

ON FINDING THE MATRIX OF THE  $a + a' \rightarrow b + b'$  REACTION

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We consider the conditions necessary for finding the matrix of the  $a + a' \rightarrow b + b'$  reaction from experimental data. Invariance under rotation and reflection in space is used to find the number of complex scalar functions that define the reaction matrix M. Invariance of the reaction under time reversal gives relations between the polarization effects in the direct and inverse reactions. We find the number of experiments necessary for the complete determination of the reaction matrix.

WOLFENSTEIN and Ashkin<sup>1</sup> have used invariance under rotation and reflection in space and under time reversal to obtain general expressions for the scattering amplitudes of particles with spin 0 and  $\frac{1}{2}$  on particles with spin  $\frac{1}{2}$ . These expressions have been used<sup>2</sup> together with the fact that the S matrix is unitary to discuss the problem of how many experiments are necessary to completely determine the scattering amplitudes for these cases.

In the present note we perform a similar analysis for the general case of the  $a + a' \rightarrow b + b'$  reaction. We first construct the amplitude for the process, and then find a relation between the direct and inverse reaction matrices which follows from invariance under time reflection. The two possible directions of the reaction are then used as an example to establish the number of experiments necessary to find the amplitude of the process in general.

We introduce the complete set of matrices

$$T^{JM}(j_b, j_a) = A(j_b, j_a, J) \sum_{m_a, m_b} (j_a m_a M | j_a J j_b m_b) \chi_{j_b m_b} \chi_{j_a m_a}^+ \quad (1)$$

Here  $\chi_{j_a m_a}$  and  $\chi_{j_b m_b}$  are the spin functions of the initial and final states, and  $j_a, j_b, m_a,$  and  $m_b$  are the total spins and their z-components.

The  $T^{JM}$  operators satisfy the condition

$$\overset{\pm}{T}^{JM}(j_b, j_a) = (-1)^{M} T^{J, -M}(j_a, j_b).$$

From this we obtain

$$A^*(j_b, j_a, J) \sqrt{2j_b + 1} = (-1)^{j_a - j_b} \sqrt{2j_a + 1} A(j_a, j_b, J).$$

Let us expand the reaction matrix in terms of the complete set of  $T^{JM}$  operators. We have

$$M_{b_a}(n_b, n_a) = \sum_{JM} \Phi_{j_b j_a}^{JM}(n_b, n_a) \overset{\pm}{T}^{JM}(j_b, j_a), \quad (2)$$

where  $n_a$  and  $n_b$  are unit vectors in the directions of the relative momenta. From invariance under rotation we may conclude that the  $\Phi_{j_b j_a}^{JM}$  functions are of the form

$$\Phi_{j_b j_a}^{JM}(n_b, n_a) = \sum_{\lambda} a_{j_b j_a}^{\lambda} \psi_{JM}^{\lambda}(n_b, n_a), \quad (3)$$

where  $a_{j_b j_a}^{\lambda}$  are arbitrary complex scalar functions of the energy and the angle between  $n_a$  and  $n_b$ , and

$$\psi_{JM}^{\lambda}(n_b, n_a) = \sum_{\mu, \mu'} (r - \lambda, r + \lambda, \mu, \mu' | r - \lambda, r + \lambda, J, M) Y_{r - \lambda, \mu}(n_a) Y_{r + \lambda, \mu'}(n_b). \quad (4)$$

Thus the amplitude is entirely determined by the

$a_{j_b j_a}^{\lambda}$  functions. The number of such functions can be found from Eqs. (1) to (4). If all the particles involved in the reaction have integral spin, this number is

$$N_0(i_1, i_2, I_1, I_2) = \pm^{1/2} (2i_1 + 1)(2i_2 + 1)(2I_1 + 1)(2I_2 + 1) (-1)^{i_1 + i_2 + I_1 + I_2}, \quad (5)$$

where  $i_1, i_2, I_1,$  and  $I_2$  are the spins of the particles (the minus sign refers to the case in which internal parity changes, and the plus sign to the case in which there is no change of internal parity). In other cases  $N_0$  is given by Eq. (5) without the last term.

Invariance of the theory under time reversal gives the relation

$$k_a M_{b_a}^{\pm}(n_b, n_a) = k_b T M_{a_b}(-n_a, -n_b) T. \quad (6)$$

Here T is the time reversal operator, whose action on the simultaneous eigenfunction of the angular momentum and its z-component is given by

$$T \psi_{JM} = (-1)^{J-M} \psi_{J, -M}.$$

If  $b_{ja}^\lambda$  are scalar functions in the expansion of the inverse reaction matrix  $M_{ab}(\mathbf{n}_a, \mathbf{n}_b)$  in a form analogous to (2), Eq. (6) gives

$$k_a a_{ib}^\lambda = k_b b_{ia}^{-\lambda}. \quad (7)$$

Equation (6) can also be used for a general reaction to establish the relation between the quantities observed in the direct and inverse reactions. Assume the operators  $T^{q_a \kappa_a}$ ,  $T^{q_a' \kappa_a'}$ , ... to act on the spin variables of particles  $a$ ,  $a'$ , ..., respectively. We then have

$$\sigma_{ba} \langle T^{q_b \kappa_b} T^{q_b' \kappa_b'} \rangle = \sum_{q_a q_a' \kappa_a \kappa_a'} \langle T^{q_a \kappa_a} T^{q_a' \kappa_a'} \rangle \quad (8)$$

$$\times \text{Sp} T^{q_b \kappa_b} T^{q_b' \kappa_b'} M_{ba}(\mathbf{n}_b, \mathbf{n}_a) \hat{T}^{q_a \kappa_a} \hat{T}^{q_a' \kappa_a'} M_{ba}^+(\mathbf{n}_b, \mathbf{n}_a)$$

and a similar expression for the inverse reaction with an interchange of  $a$  and  $b$ . Condition (6) then gives<sup>3</sup>

$$\begin{aligned} & k_a^2 \text{Sp} T^{q_b \kappa_b} T^{q_b' \kappa_b'} M_{ba}(\mathbf{n}_b, \mathbf{n}_a) \hat{T}^{q_a \kappa_a} \hat{T}^{q_a' \kappa_a'} M_{ba}^+(\mathbf{n}_b, \mathbf{n}_a) \\ &= (-1)^{q_a + q_a' + q_b + q_b' + \kappa_a + \kappa_a' + \kappa_b + \kappa_b'} k_b^2 \text{Sp} T^{q_a \kappa_a} T^{q_a' \kappa_a'} \\ &\times M_{ab}(-\mathbf{n}_a, -\mathbf{n}_b) \hat{T}^{q_b \kappa_b} \hat{T}^{q_b' \kappa_b'} M_{ab}^+(-\mathbf{n}_a, -\mathbf{n}_b). \end{aligned} \quad (9)$$

Let us now use two directions of a reaction as an example to analyze the conditions necessary to determine a reaction matrix from experimental data. The number of independent real functions

entering into  $M_{ba}(\mathbf{n}_b, \mathbf{n}_a)$  (or  $M_{ab}(\mathbf{n}_a, \mathbf{n}_b)$ ) is  $2N_0(i_1, i_2, I_1, I_2)$ . The number of functions which determine the scattering matrix  $M_{aa}(\mathbf{n}_a, \mathbf{n}_a')$  (or  $M_{bb}(\mathbf{n}_b, \mathbf{n}_b')$ ) is reduced, as a result of time reversal invariance, from  $2N_0(i_1, i_2, i_1, i_2)$  to  $N(i_1, i_2) = N_0(i_1, i_2, i_1, i_2) + (2i_1 + 1)(2i_2 + 1)$ . Therefore the total number of real functions entering into the matrix of these reactions is

$$N(i_1, i_2) + N(I_1, I_2) + 2N_0(i_1, i_2, I_1, I_2). \quad (10)$$

It can be shown further that considerations based on the fact that the  $S$  matrix is unitary further reduces by a factor of 2 the number of independent functions which must be determined from experiment.

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<sup>1</sup>L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952).

<sup>2</sup>Puzikov, Ryndin, and Smorodinskii, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 592 (1957), Soviet Phys. JETP **5**, 489 (1957).

<sup>3</sup>M. I. Shirokov, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 975 (1957), Soviet Phys. JETP **6**, 748 (1958).

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