EQUATIONS OF MOTION FOR A SYSTEM CONSISTING OF TWO TYPES OF INTERACTING SPINS

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SOLOMON[1] has found the equations of motion describing the magnetization of a system consisting of two types of interacting magnetic moments in parallel fields. Kurbatov and the author[2] have investigated the thermodynamic properties of a two-spin system, including the spin-spin and spin-lattice relaxations. The present note gives a simple thermodynamic derivation of the equations describing the behavior of such a system in a constant field $\mathbf{H}_0$ arbitrarily oriented with respect to an alternating field $\mathbf{h}$.

We shall start with the equations

$$\dot{M}_k^{(1)} = L_{ik}^{11}(H_i - H_i^{(1)}) + L_{ik}^{12}(H_i - H_i^{(2)}),$$
$$\dot{M}_k^{(2)} = L_{ik}^{21}(H_i - H_i^{(1)}) + L_{ik}^{22}(H_i - H_i^{(2)}), \quad (1)$$

where $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ are related to the magnetizations $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ of the spin subsystems by

$$\mathbf{M}^{(1)} = \chi_{01}\mathbf{H}^{(1)}, \quad \mathbf{M}^{(2)} = \chi_{02}\mathbf{H}^{(2)}. \quad (2)$$

The L_{ik} satisfy the Onsager relations. Assuming that in the absence of a field the medium is isotropic, we write

$$L_{ik}^{11} = \frac{\chi_{01}}{\tau_1}\delta_{ik} + \gamma_1\chi_{01}\varepsilon_{ikl}H_0; \quad L_{ik}^{12} = \frac{\chi_{02}}{\tau}\delta_{ik},$$
$$L_{ik}^{21} = \frac{\chi_{01}}{\tau}\delta_{ik}; \quad L_{ik}^{22} = \frac{\chi_{02}}{\tau_2}\delta_{ik} + \gamma_2\chi_{02}\varepsilon_{ikl}H_0, \quad (3)$$

where γ_1 and γ_2 are the gyromagnetic ratios for the spin subsystems, ϵ_{ikl} is the unit antisymmetric tensor, and $\mathbf{H} = \mathbf{H}_0 + \mathbf{h}(t)$. Equations (1) now become*

$$\dot{\mathbf{M}}_1 + \mathbf{M}_1/\tau_1 + \mathbf{M}_2/\tau = (\chi_{01}/\tau_1 + \chi_{02}/\tau)\mathbf{H} + \gamma_1[\mathbf{M}_1\times\mathbf{H}],$$
$$\dot{\mathbf{M}}_2 + \mathbf{M}_2/\tau_2 + \mathbf{M}_1/\tau = (\chi_{01}/\tau + \chi_{02}/\tau_2)\mathbf{H} + \gamma_2[\mathbf{M}_2\times\mathbf{H}]. \quad (4)$$

In the absence of a transverse rf field in the steady state, as may have been expected, these equations lead to the relations given by (2).

For parallel fields, i.e., if $[\mathbf{H}_0 \times \mathbf{h}(t)] = 0$, Eqs. (4) are the same as those obtained by Solomon. If the second subsystem is missing, they become

$$\dot{\mathbf{M}} + \mathbf{M}/\tau = (\chi_0/\tau)\mathbf{H} + \gamma[\mathbf{M}\times\mathbf{H}].$$

Let us now require that $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ are of equal magnitudes; then multiplying Eqs. (4) by $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$, respectively, we obtain

$$\frac{\mathbf{M}_1^2}{\tau_1} + \frac{(\mathbf{M}_1\cdot\mathbf{M}_2)}{\tau} = \left(\frac{\chi_{01}}{\tau_1} + \frac{\chi_{02}}{\tau}\right)(\mathbf{M}_1\cdot\mathbf{H}),$$
$$\frac{\mathbf{M}_2^2}{\tau_2} + \frac{(\mathbf{M}_1\cdot\mathbf{M}_2)}{\tau} = \left(\frac{\chi_{01}}{\tau} + \frac{\chi_{02}}{\tau_2}\right)(\mathbf{M}_2\cdot\mathbf{H}). \quad (5)$$

Eliminating χ_{01} and χ_{02} from (4) and (5), we obtain

$$\dot{\mathbf{M}}_1 = \gamma_1[\mathbf{M}_1\times\mathbf{H}] - \frac{\lambda_{11}}{M_1^2}[\mathbf{M}_1\times[\mathbf{M}_1\times\mathbf{H}]] - \frac{\lambda_{12}}{(\mathbf{M}_1\cdot\mathbf{M}_2)}[\mathbf{M}_1\times[\mathbf{M}_2\times\mathbf{H}]],$$
$$\dot{\mathbf{M}}_2 = \gamma_2[\mathbf{M}_2\times\mathbf{H}] - \frac{\lambda_{22}}{M_2^2}[\mathbf{M}_2\times[\mathbf{M}_2\times\mathbf{H}]] - \frac{\lambda_{21}}{(\mathbf{M}_1\cdot\mathbf{M}_2)}[\mathbf{M}_2\times[\mathbf{M}_1\times\mathbf{H}]], \quad (6)$$

where

$$\lambda_{11} = M_1^2/\tau(\mathbf{M}_1\cdot\mathbf{H}); \quad \lambda_{12} = (\mathbf{M}_1\cdot\mathbf{M}_2)/\tau(\mathbf{M}_1\cdot\mathbf{H});$$
$$\lambda_{21} = (\mathbf{M}_1\cdot\mathbf{M}_2)/\tau(\mathbf{M}_2\cdot\mathbf{H}); \quad \lambda_{22} = M_2^2/\tau(\mathbf{M}_2\cdot\mathbf{H}). \quad (7)$$

If $\lambda_{12} = \lambda_{21} = 0$ (that is in the limit as $\tau \to \infty$), Eqs. (6) go over into the Landau-Lifshitz equations for two noninteracting spin systems. They can be used to describe relaxation processes and resonance phenomena in antiferromagnets.

*Henceforth we shall write the indices denoting the subsystems as subscripts.

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159

CLEBSCH-GORDAN EXPANSION FOR INFINITE-DIMENSIONAL REPRESENTATIONS OF THE LORENTZ GROUP

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ONE of the authors has given[1] the explicit form of the Clebsch-Gordan coefficients for the expansion of the finite-dimensional representations of the Lorentz group. If one choose the basis functions of the finite-dimensional representation to be

$$\psi_{Nlm}(\alpha\vartheta\varphi)=\frac{\sinh^{l}\alpha}{\sqrt{N^2(N^2-1^2)\ldots(N^2-l^2)}}\frac{d^{l+1}\cosh N\alpha}{d\cosh^{l+1}\alpha}Y_{lm}(\vartheta\varphi),$$
$$t=\rho\cosh\alpha,\ r=\rho\sinh\alpha,\ 0\leqslant\alpha\leqslant\infty,$$
$$-\infty\leqslant\rho\leqslant\infty,\ N=0,1,2,\ldots, \tag{1}$$

the expansion has the form

$$\psi_{N_1l_1m_1}\psi_{N_2l_2m_2}=\sum_{N,l}\sqrt{\frac{N_1N_2}{4\pi N}}A(N_1l_1N_2l_2Nl)C^{lm}_{l_1m_1l_2m_2}\psi_{Nlm}, \tag{2}$$
$$A(N_1l_1N_2l_2Nl)\equiv N\sqrt{(2l_1+1)(2l_2+1)}X(j_1j_1l_1,j_2j_2l_2,jjl);$$

$2j_i+1=N_i$, and X are the Fano functions.[2] It was mentioned that if one replaces N by in, where n is real and $0\le n\le\infty$, Eq. (1) gives the basis functions of one of the irreducible unitary infinite-dimensional representations of the Lorentz group:

$$\psi_{nlm}(\alpha,\vartheta,\varphi)=\frac{\sinh^{l}\alpha}{\sqrt{n^2(n^2+1)\ldots(n^2+l^2)}}\frac{d^{l+1}\cos n\alpha}{d\cosh^{l+1}\alpha}Y_{lm}(\vartheta,\varphi). \tag{3}$$

The functions ψ_{nlm} are orthogonal and are normalized by the condition

$$\int_0^\infty\sinh^2\alpha d\alpha\int d\Omega\psi^*_{n_1l_1m_1}(\alpha,\vartheta,\varphi)\psi_{n_2l_2m_2}(\alpha,\vartheta,\varphi)=\frac{\pi}{2}\delta(n_1-n_2)\delta_{l_1l_2}\delta_{m_1m_2}. \tag{4}$$

Let us find the Clebsch-Gordan expansion for the ψ_{nlm}. We shall look for an expansion of the form

$$\psi_{n_1l_1m_1}\psi_{n_2l_2m_2}=\sum_l\int_0^\infty dn\, i^{l-l_1-l_2}\sqrt{in_1n_2/4\pi n}\,C^{lm}_{l_1m_1l_2m_2}\times B(n_1n_2n,\ l_1l_2l)\,A(n_1l_1n_2l_2nl)\,\psi_{nlm}. \tag{5}$$

$A(n_1l_1n_2l_2nl)$ is expressed in terms of j_1, j_2 and j exactly as in the case of Eq. (2), except that instead of using integer or half-integer values of j_1, j_2 and j, we must take

$$j_1=\tfrac{1}{2}(in_1-1),\ j_2=\tfrac{1}{2}(in_2-1)\ \text{and}\ j=\tfrac{1}{2}(in-1).$$

The recursion relations for the ψ_{nlm} enable us to obtain equations relating the $B(n_1n_2n,\ l_1l_2l)$ for different values of l_1, l_2 and l. These equations are satisfied if $B(n_1n_2n,\ l_1l_2l)\equiv B(n_1n_2n)$ does not depend on l_1, l_2 and l:

$$B(n_1n_2n)=\sinh\pi n_1\sinh\pi n_2\sinh\pi n\Big[\cosh\frac{\pi}{2}(n_1+n_2+n)\times\cosh\frac{\pi}{2}(n_1-n_2-n)\cosh\frac{\pi}{2}(n_1+n_2-n)\times\cosh\frac{\pi}{2}(n_1-n_2+n)\Big]^{-1}. \tag{6}$$

Using the fact that

$$\sum_{l_1l_2}A(n_1l_1n_2l_2nl)\,A(n_1l_1n_2l_2n'l)=\delta(n-n'),$$
$$A(n_10n_20n0)=\sqrt{n/in_1n_2}, \tag{7}$$

we get the inverse Clebsch-Gordan series:

$$\psi_{nlm}=\sqrt{4\pi n/in_1n_2}\,[B(n_1n_2n)]^{-1}\sum_{l_1l_2m_1m_2}i^{l_1+l_2-l}\times A(n_1l_1n_2l_2nl)\,C^{lm}_{l_1m_1l_2m_2}\psi_{n_1l_1m_1}\psi_{n_2l_2m_2}. \tag{8}$$

For complex n, formula (3) gives the basis functions of an infinite-dimensional irreducible nonunitary representation. Such functions occur in the expansion of the product of (1) and (3) in a Clebsch-Gordan series:

$$\psi_{nl_1m_1}\psi_{Nl_2m_2}=\sum_{l,\nu}i^{l_1-l}\sqrt{nN/4\pi\nu}\,A(n_1l_1Nl_2\nu l)\,C^{lm}_{l_1m_1l_2m_2}\psi_{\nu lm}, \tag{9}$$
$$\nu=n_1-i\varkappa;\ \varkappa=-N+1,\ -N+2,\ldots,N-1;$$
$$i\nu=2j+1,\ N=2j_2+1.$$

Using Eqs. (22) and (29) of reference 1 and Eq. (9), one can obtain the expansion of derivatives of the ψ_{nlm} in terms of irreducible representations:

$$\partial_{\alpha\beta}G_n(\rho)\psi_{nlm}=\sum_{\nu,f,L}i^{l-L}\sqrt{(2f+1)n/2\nu}\,C^{1/2\alpha}_{1/2\beta f-\varphi}\times C^{L\Lambda}_{lmf\varphi}A(nl,2f,\nu L)\Big[\frac{\partial}{\partial\rho}-\varkappa\frac{in-\varkappa}{\rho}\Big]G_n(\rho)\psi_{\nu L\Lambda}, \tag{10}$$

$i\nu=in+\kappa$, $\kappa=\pm1$, $G_n(\rho)$ depends only on ρ, $\partial_{\pm1/2,\pm1/2}=\partial/\partial t\mp\partial/\partial z$, $\partial_{\pm1/2,\mp1/2}=\pm(\partial/\partial x\mp i\partial/\partial y)$.

All the formulas given here apply to the case where $\rho^2=t^2-r^2>0$, i.e., when all the basis functions are timelike. To go over to the case of $r=\rho\cosh\alpha$, $t=\rho\sinh\alpha$, $-\infty\le\alpha\le\infty$, $0\le\rho\le\infty$, one should make the substitution $\alpha\to\alpha-i\pi/2$ in all the formulas.

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160