POLARIZATION OF INTERNAL CONVERSION ELECTRONS FOLLOWING BETA DECAY

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The correlation between the polarization of conversion electrons and the direction of electron emission in the preceding β decay is treated. The entire calculation is done taking account of the electric field of the nucleus. Conversion can occur in any atomic shell. In particular, the nuclear electric field has a significant effect in K conversion. It causes the appearance of an appreciable transverse component of the polarization in magnetic transitions, and significantly increases the magnitude of the polarization for electric transitions.

BECAUSE of non-conservation of parity in β decay, the residual nucleus will be polarized in the direction of emission of the electron. The β-decaying nucleus is assumed to be unpolarized and the direction of emission of the neutrino is not recorded. Thus if the β decay is followed by internal conversion, the conversion electrons should have a definite polarization. This phenomenon, for the case of K conversion and omitting effects of the nuclear electric field, was treated by Berestetskii and Rudik.[1] The electric field of the nucleus has a marked effect on the internal conversion process and causes a large change in the conversion coefficient. The nuclear electric field should therefore also have a marked effect on the polarization of the conversion electrons.

We write the initial wave function of the electron which undergoes conversion in the form:

$$\psi_1 = \begin{pmatrix} ig_{\varkappa_1}(r)\ \Omega^{(l_1)}_{j_1M_1}(\mathbf{r}/r) \\ f_{\varkappa_1}(r)\ \Omega^{(l'_1)}_{j_1M_1}(\mathbf{r}/r) \end{pmatrix}, \qquad (1)$$

where $\Omega^{(l)}_{jM}(\mathbf{n})$ is a spinor surface harmonic having components

$$[\Omega^{(l)}_{jM}(\mathbf{n})]_\mu = C^{jM}_{l_2\mu;\ l,\ M-\mu} Y_{l,\ M-\mu}(\mathbf{n}),$$

$$l' = 2j - l,\quad \varkappa = (l - l')(j + 1/2),$$

in which $C^{c\gamma}_{a\alpha;b\beta}$ are Clebsch-Gordan coefficients and Y_{lm} is a spherical harmonic. Throughout the paper, we use the system of units in which $\hbar = c = 1$.

The wave function of the emitted electron is a solution of the Dirac equation which, at infinity, is a superposition of plane and outgoing spherical waves. This function was found in reference 2, and we shall write it in the form

$$\psi_2 = \frac{(2\pi)^{3/2}}{\sqrt{p_2\varepsilon_2}} \sum_{j_2l_2M_2} [\Omega^{(l_2)}_{j_2M_2}(\mathbf{n})]^*_\xi \begin{pmatrix} ig_{\varkappa_2}(r)\ \Omega^{(l_2)}_{j_2M_2}(\mathbf{r}/r) \\ f_{\varkappa_2}(r)\ \Omega^{(l'_2)}_{j_2M_2}(\mathbf{r}/r) \end{pmatrix} \exp\{-i\delta_{\varkappa_2}\}, \qquad (2)$$

where $\mathbf{n}$ is a unit vector in the direction of the emerging electron, ξ is the polarization of the electron; $g_{K_2}(r)$ and $f_{K_2}(r)$ are the radial parts of the Coulomb functions, normalized to a δ-function in the energy scale; the phase δ_{K_2} approaches $-\pi(l_2 - 1)/2$ as $z \to 0$.

The matrix element for the conversion process can be written in the following form:[3]

$$\mathfrak{M}_{m_1m_2} = \sum_M (I_1m_1 \mid Q^{(\lambda)}_{jM} \mid I_2m_2)^* (B^{(\lambda)}_{jM})_{21}; \qquad (3)$$

$$(B^{(\lambda)}_{jM})_{21} = \int \psi_2^* B^{(\lambda)}_{jM} \psi_1\, dv. \qquad (3a)$$

Here $(I_1m_1 \mid Q^{(\lambda)}_{jM} \mid I_2m_2)$ is the nuclear matrix element for the transition, I_1m_1 are the spin and spin projection of the nucleus before conversion, I_2m_2 the spin and spin projection of the nucleus after the conversion, I is the nuclear spin before the β-decay, $B^{(\lambda)}_{jM}$ is the operator of the interaction of the electron with the field of the multipole. This operator has the following form:

$$B^{(0)}_{jM} = G_j(\omega r)\,\boldsymbol{\alpha}\mathbf{Y}_{jjM}(\mathbf{r}/r),$$

$$B^{(1)}_{jM} = \sqrt{\frac{j}{j+1}}\, G_j(\omega r)\, Y_{jM}(\mathbf{r}/r) + \sqrt{\frac{2j+1}{j+1}}\, G_{j+1}(\omega r)\,\boldsymbol{\alpha}\mathbf{Y}_{j,\ j-1,\ M}(\mathbf{r}/r). \qquad (4)$$

$\lambda = 0$ for magnetic transitions and $\lambda = 1$ for electric transitions. In addition, $\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}$, where the σ's are the Pauli matrices; ω is the energy of the transition;

$$G_j(x) = (2\pi)^{3/2} i^j H^{(1)}_{j+1/2}(x) / \sqrt{x}$$

($H^{(1)}$ is a Hankel function); $\mathbf{Y}_{jlM}$ is a vector spherical harmonic with contravariant components equal to

$$(\mathbf{Y}_{jlM})^\mu = C^{jM}_{l,\ M-\mu;1\mu} Y_{l,\ M-\mu}.$$

The properties of vector spherical harmonics were studied in Ref. 3.

The density matrix with respect to the spin states of the conversion electron has the form

$$P = \sum_{\substack{m_1, m_1', m_2 \\ M, M'}} \rho_{m_1 m_1'} (I_1 m_1 | Q_{jM}^{(\lambda)} | I_2 m_2)^* \times (I_1 m_1' | Q_{jM'}^{(\lambda)} | I_2 m_2) (B_{jM}^{(\lambda)})_{21} (B_{jM'}^{(\lambda)})_{21}^*. \quad (5)$$

The expression for the matrix $\rho_{m_1 m_1'}$ which characterizes the state of polarization of the nucleus after β-decay was found in reference 1:

$$\rho_{m_1 m_1'} = \frac{1}{2I_1+1}\left\{\delta_{m_1 m_1'} + \sqrt{\frac{I_1+1}{I_1}}\alpha \sum_\mu C_{I_1 m_1'; 1\mu}^{I_1 m_1} v^\mu\right\}, \quad (6)$$

where v^μ are the spherical components of the velocity vector of the β particle:

$$v^{(0)} = v_z, \quad v^{\pm 1} = \mp \frac{1}{\sqrt{2}}\left(v_x \mp i v_y\right)/\sqrt{2};$$

and α is a constant which determines the angular distribution of β particles in the decay of polarized nuclei with spin I_1 and average value $\langle I_{1z}\rangle$ for the spin projection, which make a transition to a state with spin I.

For allowed transitions, and for the S, T, A and V interactions, the value of α is:[4]

$$\alpha = [c_1 + (Ze^2/\varepsilon v) c_2]/(1 + b/\varepsilon),$$

$$c_1\zeta = 2\,\mathrm{Re}\Big\{(C_T C_T'^* - C_A C_A'^*)|M_{GT}|^2 \lambda_{I_1 I} + \sqrt{\frac{I_1}{I_1+1}}(C_T C_S'^* + C_T' C_S^* - C_A C_V'^* - C_A' C_V^*) M_F M_{GT}^*\Big\};$$

$$c_2\zeta = 2\,\mathrm{Im}\Big\{(C_T C_A'^* + C_T' C_A^*)|M_{GT}|^2 \lambda_{I_1 I} + \sqrt{\frac{I_1}{I_1+1}}(C_T C_V'^* + C_T' C_V^* - C_A C_S'^* - C_A' C_S^*) M_F M_{GT}^*\Big\};$$

$$b\zeta = 2\sqrt{1-(Ze^2)^2}\,\mathrm{Re}\{(C_T C_A'^* + C_T' C_A^*)|M_{GT}|^2 + (C_S C_V^* + C_S' C_V'^*)|M_F|^2\};$$

$$\zeta = (|C_T|^2 + |C_T'|^2 + |C_A|^2 + |C_A'|^2)|M_{GT}|^2 + (|C_S|^2 + |C_S'|^2 + |C_V|^2 + |C_V'|^2)|M_F|^2;$$

$$\lambda_{I_1 I} = \begin{cases} 1/(I+1) & \text{if } I_1 = I \\ 1 & \text{"} \ I_1 = I+1 \\ -(I-1)/I & \text{"} \ I_1 = I-1. \end{cases}$$

The matrix element of the multipole moment can be expressed as

$$(I_1 m_1 | Q_{jM}^{(\lambda)} | I_2 m_2) = Q^{(\lambda)} C_{I_2 m_2; jM}^{I_1 m_1}. \quad (7)$$

The general expression for the polarization vector $\langle\sigma\rangle$ of the conversion electron must have the following form:

$$\langle\sigma\rangle = a(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n} + b\{\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n}\}, \quad (8)$$

where a and b are constants.

We note that the matrix $\rho_{m_1 m_1'}$ can be written as in Eq. (6) and the polarization expressed in the form (8) only for allowed transitions and for first forbidden Coulomb transitions.

MAGNETIC TRANSITIONS

Substituting in (3a) the electron wave functions ψ_1 and ψ_2 from (1) and (2) and the operator $B_{jM}^{(0)}$, we get

$$(B_{jM}^{(0)})_{21} = -\frac{(2\pi)^{3/2} i}{\sqrt{p_2\varepsilon_2}} \sum_{j_2 l_2 M_2} [\Omega_{j_2 M_2}^{(l_2)}(\mathbf{n})]_\xi e^{i\delta_{\varkappa_2}} \Big\{R_{1\varkappa_2} \int \Omega_{j_2 M_2}^{(l_2')*}\left(\frac{\mathbf{r}}{r}\right) \sigma \mathbf{Y}_{jM}^{(0)}\left(\frac{\mathbf{r}}{r}\right) \times \Omega_{j_1 M_1}^{(l_1)}\left(\frac{\mathbf{r}}{r}\right) do - R_{2\varkappa_2}\int \Omega_{j_2 M_2}^{(l_2)*}\left(\frac{\mathbf{r}}{r}\right)\sigma\mathbf{Y}_{jM}^{(0)}\left(\frac{\mathbf{r}}{r}\right)\Omega_{j_1 M_1}^{(l_1')}\left(\frac{\mathbf{r}}{r}\right) do\Big\}. \quad (9)$$

We have introduced the following notation for the radial integrals:

$$R_{1\varkappa_2} = \int f_{\varkappa_2}(r)\, g_{\varkappa_1}(r)\, G_j(\omega r)\, r^2\, dr, \qquad R_{2\varkappa_2} = \int g_{\varkappa_2}(r)\, f_{\varkappa_1}(r)\, G_j(\omega r)\, r^2\, dr. \quad (10)$$

We shall use the system of mutually orthogonal vector spherical harmonics

$$\mathbf{Y}_{jM}^{(0)} = \mathbf{Y}_{jjM},$$

$$\mathbf{Y}_{jM}^{(1)} = \sqrt{\frac{j}{2j+1}}\,\mathbf{Y}_{j,\,j+1,\,M} + \sqrt{\frac{j+1}{2j+1}}\,\mathbf{Y}_{j;\,j-1,\,M}, \quad (11)$$

$$\mathbf{Y}_{jM}^{(-1)} = \mathbf{n}\mathbf{Y}_{jM}.$$

$\mathbf{Y}_{jM}^{(0)}$ and $\mathbf{Y}_{jM}^{(1)}$ are transverse vectors, so that $\mathbf{Y}_{jM}^{(0)}(\mathbf{n})\cdot\mathbf{n} = \mathbf{Y}_{jM}^{(1)}(\mathbf{n})\cdot\mathbf{n} = 0$. They are related to one another by

$$i\mathbf{Y}_{jM}^{(1)} = [\mathbf{n}\times\mathbf{Y}_{jM}^{(0)}]. \quad (12)$$

The following useful formula holds for $\mathbf{Y}_{jM}^{(0)}$:

$$\mathbf{Y}_{jM}^{(0)} = \mathbf{L}Y_{jM}/\sqrt{j(j+1)} \quad (\mathbf{L} = -i[\mathbf{r}\times\nabla]). \quad (13)$$

In calculating the angular integrals in (9), we make use of formula (13), the self-adjoint character of the operator $\sigma\cdot\mathbf{L}$, and the formulas:

$$\sigma\cdot\mathbf{L}\Omega_{jM}^{(l)} = \left\{j(j+1) - l(l+1) - \frac{3}{4}\right\}\Omega_{jM}^{(l)}; \quad (14)$$

$$\sigma\cdot\mathbf{n}\Omega_{jM}^{(l)}(\mathbf{n}) = \Omega_{jM}^{(l')}(\mathbf{n}); \quad (15)$$

$$Y_{l_1 m_1}(\mathbf{n})\,Y_{l_2 m_2}(\mathbf{n}) = \sum_{LM}\sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}}\,C_{l_1 0;\, l_2 0}^{L0}\, C_{l_1 m_1;\, l_2 m_2}^{LM}\, Y_{LM}(\mathbf{n}). \quad (16)$$

In carrying out the summation in (9) we use Racah's formula (cf., for example, reference 5):

$$\sum_\beta C_{a\alpha;\, b\beta}^{e,\,\alpha+\beta}\, C_{e,\,\alpha+\beta;\, d,\,\gamma-\alpha-\beta}^{c\gamma}\, C_{b\beta;\, d,\,\gamma-\alpha-\beta}^{f,\,\gamma-\alpha} = (2e+1)^{1/2}(2f+1)^{1/2}\, C_{a\alpha;\, f,\,\gamma-\alpha}^{c\gamma}\, W(abcd;\, ef) \quad (17)$$

in which W (abcd; ef) are Racah coefficients. After performing the integration and summation in (9), we get

$$(B_{jM}^{(0)})_{21} = \sum_{j_2 l_2 M_2} a_{\varkappa_2} W(l_2 j_2 l_1' j_1;\ {}^1\!/_2\, j) \times C_{j0,\ l_1' 0}^{l_2 0}\, C_{jM;\ j_1 M_1}^{j_2 M_2}\, C_{l_2,\ M_2-\xi;\ {}^1\!/_2\, \xi}^{j_2 M_2}\, Y_{l_2,\ M_2-\xi}(\mathbf{n}); \quad (18)$$

$$a_{\varkappa_2} = -\pi i \sqrt{\frac{2(2j+1)(2l_1'+1)(2j_1+1)}{j(j+1)\,p_2\varepsilon_2}}\, e^{i\delta_{\varkappa_2}}(R_{1\varkappa_2} + R_{2\varkappa_2}).$$

Substituting from (6), (7) and (18) into formula (5), dropping irrelevant common factors, and summing over all values of M, M′, M_1, M_2, M_3, μ, we find for the density matrix:

$$P_{\xi\xi'} = (A + 2A_1\xi)\,\delta_{\xi\xi'} + 2\sqrt{2\pi}\, A_4 C_{2,\ \xi'-\xi;\ {}^1\!/_2\, \xi}^{{}^1\!/_2\, \xi'}\, Y_{2,\ \xi'-\xi}(\mathbf{n}), \quad (19)$$

where

$$A = \sum_{j_2 l_2} |a_{\varkappa_2}|^2 (2j_2+1)(C_{j0;\ l_1'0}^{l_2 0})^2\, W^2(l_2 j_2 l_1 j_1;\ {}^1\!/_2\, j),$$

$$A_1 = \sqrt{\frac{2j+1}{6}}\, \alpha\, \frac{j(j+1) + I_1(I_1+1) - I_2(I_2+1)}{I_1\sqrt{j(j+1)}} \times \sum_{\substack{j_2\ l_2,\ j_3\\ l_3 = l_2}} (-1)^{j_2 - j_3 - 1}\, \alpha_{\varkappa_2}\alpha_{\varkappa_3}^* \times (2j_2+1)(2j_3+1)(C_{j0;\ l_1'0}^{l_2 0})^2\, W(l_2 j_2 l_1' j_1;\ {}^1\!/_2\, j)$$
$$\times W(l_2 j_3 l_1' j_1;\ {}^1\!/_2\, j)\, W(j j j_2 j_3;\ 1 j)\, W(j_3 j_2\, {}^1\!/_2\, {}^1\!/_2;\ 1 l_2), \quad (20)$$

$$A_2 = \sqrt{\frac{2j+1}{3}}\, \alpha\, \frac{j(j+1) + I_1(I_1+1) - I_2(I_2+1)}{I_1\sqrt{j(j+1)}} \times \sum_{j_2,\ l_2,\ j_3,\ l_3} (-1)^{j_2 - j_3 + l_2 + 1}\, \alpha_{\varkappa_2}\alpha_{\varkappa_3}^*$$
$$\times \sqrt{(2l_2+1)(2l_3+1)}(2j_2+1)(2j_3+1)\, C_{j0;\ l_1'0}^{l_2 0}\, C_{j0;\ l_1'0}^{l_3 0}\, C_{l_2 0;\ l_3 0}^{20}$$
$$\times W(l_2 j_2 l_1' j_1;\ {}^1\!/_2\, j)\, W(l_3 j_3 l_1' j_1;\ {}^1\!/_2\, j)$$
$$\times W(j_3 j_2\, {}^1\!/_2\, {}^3\!/_2;\ 1 l_3)\, W(l_2 j_2\, 2\, {}^3\!/_2;\ {}^1\!/_2\, l_3)\, W(j j j_2 j_3;\ 1 j_1).$$

We choose the z axis to be along the direction of the velocity of the β particle. Then formula (19) can be rewritten as

$$P_{\xi\xi'} = \quad (21)$$
$$\begin{pmatrix} A + A_1 - A_2(3\cos^2\theta - 1), & -3A_2\sin\theta\cos\theta \\ -3A_2\sin\theta\cos\theta, & A - A_1 + A_2(3\cos^2\theta - 1) \end{pmatrix},$$

where θ is the angle between the directions of emergence of the β particle and the conversion electron. Knowing the density matrix $P_{\xi\xi'}$, it is easy to find the polarization of the conversion electrons:

$$\langle\sigma\rangle = \mathrm{Sp}\, P\sigma / \mathrm{Sp}\, P; \quad (22)$$

$$\langle\sigma\rangle = a(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n} + b\{\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n}\},$$
$$a = (A_1 - 2A_2)/vA, \quad b = (A_1 + A_2)/vA. \quad (23)$$

For the K shell ($l_1 = 0$; $j_1 = \frac{1}{2}$, $\kappa_1 = -1$), using formulas (23) and (20) and inserting the values of the Clebsch-Gordan and Racah coefficients, we find

$$\langle\sigma\rangle = \alpha\, \frac{j(j+1) + I_1(I_1+1) - I_2(I_2+1)}{2j(j+1)\, I_2(1 + |\eta_k^0|^2)} \times \{(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n} + \sqrt{j(j+1)}\, \mathrm{Re}\, \eta_k^{(0)}(\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n})\},$$

$$\eta_k^{(0)} = \frac{\sqrt{j(j+1)}\{R_{1,\ -j} + R_{2,\ -j} - (R_{1,\ j+1} + R_{2,\ j+1})\exp(i(\delta_{j+1} - \delta_{-j}))\}}{(j+1)(R_{1,\ -j} + R_{2,\ -j}) + j(R_{1,\ j+1} + R_{2,\ j+1})\exp(i(\delta_{j+1} - \delta_{-j}))}. \quad (24)$$

Let us consider two limiting cases.

(1) Small Z. In this case the electric field of the nucleus can be neglected. Then the radial integrals are calculated explicitly and, using (24), we get the same value of the polarization as was given in reference 1.

(2) Large Z and low energy of the conversion electron, so that $p_2/mZe^2 \ll 1$. In this case, $|R_{1,\ j+1} + R_{2,\ j+1}| \ll |R_{1,\ -j} + R_{2,\ -j}|$, and we find for the polarization the expression

$$\langle\sigma\rangle = \alpha\, \frac{j(j+1) + I_1(I_1+1) - I_2(I_2+1)}{2j(2j+1)\, I_1} \times \{(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n} + j(\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n})\}. \quad (25)$$

We note that the formulas (24) are valid for any shell, so long as $\kappa_1 = -1$ for the electrons in the shell. In particular they are applicable to conversion in the L I shell. We also give the formulas for conversion in the L II shell (or in any shell with $\kappa_1 = 1$; $j_1 = \frac{1}{2}$; $l_1 = 1$):

$$\langle\sigma\rangle = \alpha\, \frac{j(j+1) + I_1(I_1+1) - I_2(I_2+1)}{2j(j+1)\, I_2(1 + |\eta_{L\,\mathrm{II}}^{(0)}|^2)} \times \{(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n} + \sqrt{j(j+1)}\, \mathrm{Re}\, \eta_{L\,\mathrm{II}}^{(0)}(\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n})\},$$

$$\eta_{L\,\mathrm{II}}^{(0)} = \sqrt{j(j+1)} \times \frac{(R_{1,\ -j-1} + R_{2,\ -j-1}) - (R_{1,\ j} + R_{2,\ j})\exp\{i(\delta_j - \delta_{-j-1})\}}{j(R_{1,\ -j-1} + R_{2,\ -j-1}) + (j+1)(R_{1,\ j} + R_{2,\ j})\exp\{i(\delta_j - \delta_{-j-1})\}} \quad (26)$$

ELECTRIC TRANSITIONS

Substituting the electron wave functions ψ_1 and ψ_2 from (1) and (2) and the operator $B_{jM}^{(1)}$ into (3a), we get

$$(B_{jM}^{(1)})_{21} = \frac{(2\pi)^{3/2}}{\sqrt{p_2\varepsilon_2}} \sum_{j_2 l_2 M_2} [\Omega_{j_2 M_2}^{(l_2)}(\mathbf{n})]_\xi \exp\{i\delta_{\varkappa_2}\} \cdot \Bigg\{\sqrt{\frac{j}{j+1}} \times \Big[R_{3,\ \varkappa_2}\int \Omega_{j_2 M_2}^{(l_2)*} Y_{jM}\Omega_{j_1 M_1}^{(l_1)}\, do + R_{4,\ \varkappa_2}\int \Omega_{j_2 M_2}^{(l_2')} Y_{jM}\Omega_{j_1 M_1}^{(l_1')}\, do\Big]$$
$$- \sqrt{\frac{2j+1}{j+1}}\Big[R_{5,\ \varkappa_2}\int \Omega_{j_2 M_2}^{(l_2')*}\sigma\cdot\mathbf{Y}_{j,\ j-1,\ M}\Omega_{j_1 M_1}^{(l_1)}\, do - R_{6,\ \varkappa_2}\int \Omega_{j_2 M_2}^{(l_2)*}\sigma\cdot\mathbf{Y}_{j,\ j-1,\ M}\Omega_{j_1 M_1}^{(l_1')}\, do\Big]\Bigg\}. \quad (27)$$

We have introduced the following notation for the radial integrals:

$$\begin{aligned} R_{3,\varkappa_2} &= \int G_j(\omega r)\, g_{\varkappa_2}(r)\, g_{\varkappa_1}(r)\, r^2\, dr, \\ R_{4,\varkappa_2} &= \int G_j(\omega r)\, f_{\varkappa_2}(r)\, f_{\varkappa_1}(r)\, r^2\, dr, \\ R_{5,\varkappa_2} &= -i\int G_{j-1}(\omega r)\, f_{\varkappa_2}(r)\, g_{\varkappa_1}(r)\, r^2\, dr, \\ R_{6,\varkappa_2} &= -i\int G_{j-1}(\omega r)\, g_{\varkappa_2}(r)\, f_{\varkappa_1}(r)\, r^2\, dr. \end{aligned} \tag{28}$$

Now let us calculate the angular integrals. For the first pair of integrals over angle, we get:

$$\int \Omega^{(l_2)*}_{j_2M_2} Y_{jM} \Omega^{(l_1)}_{j_1M_1}\, do = \int \Omega^{(l_2')}_{j_2M_2} Y_{jM} \Omega^{(l_1')}_{j_1M_1}\, do = \frac{1}{\sqrt{4\pi}}\sqrt{(2j+1)(2j_1+1)(2l_1+1)} \times C^{l_20}_{l_10;\,j0} C^{j_2M_2}_{j_1M_1;\,jM}\, W(jl_1j_2\,{}^1/_2;\; l_2j_1). \tag{29}$$

To calculate the second pair of integrals over angle, we write

$$\mathbf{Y}_{j,\,j-1,\,M} = \sqrt{\frac{j}{2j+1}}\,\mathbf{Y}^{(-1)}_{jM} + \sqrt{\frac{j+1}{2j+1}}\,\mathbf{Y}^{(1)}_{jM}. \tag{30}$$

Using formula (13) and the commutation relations for the Pauli matrices, we find:

$$\begin{aligned} (\sigma\cdot\mathbf{Y}_{j,\,j-1,\,M})\left(\sigma\frac{\mathbf{r}}{r}\right) &= \sqrt{\frac{j}{2j+1}}\,Y_{jM} + \frac{1}{\sqrt{j(2j+1)}}\,\sigma\cdot\mathbf{L}Y_{jM}, \\ \left(\sigma\frac{\mathbf{r}}{r}\right)(\sigma\cdot\mathbf{Y}_{j,\,j-1,\,M}) &= \sqrt{\frac{j}{2j+1}}\,Y_{jM} - \frac{1}{\sqrt{j(2j+1)}}\,\sigma\cdot\mathbf{L}Y_{jM}. \end{aligned} \tag{31}$$

Using formulas (31), (14), (15), and (16) along with the fact that the operator $\sigma\cdot\mathbf{L}$ is self-adjoint, we find for the second pair of angle integrals:

$$\begin{aligned} &\int \Omega^{(l_2')*}_{j_2M_2}\sigma\cdot\mathbf{Y}_{j,\,j-1,\,M}\Omega^{(l_1)}_{j_1M_1}\, do = \frac{1}{\sqrt{4\pi}}\sqrt{j(2j_1+1)(2l_1+1)}\left(1+\frac{\varkappa_2-\varkappa_1}{j}\right) \times C^{l_20}_{l_10;\,j0} C^{j_2M_2}_{j_1M_1;\,jM} W(jl_1j_2\,{}^1/_2;\; l_2j_1); \\ &\int \Omega^{(l_2)*}_{j_2M_2}\sigma\cdot\mathbf{Y}_{j,\,j-1,\,M}\Omega^{(l_1')}_{j_1M_1}\, do = \frac{1}{\sqrt{4\pi}}\sqrt{j(2j_1+1)(2l_1+1)}\left(1-\frac{\varkappa_2-\varkappa_1}{j}\right) \times C^{l_20}_{l_10;\,j0} C^{j_2M_2}_{j_1M_1;\,jM} W(jl_1j_2\,{}^1/_2;\; l_2j_1). \end{aligned} \tag{32}$$

Substituting the results of the angular integrations in (27) and using the symmetry properties of the Clebsch-Gordan and Racah coefficients, we find an expression for $(B^{(1)}_{jM})_{21}$ which differs from (18) only in the replacement of l_1' by l_1 and of $a_{\varkappa_2}$ by

$$b_{\varkappa_2} = \pi\sqrt{\frac{2j(2j+1)(2j_1+1)(2l_1+1)}{(j+1)p_2\varepsilon_2}}\Big[R_{3,\varkappa_2} + R_{4,\varkappa_2} - \left(1+\frac{\varkappa_2-\varkappa_1}{j}\right)R_{5,\varkappa_2} + \left(1-\frac{\varkappa_2-\varkappa_1}{j}\right)R_{6,\varkappa_2}\Big]\exp\{i\delta_{\varkappa_2}\}. \tag{33}$$

We can therefore immediately write the formula for the polarization of conversion electrons in an electric transition:

$$\langle\sigma\rangle = a(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n} + b\{\mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n}\};\quad a = (B_1 - 2B_1)/vB,\quad b = (B_1 + B_2)/vB, \tag{34}$$

where the quantities B, B_1, and B_2 are gotten from A, A_1, and A_2 by making the substitution described above.

For the polarization of conversion electrons ejected from the K-shell, we get the expression

$$\langle\sigma\rangle = \alpha\,\frac{j(j+1)+l_1(l_1+1)-l_2(l_2+1)}{2j(j+1)\,l_1\,(1+|\eta^{(1)}_k|^2)} \times \{(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n} + \sqrt{j(j+1)}\,\mathrm{Re}\,\eta^{(1)}_k(\mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n})\}; \tag{35}$$

$$\eta^{(1)}_k = \frac{1}{\sqrt{j(j+1)}}\,\frac{(j+1)(R_{3,-j-1}+R_{4,-j-1}+2R_{6,-j-1}) + j\left(R_{3,j}+R_{4,j}-\frac{2j+1}{j}R_{5,j}-\frac{1}{j}R_{6,j}\right)\exp\{i(\delta_j-\delta_{-j-1})\}}{(R_{3,-j-1}+R_{4,-j-1}+2R_{6,-j-1}) - \left(R_{3,j}+R_{4,j}-\frac{2j+1}{j}R_{5,j}-\frac{1}{j}R_{6,j}\right)\exp\{i(\delta_j-\delta_{-j-1})\}}. \tag{36}$$

In the free electron approximation, the radial integrals can be calculated explicitly:

$$\begin{aligned} R_{3,\varkappa_2} &= (2\pi Ze^2m)^{3/2}\sqrt{\frac{2(\varepsilon_2+m)}{\omega}}\,I_j i^j; \\ R_{4,\varkappa_2} &= R_{6,\varkappa_2} = 0; \\ R_{5,j} &= -(2\pi Ze^2m)^{3/2}\sqrt{\frac{2(\varepsilon_2-m)}{\omega}}\,I_{j-1} i^j;\quad \delta_j - \delta_{-j-1} = 0. \end{aligned}$$

Substituting these values for the radial integrals into (36), we get

$$\eta^{(1)}_k = -\sqrt{\frac{j}{j+1}}\,\frac{2\varepsilon_2}{\varepsilon_2-m}, \tag{37}$$

after which we find from (35) the same value for the polarization as in Ref. 1.

For high Z the radial integrals cannot be found explicitly, and must be computed numerically. We give, for Z = 80, a table of the values of $\sqrt{j(j+1)} \times \eta^{(1)}_k$, calculated using values of the radial integrals which were kindly communicated to us by L. A. Sliv.

As we see from a comparison of the table with formula (37), the electric field of the nucleus also has a significant effect on the polarization of the

j \ ω/m	0.3	0.5	0.7	1
1	4.6	3.3	2.5	2.0
2	3.7	2.5	2.0	1.4
3	3.2	2.2	1.6	1.04–0.27i

conversion electrons in the case of electric transitions. Formulas (35) and (36) are applicable to conversion in any shell with $\kappa_2 = -1$. In particular they are applicable to conversion in the L I shell, except that the radial integrals will be different. For conversion in the L II shell (or any shell with $\kappa_1 = 1$), we find for the polarization of the conversion electrons

$$\langle\sigma\rangle = \alpha \frac{j(j+1) + I_1(I_1+1) - I_2(I_2+1)}{2j(j+1) I_1 (1 + |\eta^{(1)}_{L II}|^2)} \times \{(\mathbf{v}\cdot\mathbf{n})\mathbf{n} + \sqrt{j(j+1)}\,\mathrm{Re}\,\eta_{L II}(\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n})\}; \quad (38)$$

$$\eta_{L II} = \frac{1}{\sqrt{j(j+1)}} \times \frac{j\left(R_{3,-j} + R_{4,-j} + \frac{1}{j} R_{5,-j} + \frac{2j+1}{j} R_{6,-j}\right) + (j+1)(R_{3,j+1} + R_{4,j+1} - 2R_{5,j+1}) \exp\{i(\delta_{j+1} - \delta_{-j})\}}{\left(R_{3,-j} + R_{4,-j} + \frac{1}{j} R_{5,-j} + \frac{2j+1}{j} R_{6,-j}\right) - (R_{3,j+1} + R_{4,j+1} - 2R_{5,j+1}) \exp\{i(\delta_{j+1} - \delta_{-j})\}}.$$

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