0}\delta_{m'0}\times\left[\cos\theta_k N_{n1}^{m0} + \frac{e^{i\varphi_k}}{2}\sin\theta_k N_{n,1}^{m,-1} + \frac{e^{-i\varphi_k}}{2}\sin\theta_k N_{n1}^{m1}\right]$$
$$+ \frac{\omega v_0}{3}\left(1 + \frac{A_1}{3}\right)\delta_{n'1}(\delta_{m'0} + \delta_{m'1} + \delta_{m',-1})N_{nn'}^{mm'}\Bigg\}\Bigg| = 0, \tag{12}$$

where $\Omega = evH_0/cp$, θ_k and φ_k are the angles fixing the direction of the vector $\mathbf{k}$, and, finally,

$$N_{nn'}^{mm'} = \int do\,Y_n^m(\theta, \varphi) \times \left\{\exp\left[-\frac{2\pi i}{\Omega}\left(\omega + \frac{i}{\tau} - kv\cos\theta_k\cos\theta\right)\right] - 1\right\}^{-1}\frac{1}{\Omega}\int_\varphi^{2\pi+\varphi}d\varphi'$$
$$\times\exp\left\{-\frac{1}{\Omega}\left(\left[-i\omega + \frac{1}{\tau} + ikv_0\cos\theta_k\cos\theta\right](\varphi - \varphi') - ikv_0\sin\theta_k\sin\theta\,[\sin(\varphi_k - \varphi) - \sin(\varphi_k - \varphi')]\right)\right\}Y_{n'}^{m'}(\theta, \varphi'). \tag{13}$$

Equation (12) has a simple solution in the long-wave region, where the condition $kv_0 \ll \Omega_0$ holds. In this case we get*

$$\omega_{nm} = m\Omega\left(1 + \frac{A_n}{2n+1}\right) + O(k^2), \quad (1 < |m| \leqslant n). \tag{14}$$

The frequencies ω_{nm} correspond to definite spherical harmonics, which are approximately independent of each other. With increasing values of kv_0, however, the different harmonics are

*Hereafter $\mathbf{H}$ is the alternating magnetic field.

*In what follows we suppose τ sufficiently large and neglect the frequency of collisions. We note that the frequency corresponding to $n = 1$ is the resonance frequency of the cyclotron resonance in the region of the normal skin effect.

more strongly coupled to each other, and it is then impossible to find solutions like Eq. (14).

Let us analyze the simplest case of oscillations under the condition $H_0 = 0$. Then by orienting the polar axis parallel to **k** and taking $\tau \to \infty$, we get instead of Eq. (12)

$$\Big| \delta_{nn'}\delta_{mm'} + ikv_0 \frac{A_{n'}}{2n'+1}\left[\sqrt{\frac{(n'+1)^2-m'^2}{4(n'+1)^2-1}} N^{mm'}_{n,n'+1} + \sqrt{\frac{n'^2-m'^2}{4n'^2-1}} N^{mm'}_{n,n'-1}\right] + \frac{4\pi i e^2 c^2}{\omega^2 - c^2k^2}\frac{8\pi p_0^2}{(2\pi\hbar)^3}\left[\frac{k}{\sqrt{3}}\delta_{n'0}\delta_{m'0}N^{m0}_{n1} + \frac{\omega v_0}{3}\left(1+\frac{A_1}{3}\right)\delta_{n'1}(\delta_{m'0}+\delta_{m'1}+\delta_{m',-1})N^{mm'}_{nn'}\right]\Big| = 0, \quad (15)$$

and here

$$iN^{mm'}_{nn'} = \delta_{mm'}\int \frac{do Y^{m*}_n(\theta,\varphi) Y^{m'}_{n'}(\theta,\varphi)}{-\omega + kv\cos\theta}. \quad (16)$$

From this we see that the waves with different values of m are independent. Furthermore, the terms in Eq. (15) that arise from the charge of the electrons are important only for waves with $m \leq 1$ and play no part for the oscillations with $m > 1$. Therefore the theory of the zeroth sound in the electron fluid for waves with $m > 1$ is just like that for the case of He^3.[7,8] In particular, if for $i > 2$ all the A_i are zero, then the dispersion equation for the zeroth sound with $m = 2$, which determines the speed of sound $sv_0 = \omega/k$, has the form

$$\frac{4}{A_2} - \frac{1}{5} = -\frac{3}{2}(s^2-1)\left\{\frac{1}{3} + (s^2-1)\left[1 - \frac{s}{2}\ln\frac{s+1}{s-1}\right]\right\}. \quad (17)$$

The case $m = 0$ corresponds to the plasma oscillations considered in reference 10. For them let us here look in more detail into the case in which only A_1 is different from zero. In this case we get from Eq. (15):

$$1 + \frac{A_1}{3} + \left\{A_1 s^2 + \frac{32\pi^2 e^2 p_0^2}{k^2 v_0 3(2\pi\hbar)^3}\left(1+\frac{A_1}{3}\right)\right\}\left[1 - \frac{s}{2}\ln\frac{s+1}{s-1}\right] = 0. \quad (18)$$

In the region $\omega \gg kv_0$ we get from this the dispersion equation of the plasma oscillations:

$$\omega^2 = \omega_0^2 + \frac{3}{5}\left(1+\frac{A_1}{3}\right)k^2v_0^2, \quad \omega_0^2 = \frac{32\pi^2 e^2 p_0^2 v_0}{3(2\pi\hbar)^3}\left(1+\frac{A_1}{3}\right). \quad (19)$$

It is important to consider the question of the possibility of undamped longitudinal oscillations with frequency much smaller than the plasma frequency ω_0. For this purpose we can write Eq. (18) approximately in the form

$$\frac{k^2v_0^2}{3\omega_0^2}\left(1+\frac{A_1}{3}\right) = \frac{s}{2}\ln\frac{s+1}{s-1} - 1 \equiv \eta(s). \quad (20)$$

Because of the fact that the left side is small and the right side can be small only in the region $s \gg 1$, where it is $\sim (1/3s^2)$, Eq. (20) has no solutions with $\omega \ll \omega_0$. Here it is supposed throughout that A_1 is not too large in comparison with unity.

Let us now consider the transverse oscillations $m = 1$, again supposing that only A_1 is different from zero and that $\tau \to \infty$. Here we have from Eq. (15):

$$1 - \frac{1}{2}A_1\left\{\frac{1}{3} - (s^2-1)\eta(s)\right\} - \frac{3}{2}\frac{\omega_0^2}{\omega^2 - c^2k^2}\{1 - (s^2-1)\eta(s)\} = 0. \quad (21)$$

In the case in which the phase velocity considerably exceeds the speed of the electrons on the Fermi surface $(s \gg 1)$ we get $\omega^2 = \omega_0^2 + c^2k^2$, which corresponds to propagation of transverse electromagnetic waves in the electron fluid with the dielectric constant $\epsilon(\omega) = 1 - \omega_0^2/\omega^2$.

Let us now examine the possibility of propagation of waves with frequency much smaller than the plasma frequency (ω_0) and with phase velocity much smaller than the speed of light. Then Eq. (21) gives

$$(s^2-1)\eta(s) = 1 + \left(1+\frac{A_1}{3}\right)\left[\frac{3}{2}\frac{v_0^2}{c^2}\frac{\omega_0^2}{\omega^2}s^2 - \frac{A_1}{2}\right]^{-1}. \quad (22)$$

Because of the fact that the right member of Eq. (22) is positive and bounded $(\lesssim 0.3)$, undamped solutions are not always possible. In particular, there are no solutions for $\omega \ll (v_0/c)\omega_0$. The situation is particularly simple in the opposite case, in which the inequalities

$$\omega_0 \gg \omega \gg (v_0/c)\omega_0 A_1^{-1/2}, \quad (23)$$

hold and Eq. (22) takes the form

$$(s^2-1)\eta(s) = 1/3 - 2/A_1, \quad (24)$$

corresponding to oscillations of an uncharged fluid. Equation (24) has solutions only for $A_1 > 6$. Calculations of the coefficient A_1 for several metals,* carried out in reference 11, give $A_1 \lesssim 3$. Therefore we can suppose that for such metals a transverse zeroth sound can scarcely be possible. In principle, however, the possibility of the existence of solutions of Eq. (24) is of interest in connection with the possibility that in the range of frequencies given by the inequalities (23) there may be a band of transparency of the metal, with a real index of refraction.

If the constant magnetic field is not zero, and H_0 is parallel to k, the dispersion equation (12) can be put in the form:

*The quantity A given by Eq. (19) of reference 11 is connected with A_1 by the equation $A_1 = 3A$.

$$\Big|\delta_{nn'}\delta_{mm'} + i\frac{A_{n'}}{2n'+1}\Big\{m'\Omega N_{nn'}^{mm'} + kv_0\Big[\sqrt{\frac{(n'+1)^2-m'^2}{4(n'+1)^2-1}}N_{n,n'+1}^{mm'} + \sqrt{\frac{n'^2-m'^2}{4n'^2-1}}N_{n,n'-1}^{mm'}\Big]\Big\}$$
$$+\frac{4\pi i e^2c^2}{\omega^2-c^2k^2}\frac{8\pi p_0^2}{(2\pi\hbar)^3}\Big[\frac{k}{\sqrt{3}}\delta_{n'0}\delta_{m'0}N_{n,1}^{m0} + \frac{\omega v_0}{3}\Big(1+\frac{A_1}{3}\Big)\delta_{n'1} \times(\delta_{m'0}+\delta_{m'1}+\delta_{m'-1})N_{nn'}^{mm'}\Big]\Big| = 0, \qquad (25)$$

where

$$iN_{nn'}^{mm'} = \delta_{mm'}\int\frac{do Y_n^{m*}(\theta,\varphi)\,Y_{n'}^{m'}(\theta,\varphi)}{-\omega+kv_0\cos\theta+m\Omega}. \qquad (26)$$

To bring out the part played by the constant magnetic field let us consider oscillations corresponding to the conditions under which the dispersion relation (17) holds in the absence of a magnetic field. Then for the oscillations with $m = 2$ we get:

$$-\frac{1}{A_2} = \frac{1}{5} + \frac{3\omega}{8kv_0}\Big\{\Big(\frac{\omega-2\Omega}{kv_0}\Big)^3 - \frac{5}{3}\Big(\frac{\omega-2\Omega}{kv_0}\Big) + \frac{1}{2}\Big[\Big(\frac{\omega-2\Omega}{kv_0}\Big)^2-1\Big]^2\ln\Big|\frac{\omega-2\Omega-kv_0}{\omega-2\Omega+kv_0}\Big|\Big\}. \qquad (27)$$

In the long-wave region ($kv_0 \ll 2\Omega$) we have from this

$$\omega = 2\Omega\Big(1+\frac{A_2}{5}\Big)\Big\{1+\frac{5}{7A_2}\Big(\frac{kv_0}{2\Omega}\Big)^2\Big\}. \qquad (28)$$

In order for Eq. (27) to have undamped solutions, it is necessary that the inequality

$$|\omega-2\Omega| > |k|v_0 \qquad (29)$$

be satisfied. From this and Eq. (27) it follows that for negative A_2, when Eq. (17) has no solutions, Eq. (27) also will have no solutions for wavelengths smaller than a critical value, which is of the order of magnitude v_0/Ω. The same assertion also holds for cases less restricted than that in which only A_2 is different from zero.

4. Let us now consider oscillations of the spin distribution function. Here we turn first to the case in which there are changes of the component of the vector function σ parallel to the constant magnetic field H_0. Neglecting the collisions, we then have from Eq. (5′):

$$\frac{\partial\delta\sigma_z}{\partial t} + \Big(\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\Big)\Big(\delta\sigma_z - \frac{\partial f_0}{\partial\varepsilon}\delta\varepsilon_{2z}\Big) + \frac{e}{c}\Big([\mathbf{v}\times\mathbf{H}_0]\frac{\partial}{\partial\mathbf{p}}\Big)\Big(\delta\sigma_z - \frac{\partial f_0}{\partial\varepsilon}\delta\varepsilon_{2z}\Big) = 0. \qquad (30)$$

This equation is similar to Eq. (3), differing in the absence of terms coming from the alternating magnetic field, and also in the fact that $\delta\epsilon_{2Z}$ appears instead of $\delta\epsilon_1$ and δf is replaced by $\delta\sigma_Z$. Therefore it is easy to verify that Eq. (30) yields the following dispersion relation:

$$\Big|\delta_{nn'}\delta_{mm'} + i\frac{B_{n'}}{2n'+1}\Big\{m'\Omega N_{nn'}^{mm'} + kv_0\Big[\cos\theta_k\Big(\sqrt{\frac{(n'+1)^2-m'^2}{4(n'+1)^2-1}}N_{n,n'+1}^{mm'} + \sqrt{\frac{n'^2-m'^2}{4n'^2-1}}N_{n,n'-1}^{m,m'}\Big)$$
$$+\frac{1}{2}\sin\theta_k e^{-i\varphi_k}\Big(\sqrt{\frac{(n'-m'-1)(n'-m')}{4(n'+1)^2-1}}N_{n,n'-1}^{m,m'+1} - \sqrt{\frac{(n'+m'+1)(n'+m'+2)}{4(n'+1)^2-1}}N_{n,n'+1}^{m,m'+1}\Big)$$
$$+\frac{1}{2}\sin\theta_k e^{i\varphi_k}\Big(\sqrt{\frac{(n'-m'+1)(n'-m'+2)}{4(n'+1)^2-1}}N_{n,n'+1}^{m,m'-1} - \sqrt{\frac{(n'+m')(n'+m'-1)}{4n'^2-1}}N_{n,n'-1}^{m,m'-1}\Big)\Big]\Big\}\Big| = 0, \qquad (31)$$

where $N_{nn'}^{mm'}$ is given by Eq. (13) for $\tau \to \infty$, and the coefficients B_n are defined by the relation

$$\frac{2}{(2\pi\hbar)^3}\frac{4\pi p_0^2}{v_0}\psi = \Psi = \sum_n B_nP_n(\cos\chi). \qquad (32)$$

In the long-wave region we get from this a simple expression for the frequencies of the spin oscillations

$$\omega_{n,m} = m\Omega\Big(1+\frac{B_n}{2n+1}\Big) + O\Big(\frac{k^2v_0^2}{\Omega}\Big). \qquad (33)$$

In the region of shorter waves, for example in the case in which the wave vector is parallel to the magnetic field, the dispersion relation is like that obtained for the oscillations of the distribution function, with replacement of A_n by B_n [cf. e.g., Eqs. (27) to (29)].

Let us now turn to the case in which there are vibrations of the components of σ perpendicular to the constant magnetic field. Writing $\sigma^{(\pm)} = \sigma_X \pm i\sigma_y$, we get the following equation:

$$\frac{\partial}{\partial t}\delta\sigma^{(\pm)} + \Big\{\Big(\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\Big) + \frac{e}{c}\Big([\mathbf{v}\times\mathbf{H}_0]\frac{\partial}{\partial\mathbf{p}}\Big) \pm i\frac{2\gamma(p)}{\hbar}H_0\Big\} \times\Big(\delta\sigma^{(\pm)} - \frac{\partial f_0}{\partial\varepsilon}\delta\varepsilon_2^{(\pm)}\Big) = 0, \qquad (34)$$

which leads to a dispersion relation differing from Eq. (31) only by the replacement of $m'\Omega$ by $m'\Omega \pm \Omega_0$, where $\Omega_0 = 2\gamma(p_0)H_0/\hbar$. Therefore in the special case of long waves we have at once, as in Eq. (33):*

$$\omega_{m,n} = (\pm\Omega_0 + m\Omega)\Big(1+\frac{B_n}{2n+1}\Big) + O\Big(\frac{k^2v_0^2}{\Omega}\Big). \qquad (35)$$

In particular, for $m = 0$ we have $\omega_{00} = \pm 2\beta H_0/\hbar$, which corresponds to the ordinary Bloch frequency.

Let us examine in somewhat greater detail the

*For $\Omega = 0$ the spectrum obtained here corresponds to the spectrum of the spin waves of a paramagnetic fluid, as treated in reference 9 in the approximation of small B_n. I take this occasion to point out that in Eq. (2.16) of that paper the coefficients α_n should be taken with the opposite signs.

case of short waves, on the assumption that only B_0 is different from zero and that the wave vector **k** is parallel to the direction of the constant magnetic field H_0. The dispersion equation takes the form:

$$-\frac{1}{B_0} = 1 - \frac{s}{2} \ln\left|\frac{s \mp s_0 + 1}{s \mp s_0 - 1}\right|,$$

where $s = \omega/kv_0$, $s_0 = \Omega_0/kv_0$.

In the region of small wavelengths we have from this

$$\omega = kv_0 s = \pm \Omega_0 (1 + B_0) \{1 + k^2 v_0^2 / 3B_0 \Omega_0^2\}. \quad (36)$$

For positive B_0 the frequency increases with increasing k.

In the short-wave region $kv_0 \gg \Omega_0$ the dispersion equation takes the form $1/B_0 = \eta(s)$ and has solutions only for positive B_0. In the case of negative B_0, on the other hand, it follows from Eq. (36) that with decrease of the wavelength the absolute value of the frequency decreases. Furthermore, in the case $1 + 1/B_0 < 0$ the frequency of the spin wave goes to zero for $|k| = \Omega_0/v_0$, according to the formula

$$\omega = \pm \left\{\Omega_0 - |k| v_0 \left[1 + \exp\left(\frac{1 + B_0}{B_0} \frac{2|k|v_0}{\Omega_0 - |k|v_0}\right)\right]\right\}. \quad (37)$$

5. The oscillations of a degenerate electron fluid considered here are in a certain sense similar to plasma waves, and correspond to excitations that obey Bose statistics. The same applies also in particular to the excitations of an electron fluid in a magnetic field, with the spectra given by Eqs. (14), (33), and (35). Therefore the contributions of such excitations to the partition function will be decidedly different from the contribution of the single-particle states, which obey Fermi statistics.

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