SPIN WAVES IN A FERROMAGNETIC METAL

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Using the theory of a Fermi liquid, developed by Landau,[1] we consider spin waves in a ferromagnetic metal.

It is well known that the energy spectrum of a ferromagnetic contains elementary excitations corresponding to oscillations of the magnetic moment. These are called spin waves and have a characteristic dispersion law $\epsilon \sim k^2$. The spin waves were first obtained by Bloch[2] starting from the Heisenberg model of a ferromagnet. Later Landau and Lifshitz[3,4] constructed a phenomenological theory of spin waves and showed that the quadratic dispersion law is by no means particular to the Heisenberg model, but arises from the general properties of the exchange interaction.

Both the Bloch and the Landau-Lifshitz theories considered a ferromagnet as a system of spins, rigidly fixed to the crystalline lattice. This is at any rate not the case in metals where the electrons can freely move through the crystal. It is thus of interest to consider how the motion of the electrons influences the general conclusions of the phenomenological theory. In the present paper we shall consider this problem, using the theory of a Fermi liquid, constructed by Landau.[1]

The electron spectrum of a metal is described by means of quasi-particles, obeying Fermi statistics, conventionally called "electrons." Since the energy of these quasi-particles depends on the orientation of their spin, we shall introduce, in accordance with Landau's theory, the energy operator of the quasi-particles $\epsilon(\mathbf{p}, \sigma)$ (where $\mathbf{p}$ is the quasi-momentum and σ the spin operator) by the formula

$$\delta E = \frac{1}{2}\mathrm{Sp}_\sigma \int \varepsilon(\mathbf{p}, \sigma)\,\delta n(\mathbf{p}, \sigma)\,d\tau,$$

$$d\tau = 2dp_x\,dp_y\,dp_z/(2\pi\hbar)^3,$$

where E is the energy per unit volume, n the distribution function occurring in the spin statistical operator relation. The quantity $\epsilon(\mathbf{p}, \sigma)$ is in general a complicated function of the quasi-momentum and is an operator, as is n, depending on the spin operators (here and henceforth we shall mean by σ_i the Pauli matrices). The electrons in a metal interact with one another and form the so-called electron Fermi liquid. According to Landau's theory, their interaction displays itself in particular in the fact that the energy $\epsilon(\mathbf{p}, \sigma)$ of each electron depends on the state of the other electrons and is a functional of the distribution function.

If the distribution function is slightly changed the electron energy is increased by

$$\delta\varepsilon(\mathbf{p}, \sigma) = \frac{1}{2}\mathrm{Sp}_{\sigma'} \int f(\mathbf{p}, \sigma;\ \mathbf{p}', \sigma')\,\delta n(\mathbf{p}', \sigma')\,d\tau', \tag{1}$$

where the function

$$f(\mathbf{p}, \sigma; \mathbf{p}', \sigma') = \delta\varepsilon(\mathbf{p}, \sigma)/\delta n(\mathbf{p}', \sigma')$$

is symmetrical with respect to an interchange of $\mathbf{p}, \sigma$ with $\mathbf{p}', \sigma'$ and in general can be written in the form

$$f(\mathbf{p}, \sigma; \mathbf{p}', \sigma') = \varphi(\mathbf{p}, \mathbf{p}') + \psi(\mathbf{p}, \mathbf{p}')(\sigma + \sigma') + \psi_{ik}(\mathbf{p}, \mathbf{p}')\,\sigma_i\sigma'_k. \tag{2}$$

In a ferromagnetic metal where basically the electron interaction is purely an exchange interaction, the energy operator will depend only on the orientation of the spin with respect to the total magnetic moment. The electron energy is thus of the form

$$\varepsilon(\mathbf{p}, \sigma) = \alpha(\mathbf{p}) - \beta(\mathbf{p})(\mathbf{m}\cdot\sigma), \tag{3}$$

where $\mathbf{m}$ is a unit pseudovector in the direction of the magnetic moment of the crystal. Expression (3) does not take the anisotropy energy into account. However, in view of the smallness of the latter, we shall neglect the anisotropy terms, since they have no special interest for us.

According to Eq. (3) the energy of electrons with σ parallel to $\mathbf{m}$ is equal to $\alpha - \beta$ and the corresponding equilibrium distribution function is $n^+ = n_F(\alpha - \beta)$ (n_F is the usual Fermi function). The energy of electrons with σ antiparallel to $\mathbf{m}$ is equal to $\alpha + \beta$ and their equilibrium distribution function is $n^- = n_F(\alpha + \beta)$. One sees easily that the eigenvalues $n_F(\alpha \pm \beta)$ belong to the operator

$$n_0 = {}^1/_2(n^+ + n^-) + {}^1/_2(n^+ - n^-)(\mathbf{m}\cdot\sigma), \qquad (4)$$

which also follows from considering the equilibrium distribution function.

Neglecting anisotropy effects, Eq. (2) also simplifies and takes on the form

$$f(\mathbf{p}, \sigma; \mathbf{p}', \sigma') = \psi(\mathbf{p}, \mathbf{p}') + \varphi(\mathbf{p}, \mathbf{p}')\,\mathbf{m}\cdot(\sigma + \sigma') + \zeta(\mathbf{p}, \mathbf{p}')(\sigma\cdot\sigma') + \xi(\mathbf{p}, \mathbf{p}')(\mathbf{m}\cdot\sigma)(\mathbf{m}\cdot\sigma'). \qquad (2')$$

There is a connection which determines β of Eq. (3) from ζ in (2′). To establish this connection* we consider how the electron energy changes when the magnetic moment is rotated over an angle $\delta\boldsymbol{\theta}$ with respect to the crystalline axes. Then $\delta\mathbf{m} = [\delta\boldsymbol{\theta} \times \mathbf{m}]$ and we have according to (3)

$$\delta\varepsilon = -\beta\,[\mathbf{m}\times\sigma]\,\delta\boldsymbol{\theta}. \qquad (5)$$

On the other hand, the equilibrium distribution function n_0 changes also when $\mathbf{m}$ changes:

$$\delta n_0 = {}^1/_2(n^+ - n^-)[\mathbf{m}\times\sigma]\,\delta\boldsymbol{\theta},$$

and with it, according to (1), the energy; namely,

$$\delta\varepsilon = {}^1/_2\,\mathrm{Sp}_{\sigma'}\int f\,\frac{n^+ - n^-}{2}\,[\mathbf{m}\times\sigma']\,\delta\boldsymbol{\theta}\,d\tau'. \qquad (6)$$

Comparing Eqs. (5) and (6) for arbitrary $\delta\boldsymbol{\theta}$ we get

$$-\beta\,[\mathbf{m}\times\sigma] = {}^1/_2\,\mathrm{Sp}_{\sigma'}\int f\,\frac{n^+ - n^-}{2}\,[\mathbf{m}\times\sigma']\,d\tau',$$

whence we find

$$\beta(\mathbf{p}) = -{}^1/_2\int \zeta(\mathbf{p}, \mathbf{p}')(n^+ - n^-)\,d\tau'. \qquad (7)$$

To obtain the magnetic spectrum we shall consider the oscillations of a system of electrons. They are described by the transport equation

$$\frac{\partial n}{\partial t} + \frac{\partial\varepsilon}{\partial\mathbf{p}}\frac{\partial n}{\partial\mathbf{r}} - \frac{\partial\varepsilon}{\partial\mathbf{r}}\frac{\partial n}{\partial\mathbf{p}} + i\,[\varepsilon, n] = 0. \qquad (8)$$

Here $[\epsilon, n]$ denotes a commutator and we have put $\hbar = 1$. The difference between Eq. (8) and the usual transport equation consists in the fact that ϵ is here a functional of the distribution function. This must be taken into account when one differentiates ϵ with respect to $\mathbf{p}$ and $\mathbf{r}$.

Let $n = n_0 + \delta n$ where n_0 is the equilibrium function (4) and where δn depends on the coordinates and the time in the form $\exp\{i[(\mathbf{k}\cdot\mathbf{r}) - \omega t]\}$. We shall write δn as a sum of a term depending on and one independent of the spin,

$$\delta n = \nu(\mathbf{p}) + (\boldsymbol{\nu}(\mathbf{p})\cdot\sigma). \qquad (9)$$

*The idea of the derivation given here stems from L. D. Landau.

Substituting (9) into (8) and taking into account that according to (1) the energy of the excitations is equal to

$$\varepsilon + \frac{1}{2}\,\mathrm{Sp}_{\sigma'}\int f\,\delta n\,d\tau',$$

with ϵ from (3), we get after a simple transformation, retaining only the terms of first order in δn a set of equations for ν and $\boldsymbol{\nu}$.

This set of equations falls into two parts.* The first one corresponds to oscillations of the electron density and the oscillations of the spin component along the direction of the magnetic moment, $\nu_z = (\boldsymbol{\nu}\cdot\mathbf{m})$ connected with them. However, since oscillations of this kind are accompanied by a change in charge density, their occurrence will be connected with the appearance of strong electrostatic forces. Consequently excitation of such oscillations requires in actual fact a large energy and we need not consider them. Only the second set of equations is of interest to us; it corresponds to oscillations of the transverse components of the total spin.

We note that in non-ferromagnetic metals when the terms in $\mathbf{m}$ are absent, oscillations of the quantity $\boldsymbol{\nu}$ are not connected with density oscillations and satisfy the equation found by Landau[5]

$$-i\omega\boldsymbol{\nu} + i(\mathbf{k}\cdot\mathbf{u})\boldsymbol{\nu} + i(\mathbf{k}\cdot\mathbf{u})\,\delta(\alpha - \eta)\int \zeta\boldsymbol{\nu}\,d\tau' = 0,$$

where $\mathbf{u} = \partial\alpha/\partial\mathbf{p}$ and η is the limiting Fermi energy. From this equation it is clear that in such metals the magnetic spectrum is described by excitations with a dispersion law $\epsilon \sim k$.

Instead of the transverse components ν_x and ν_y we introduce the quantities $\nu_\pm = \nu_x \pm i\nu_y$. From the set of equations for ν_x and ν_y we get then for ν_+

$$-\omega\nu_+ + (\mathbf{k}\cdot\mathbf{u})\nu_+ + {}^1/_2\{\mathbf{k}\cdot(\mathbf{u}-\mathbf{v})\,\delta(\alpha - \beta - \eta) + \mathbf{k}\cdot(\mathbf{u}+\mathbf{v}) \times \delta(\alpha + \beta - \eta)\}\int \zeta\nu_+ d\tau' - 2\beta\nu_+ - (n_+ - n_-)\int \zeta\nu_+ d\tau' = 0, \qquad (10)$$

where $\mathbf{v} = \partial\beta/\partial\mathbf{p}$. The spectrum of this equation gives the dispersion law for the magnetic excitations of a ferromagnetic metal (the equation for ν_- differs in the sign of the least two terms and leads to the same spectrum).

We shall solve Eq. (10) by successive approximations. For $\mathbf{k} = 0$ we have

$$\omega^{(0)}\nu^{(0)} + 2\beta\nu^{(0)} + (n_+ - n_-)\int \zeta\nu^{(0)} d\tau' = 0. \qquad (11)$$

Integrating (11) over $d\tau$ and taking the connection (7) between β and ζ into account we get $\omega^{(0)} = 0$ and

$$\nu^{(0)} = A(n_+ - n_-),$$

*It was shown by Silin[6] that the same situation occurs when one considers spin waves in a Fermi liquid when an external magnetic field is present.

where A is a constant. The change in the distribution function is in this order thus proportional to $(n^+ - n^-)\sigma_x$, or $(n^+ - n^-)\sigma_y$. Comparing this with Eq. (4) we see easily that in this case the oscillations lead to a rotation of the total magnetic moment.

We now write down the equation for the next approximation

$$A\omega^{(1)}(n^+ - n^-) + 2\beta\nu^{(1)} + (n^+ - n^-)\int \zeta\nu^{(1)}d\tau' = A(\mathbf{k}\cdot\mathbf{u})(n^+ - n^-) - A\beta\{\mathbf{k}\cdot(\mathbf{u}-\mathbf{v})\delta(\alpha - \beta - \eta) + \mathbf{k}\cdot(\mathbf{u}+\mathbf{v})\delta(\alpha + \beta - \eta)\}. \quad (12)$$

We integrate this over $d\tau$, using again condition (7) and the fact that α and β are even functions of the quasi-momentum. This leads to $\omega^{(1)} = 0$, i.e., ω is at least of second order in k.

According to Eq. (12), the change in the distribution function consists of three parts. One differs from zero in the same interval as $n_+ - n_-$, and the other two are different from zero only on the corresponding Fermi surface and are proportional to $\delta(\alpha - \beta - \eta)$ and $\delta(\alpha + \beta - \eta)$, respectively. If we denote the quantity $\nu^{(1)}/A$ (this quantity does not depend on A) by ν_1 and write ν_1 in the form

$$\nu_1 = B(\mathbf{p})(n_+ - n_-) - {}^1/_2\{\mathbf{k}\cdot(\mathbf{u}-\mathbf{v})\delta(\alpha - \beta - \eta) + \mathbf{k}\cdot(\mathbf{u}+\mathbf{v})\delta(\alpha + \beta - \eta)\}, \quad (13)$$

we get for the function $B(\mathbf{p})$ the equation

$$2\beta B + \int \zeta B'(n^+ - n^-)d\tau' = \mathbf{k}\cdot\mathbf{u} + \frac{1}{2}\int\Big[\zeta(\mathbf{p},\mathbf{p}')\mathbf{k}\cdot(\mathbf{u}'-\mathbf{v}')\frac{d\tau'}{d\varepsilon'}\Big]_{\alpha'-\beta'=\eta} + \frac{1}{2}\int\Big[\zeta(\mathbf{p},\mathbf{p}')\mathbf{k}\cdot(\mathbf{u}'+\mathbf{v}')\frac{d\tau'}{d\varepsilon'}\Big]_{\alpha'+\beta'=\eta}. \quad (14)$$

In the case where $\zeta(\mathbf{p},\mathbf{p}')$ is an arbitrary function of the quasi-momenta $\mathbf{p}$ and $\mathbf{p}'$, the last equation cannot be solved explicitly for $B(\mathbf{p})$.

To find the connection between ω and $\mathbf{k}$ we write down the equation for $\nu^{(2)}$ and integrate it over $d\tau$; this leads to

$$\omega^{(2)} = \Big\{\int \mathbf{k}\cdot\mathbf{u}\,\nu_1 d\tau + \frac{1}{2}\iint[\mathbf{k}\cdot(\mathbf{u}-\mathbf{v})\delta(\alpha - \beta - \eta) + \mathbf{k}\cdot(\mathbf{u}+\mathbf{v})\delta(\alpha + \beta - \eta)]\zeta\nu_1' d\tau d\tau'\Big\}\Big/\int(n^+ - n^-)d\tau. \quad (15)$$

Substituting Eq. (13) into this expression and transforming it by means of Eq. (14) we get

$$\omega^{(2)} = \Big\{\int \mathbf{k}\cdot\mathbf{u}(n^+ - n^-)B(\mathbf{p})d\tau - \int\Big[\beta B(\mathbf{p})\mathbf{k}\cdot(\mathbf{u}-\mathbf{v})\frac{d\tau}{d\varepsilon}\Big]_{\alpha-\beta=\eta} - \int\Big[\beta B(\mathbf{p})\mathbf{k}\cdot(\mathbf{u}+\mathbf{v})\frac{d\tau}{d\varepsilon}\Big]_{\alpha+\beta=\eta}\Big\}\Big/\int(n^+ - n^-)d\tau. \quad (16)$$

In view of the fact that B is of first order in $\mathbf{k}$ and clearly an odd function of the quasi-momentum it is evident that ω is a quadratic function of $\mathbf{k}$.

We have thus come to the rather natural conclusion that the model considered leads to the usual quadratic dispersion law.

Equations (14) and (16) determine completely the dependence $\omega(\mathbf{k})$. In the simplest case, $\zeta = \text{const}$, Eq. (14) can be solved for B. Indeed, in view of the fact that B is an odd function of the quasi-momentum and that n^+ and n^- are even functions, all integrals in Eq. (14) tend to zero. It is thus shown that $B = (\mathbf{k}\cdot\mathbf{u})/2\beta$. Substituting this into Eq. (16) and taking into account the fact that according to Eq. (7) for $\zeta = \text{const}$, β is also a relative constant and $\mathbf{v} = \partial\beta/\partial\mathbf{p} = 0$, we find

$$\omega = \Big\{\int(\mathbf{k}\cdot\mathbf{u})^2(n^+ - n^-)d\tau - \frac{\beta}{2}\int\Big[(\mathbf{k}\cdot\mathbf{u})^2\frac{d\tau}{d\varepsilon}\Big]_{\alpha=\eta+\beta} - \frac{\beta}{2}\int\Big[(\mathbf{k}\cdot\mathbf{u})^2\frac{d\tau}{d\varepsilon}\Big]_{\alpha=\eta-\beta}\Big\}\Big/2\beta\int(n^+ - n^-)d\tau. \quad (17)$$

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[1] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 1058 (1956), Soviet Phys. JETP **3**, 920 (1956).
[2] F. Bloch, Z. Physik, **61**, 206 (1930).
[3] L. D. Landau and E. M. Lifshitz, Physik. Z. Sowjetunion **8**, 153 (1935).
[4] E. M. Lifshitz, J. Exptl. Theoret. Phys. (U.S.S.R.) **15**, 97 (1945), J. Phys. (U.S.S.R.) **8**, 337 (1944).
[5] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 59 (1957), Soviet Phys. JETP **5**, 101 (1957).
[6] V. P. Silin, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 1227 (1957), Soviet Phys. JETP **6**, 945 (1958).

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