LOW-FREQUENCY OSCILLATIONS OF A PLASMA IN A MAGNETIC FIELD

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The low-frequency electron-ion longitudinal oscillations in a plasma confined by a magnetic field are considered.

IT is well known that a plasma is capable of two kinds of oscillations: high-frequency electronic oscillations, and low-frequency oscillations in which both electrons and ions take part. The theory of the low-frequency oscillations has been developed by Tonks and Langmuir,[1] Gordeev[2] and others. In the present paper we consider the low-frequency longitudinal oscillations of an unbounded plasma which is confined by a fixed, uniform, magnetic field.

1. DISPERSION EQUATION

We consider small oscillations of a plasma consisting of electrons and singly charged ions. It will be assumed that the oscillation frequency is high so that the collision integrals can be neglected in the kinetic equations which describe deviations in the electron (ion) distribution functions from the equilibrium values. Suppose that at time $t = 0$ we turn off the external effect which causes the plasma to deviate from the equilibrium state. After a sufficiently long period of time t the Fourier component of the electric field intensity will be proportional to $\exp\{-i\omega t - \gamma t\}$ where the frequency ω and the damping factor γ are determined from a dispersion equation of the form

$$An'^4 + Bn'^2 + C = 0, \quad n' = kc/\omega',$$

where $\omega' = \omega - i\gamma$ and $\mathbf{k}$ is the propagation vector. Expressions for the coefficients A and B and C are given in reference 3. If the individual terms which appear in A are much larger than $|B/n'^2|$ and $|C/n'^4|$ an approximate solution of the dispersion equation can be obtained by setting $A = 0$. The equation $A = 0$ is the dispersion equation for oscillations characterized by curl $\mathbf{E} \approx 0$ (longitudinal plasma oscillations). Thus, we have

$$A(\omega', k) = 1 + K_e + K_i = 0, \tag{1}$$

where[3,4] ($\alpha = e, i$)

$$K_\alpha = \frac{1}{k^2 a_\alpha^2} + \frac{i\omega'}{k^2 a_\alpha^2 \omega_H^\alpha} \exp(-\mu_\alpha \sin^2\theta) \times \int_0^\infty \exp\left(-\frac{1}{2}\mu_\alpha \cos^2\theta\varphi^2 + \frac{i\omega'}{\omega_H^\alpha}\varphi + \mu_\alpha \sin^2\theta\cos\varphi\right)d\varphi, \tag{2}$$

$$a_\alpha = (T_\alpha / 4\pi e^2 n_0)^{1/2}, \quad \Omega_\alpha = (4\pi e^2 n_0 / m_\alpha)^{1/2},$$

$$\mu_\alpha = (k a_\alpha \Omega_\alpha / \omega_H^\alpha)^2, \quad v_T^\alpha = (T_\alpha / m_\alpha)^{1/2}, \quad a_\alpha \Omega_\alpha = v_T^\alpha.$$

Here θ is the angle between the direction of the external magnetic field $\mathbf{H}_0$ and the propagation vector $\mathbf{k}$, $\omega_H^\alpha = eH_0/mc$ is the gyromagnetic frequency for a particle of mass m_α and charge e (the subscript $\alpha = e$ refers to electrons, $\alpha = i$ refers to ions) T_α is the temperature of the gas of particles of type α and n_0 is the equilibrium electron density.

The quantity K_α can also be given in the form

$$K_\alpha = \frac{1}{k^2 a_\alpha^2} - \frac{1}{k^2 a_\alpha^2} e^{-\mu_\alpha \sin^2\theta} \sum_{n=-\infty}^{\infty} I_n(\mu_\alpha \sin^2\theta) \frac{z_0^\alpha}{\sqrt{\pi}} \int_C \frac{e^{-t^2}}{z_n^\alpha - t} dt, \tag{3}$$

$$z_n^\alpha = (\omega' - n\omega_H^\alpha) / \sqrt{2}\, k v_T^\alpha \cos\theta,$$

where I_n is a Bessel function of imaginary argument. The integration over t in Eq. (3) is taken over the path C which goes along the real axis from $-\infty$ to $+\infty$, going around the singularities $t = z_n^\alpha$ from below for $\cos\theta > 0$ and from above for $\cos\theta < 0$. Below, it is assumed that $\cos\theta > 0$.

Equation (3) for K_α is obtained from Eq. (2) by expanding $\exp(+\mu_\alpha \sin^2\theta\cos\varphi)$ in Eq. (2) in powers of $e^{i\varphi}$ and using the relation following between the integrals over the contour C, which appear in Eq. (3), and the probability integral:[5]

$$\frac{1}{\sqrt{\pi}} \int_C \frac{e^{-t^2}}{z - t} dt = -i\sqrt{\pi} e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{t^2} dt\right). \tag{4}$$

2. INVESTIGATION OF THE DISPERSION EQUATION

The dispersion equation (1) is extremely complicated and can be solved only in certain limiting cases. If the phase velocity of the oscillations $V_{ph} = \omega/k$ is much larger than the mean thermal velocity of the electrons v_T^e in Eq. (1) K_i can be neglected compared with K_e. The resulting dispersion equation $1 + K_e = 0$ describes the high-frequency oscillations of the plasma. This equation has been investigated in detail in reference 6 and 7. Below, we study the oscillations for which

$$v_T^i \ll V_{ph} = \omega / k \ll v_T^e \tag{5}$$

(low-frequency oscillations) in three different cases: weak magnetic field, $\omega_H^e \ll kv_T^e$; strong magnetic field, $\omega_H^i \gg kv_T^i$; the intermediate case, in which $\omega_H^e \gg kv_T^e$ and $\omega_H^i \ll kv_T^i$.

In order to investigate Eq. (1) we use the asymptotic expression for the integral in (4) (large values of z)

$$\frac{1}{\sqrt{\pi}}\int_C \frac{e^{-t^2}}{z-t}\,dt \sim \frac{1}{z} + \frac{1}{2z^3} + \frac{3}{4z^5} + \ldots - i\sqrt{\pi}e^{-z^2}, \quad |\operatorname{Re} z| \gg 1, \ \operatorname{Im} z \ll 1. \tag{6}$$

At small values of z the function in (4) is conveniently expanded in powers of z, in which case

$$\frac{1}{\sqrt{\pi}}\int_C \frac{e^{-t^2}}{z-t}\,dt = -i\sqrt{\pi} + O(z), \quad |z| \ll 1. \tag{7}$$

(a) Suppose $\omega_H^e \ll kv_T^e$ (weak magnetic field). Then the following inequality holds: $\omega_H^i \ll kv_T^i$. Expanding $\cos\varphi$ in Eq. (2) in powers of φ and taking account of the inequality in (5), we have

$$K_e = \frac{1 + i\sqrt{\pi}z_e + \ldots}{k^2 a_e^2}, \quad z_e = \omega'/\sqrt{2}kv_T^e, \ |z_e| \ll 1; \tag{8}$$

$$K_i = -\frac{1}{k^2 a_i^2}\left(\frac{1}{2z_i^2} + \frac{3}{4z_i^4} + \cdots\right), \ z_i = \omega'/\sqrt{2}kv_T^i, \ |z_i| \gg 1. \tag{9}$$

Substituting Eqs. (8) and (9) in Eq. (1) and solving for ω', we have

$$\omega^2 = \omega_0^2 + 3k^2 v_T^{i2}, \ \omega_0 = \Omega_i k a_e (1 + k^2 a_e^2)^{-1/2}, \tag{10}$$

$$\gamma = \gamma_0 = \sqrt{\frac{\pi m_e}{8 m_i}} \frac{\Omega_i k a_e}{(1 + k^2 a_e^2)^2}. \tag{11}$$

The frequency equation (10) (for $v_T^i \to 0$) was given by Tonks and Langmuir.[1] The correction $3k^2 v_T^{i2}$ which takes account of the thermal spread in the ion velocities and the damping term (11), were given by Gordeev.[2] The introduction of a magnetic field results in the appearance of an additional small term in the expression for ω^2:

$$\omega_H^{i2} \sin^2\theta \cdot k^2 a_e^2 / (1 + k^2 a_e^2).$$

The condition $|z_e| \ll 1$, which must be satisfied if (8) is to hold, is always satisfied since $(m_e/m_i)^{1/2} \ll 1$. The condition of applicability for the expansion in (9), $|z_i| \gg 1$, can be used if

$$T_e \gg T_i, \tag{12}$$

i.e., the plasma must be highly non-isothermal. If this is not the case the low-frequency oscillations are highly damped: $\gamma \sim \omega_0$. According to Eqs. (10) and (11) the damping factor is small compared with the frequency $\gamma/\omega \sim (m_e/m_i)^{1/2}$. The damping factor (11) is due to the "remote" collective interactions of the particles. The plasma oscillations also are damped by "local" collisions.

(b) Intermediate case. Suppose now that the following inequalities hold: $(\omega_H^e)^2 \gg (kv_T^e)^2$ and $\omega_H^{i2} \ll k^2 v_T^{i2}$ or

$$1 \ll (\omega_H^e / kv_T^e)^2 \ll m_i T_i / m_e T_e. \tag{13}$$

Because the ratio m_i/m_e is very large (approximately 10^3 to 10^5) there are values of ω_H^e/kv_T^e (even when $T_i \ll T_e$) for which the inequality in (13) is satisfied.

If the condition in (13) is satisfied, we can use the earlier expression for K_i (9). In computing K_e, we assume that because of (13) $\mu_e \ll 1$. Hence, from Eq. (3) we find, (assuming that $|z_0^e| \ll 1$ for values of θ far from $\pi/2$)

$$K_e \approx (1 + i\sqrt{\pi}z_0^e)/k^2 a_e^2. \tag{14}$$

Substituting Eqs. (9) and (14) in Eq. (1), we obtain the earlier expression for the frequency (10), and the damping term

$$\gamma = \gamma_0 / |\cos\theta|. \tag{15}$$

As θ increases the damping term also increases. However, when θ approaches $\pi/2$, Eqs. (10) and (15) no longer hold since these equations were obtained under the condition that $|z_0^e| \ll 1$, which is satisfied only when $|\pi/2 - \theta| \gg (m_e/m_i)^{1/2}$.

We assume now that $|z_n^e| \gg 1$ $(n = 0, \pm 1, \pm 2)$. Then, by virtue of Eq. (6) we have

$$K_e = -\frac{\Omega_e^2\cos^2\theta}{\omega'^2} - \frac{\Omega_e^2\sin^2\theta}{\omega'^2-\omega_H^{e2}} - \frac{3k^2v_T^{e2}\Omega_e^2}{\omega'^4}$$
$$\times\left[\cos^4\theta + \frac{6-3u_e+u_e^2}{3(1-u_e)^3}\cos^2\theta\sin^2\theta + \frac{\sin^4\theta}{(1-u_e)(1-4u_e)}\right] \quad (16)$$
$$+ i\sqrt{\frac{\pi}{2}}\frac{\omega'}{k^2a_e^2\Omega_e|\cos\theta|}\sum_{n=-\infty}^{\infty}\frac{\mu_e^{|n|}\sin^{2|n|}\theta}{|n|!\,2^{|n|}}e^{-z_n^2}, \quad u_e = (\omega_H^e/\omega')^2.$$

Substituting Eqs. (9) and (16) in Eq. (1) and neglecting the thermal corrections we have

$$1 - \frac{\Omega_e^2\cos^2\theta}{\omega^2} - \frac{\Omega_e^2\sin^2\theta}{\omega^2-\omega_H^{e2}} - \frac{\Omega_i^2}{\omega^2} = 0. \quad (17)$$

Neglecting the last term in Eq. (17) for values of θ far from π 2, we find the resonance frequencies of the electron plasma oscillations in the "hydrodynamic" approximation:

$$\omega_{1,2}^2 = \omega_\pm^2 = \frac{1}{2}(\Omega_e^2+\omega_H^{e2}) \pm \frac{1}{2}\left[\left(\Omega_e^2+\omega_H^{e2}\right)^2 - 4\Omega_e^2\omega_H^{e2}\cos^2\theta\right]^{1/2} \quad (18)$$

When $\theta\to\pi/2$

$$\omega_1^2 = \Omega_e^2+\omega_H^{e2};\quad \omega_2^2 = \Omega_e^2\omega_H^{e2}\cos^2\theta/(\Omega_e^2+\omega_H^{e2}). \quad (19)$$

Equation (19) for ω_2 applies if $|\pi/2-\theta|^2 \gg m_e/m_i$. If, however, $|\pi/2-\theta|^2 < m_e/m_i$, from Eq. (17) we have

$$\omega_2^2 = \tilde{\omega}^2 = (\Omega_e^2\cos^2\theta+\Omega_i^2)/(1+\Omega_e^2/\omega_H^{e2}). \quad (20)$$

Taking account of the correction terms in Eqs. (9) and (16), we find the frequency and the damping term

$$\omega_2 = \tilde{\omega}(1+\tilde{\varepsilon}),\quad |\tilde{\varepsilon}|\ll 1, \quad (21)$$

$$\tilde{\varepsilon} = \frac{1}{2(1+\Omega_e^2/\omega_H^{e2})}\left\{-\frac{\tilde{\omega}^2\Omega_e^2}{\omega_H^{e4}} + \frac{3k^2v_T^{i2}\Omega_i^2}{\tilde{\omega}^4}\right.$$
$$\left.+ 3k^2v_T^{e2}\Omega_e^2\left(\frac{\cos^4\theta}{\tilde{\omega}^4} - \frac{\cos^2\theta}{3\tilde{\omega}^2\omega_H^{e2}} + \frac{1}{4\omega_H^{e4}}\right)\right\}, \quad (22)$$

$$\gamma = \sqrt{\frac{\pi}{8}}\frac{\tilde{\omega}^2}{1+\Omega_e^2/\omega_H^{e2}}\left\{\frac{\exp(-\omega_2^2/2k^2v_T^{e2}\cos^2\theta)}{k^3a_e^3\Omega_e|\cos\theta|}\right.$$
$$\left.+\frac{\exp(-\omega_2^2/2k^2v_T^{i2})}{k^3a_i^3\Omega_i}\right\}. \quad (23)$$

The inequalities $|z_i|\gg 1$, $|z_0^e|\gg 1$, $|\tilde{\varepsilon}|\ll 1$, and $\gamma_2\ll\tilde{\omega}$ are satisfied if $ka_e\ll 1$ and $ka_i\ll 1$.

(c) In the strong field case, when $\omega_H^i \gg kv_T^i$ and $\omega_H^e \gg k_T^e$, when $|z_0^e|\ll 1$ we can use Eq. (14) for K_e; when $|z_n|\gg 1$ we can use Eq. (16) for K_i (in Eq. (16) we replace T_e by T_i and m_e by m_i). Equation (1) then assumes the form

$$1+\frac{1}{k^2a_e^2} - \frac{\Omega_i^2\cos^2\theta}{\omega'^2} - \frac{\Omega_i^2\sin^2\theta}{\omega'^2-\omega_H^{i2}} - \frac{3k^2v_T^{i2}\Omega_i^2}{\omega^4}$$
$$\times\left\{\cos^4\theta + \cos^2\theta\sin^2\theta\,\frac{6-3u_i+u_i^2}{3(1-u_i)^3} + \frac{\sin^4\theta}{(1-u_i)(1-4u_i)}\right\} \quad (24)$$
$$+ i\sqrt{\frac{\pi}{2}}\frac{\omega'}{k^3a_e^3\Omega_e|\cos\theta|} = 0,\quad u_i = (\omega_H^i/\omega)^2.$$

From Eq. (24) we find the resonance frequencies $\omega_{1,2}$ of the electron-ion oscillations in a strong magnetic field:

$$\omega_{1,2} = \tilde{\omega}_\pm(1+\tilde{\varepsilon}_\pm),\quad |\tilde{\varepsilon}_\pm|\ll 1, \quad (25)$$

where

$$\tilde{\omega}_\pm^2 = \frac{1}{2}(\omega_0^2+\omega_H^{i2}) \pm \frac{1}{2}\{(\omega_0^2+\omega_H^{i2})^2 - 4\omega_0^2\omega_H^{i2}\cos^2\theta\}^{1/2} \quad (26)$$

are the resonance frequencies when the thermal motion of the ions is neglected; $\tilde{\varepsilon}_\pm$ determines the corrections for ω_1 and ω_2 due to the thermal motion of the ions

$$\tilde{\varepsilon}_\pm = \frac{3k^2v_T^{i2}}{2\tilde{\omega}_\pm^2[\cos^2\theta+\sin^2\theta\,\tilde{\omega}_\pm^4/(\tilde{\omega}_\pm^2-\omega_H^{i2})^2]}$$
$$\times\left[\cos^4\theta + \frac{6-3u_\pm+u_\pm^2}{3(1-u_\pm)^3}\cos^2\theta\sin^2\theta + \frac{\sin^4\theta}{(1-u_\pm)(1-4u_\pm)}\right], \quad (27)$$
$$u_\pm = \left(\frac{\omega_H^i}{\tilde{\omega}_\pm}\right)^2.$$

The damping factor corresponding to the frequencies $\omega_{1,2}$ is

$$\gamma_{1,2} = \sqrt{\frac{\pi}{8}}\frac{\tilde{\omega}_\pm^4}{k^3a_e^3|\cos\theta|[\cos^2\theta+\sin^2\theta\,\tilde{\omega}_\pm^4/(\tilde{\omega}_\pm^2-\omega_H^{i2})^2]\Omega_e\Omega_i^2}. \quad (28)$$

Eqs. (24) and (28) apply if the inequalities $|z_n^i| \gg 1$ and $|z_0^e|\ll 1$ hold. The inequality $|z_0^i|\gg 1$ is satisfied for values of θ far from $\pi/2$ and if the plasma is highly non-isothermal, $T_e\gg T_i$. The condition $|z_n^i|\gg 1$ $(n = 1, 2, \ldots)$ holds if $\tilde{\omega}_+$ is not close to $n\omega_H^i$. The inequality $|z_0^e|\ll 1$ is always satisfied for values of θ which are not close to $\pi/2$ and $\omega_H^i<\omega_0$ because $m_e\ll m_i$. If, however, $\omega_H^i > kv_T^e$, the inequality $|z_0^e|\ll 1$ is satisfied only for one of the solutions of Eqs. (25) to (28), $\omega=\omega_0\cos\theta$, $\gamma=\gamma_0\cos\theta$.

As $\theta\to 0$, one of the solutions of Eq. (25), $\omega_1\approx\tilde{\omega}_+$, approaches the expression given in (10) while the other $\omega_2\approx\tilde{\omega}_-$ approaches ω_H^i However, at small values of θ Eqs. (25) to (28) for ω_2 and γ_2 do not apply since the condition $|z_1^i|\gg 1$ does not hold at values of θ close to 0. If at small values of θ the inequality $|\omega_1'-\omega_H^i|\gg kv_T^i$ does not hold when $\omega_1\approx\tilde{\omega}_+$, Eq. (27) (thermal frequency correction) and Eq. (28) (damping) are not correct. When $|\tilde{\omega}-\omega_H^i|\ll kv_T^i$, the integral $\int \frac{e^{-t^2}dt}{z_1^i - t} \approx -i\pi$. Thus we find

$$\omega_1 = \frac{\Omega_i k a_e}{\sqrt{1+k^2 a_e^2}} \approx \omega_H^i,\ \gamma_1 = \gamma_0 + \frac{1}{2}\sqrt{\frac{\pi T_e}{8T_i}}\theta^2 \Omega_i k a_e. \quad (29)$$

If, at some value of θ, not close to $\pi/2$ it turns out $\tilde{\omega}_+ \approx 2\omega_H^i$, then the inequality $|z_2^i| \gg 1$ no longer holds and consequently Eq. (27) for $\tilde{\varepsilon}_+$ is no longer valid. If $|\tilde{\omega}_+ - 2\omega_H^i| \ll kv_T^i \cos\theta$, the integral which contains z_2^i in (3) is approximately equal to $-i\pi$ and gives additional damping at frequency ω_1. The additional damping also obtains in the general case when $|\omega_+ - m\omega_H^i| < kv_T^i \times \cos\theta$ $(m = 2, 3, \ldots)$

$$\gamma_1 = \sqrt{\frac{\pi}{8}} \frac{m^4}{|\cos^3\theta|\,[1+(m^2-1)^{-2} m^4 \operatorname{tg}^2\theta]} \times \left\{\frac{\omega_H^{i4}}{\Omega_i^2 \Omega_e k^2 a_e^3} + \frac{\sin^{2m}\theta k v_T^i}{2^m m!}\left(\frac{k v_T^i}{\omega_H^i}\right)^{2m-4} \exp(-z_m^{i2})\right\}, \quad (30)$$

$$z_m^i = (\omega_1 - m\omega_H^i) / (\sqrt{2}\, k v_T^i \cos\theta).$$

The inequality $\gamma_1 \ll \omega_1$ for $m = 3, 4$ is always satisfied for any value of θ excluding $\theta \sim \pi/2$ since $kv_T^i \ll \omega_H^i$. When $m = 2$ this inequality is satisfied for small values of θ.

For $\theta \to \pi/2$, from Eq. (25) we find

$$\omega_1^2 \approx \tilde{\omega}_+^2 = \omega_H^{i2} + \omega_0^2. \quad (31)$$

Eq. (31) is valid if the inequality $|z_0^e| \ll 1$ is satisfied, i.e., if

$$|\pi/2 - \theta|^2 \gg (\omega_H^i / k v_T^e)^2 + m_e/m_i,$$

from which it follows that Eq. (31) does not apply in the narrow region of angles about $\pi/2$.

We now assume that $|z_n^e| \gg 1$ and $|z_n^i| \gg 1$. In this case (16) applies for K_e and K_i. At values of θ far from $\pi/2$ K_i is small as compared with K_e and can be neglected. In this case, using Eq. (1) we obtain the results given in references 6 and 7. If, however, $\theta \sim \pi/2$, assuming that $\omega \ll \omega_H^e$, from the dispersion equation we find $\omega_{1,2} \approx \Omega_\pm$, where

$$\Omega_\mp^2 = \frac{1}{2(1+\Omega_e^2/\omega_H^{e2})}\{\omega_H^{i2} + \Omega_i^2 + \Omega_e^2\cos^2\theta \pm [(\omega_H^{i2} + \Omega_i^2 + \Omega_e^2\cos^2\theta)^2 - 4(1+\Omega_e^2/\omega_H^{e2})\Omega_e^2\omega_H^{i2}\cos^2\theta]^{1/2}\}. \quad (32)$$

The damping of waves with frequencies given by (32) turns out to be small (exponential factor).

In conclusion we may note that if the conditions

$$|K_i| \gg |B_0/n'^2|,\ |K_i| \gg |C_0/n'^4|, \quad (33)$$

are satisfied, when $H_0 \neq 0$ we can approximately isolate the low-frequency longitudinal oscillations; these conditions are satisfied in cases (b) and (c) if

$$k^2c^2 \gg \Omega_i^2.$$

(In case (c) it is assumed that $\tilde{\omega}_\pm \sim \omega_0 \sim \omega_H^i$ and $\Omega_\pm \sim \Omega_i$).

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