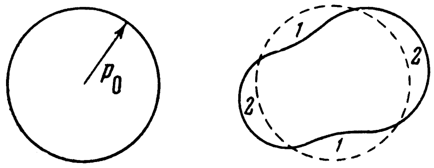

a b

(a) unperturbed distribution (all states inside occupied). (b) perturbed distribution; in regions 1, $\delta n = -1$, in regions 2, $\delta n = 1$; inside the solid line all states are occupied.

Let us expand the momentum p corresponding to the solid line of diagram b in a series of spherical surface harmonics:

$$p = p_0 + \sum_{lm} \Phi_{lm} Y_{lm}(\theta,\varphi), \quad Y_{l0}(0) = 1. \qquad (2)$$

The first term in Eq. (1) takes the following form:

$$\frac{2v_0 p_0^2}{(2\pi\hbar)^3} \int do \int_0^{p-p_0} (p' - p_0)dp' = \frac{p_0^2 v_0}{(2\pi\hbar)^3} \sum_{lm} \Phi^2_{lm} \frac{4\pi}{2l+1} \frac{(l+m)!}{(l-m)!} \qquad (3)$$

The potential energy reduces to the expression

$$2p_0^4/(2\pi\hbar)^{-6} \int\int [p(\theta_1,\varphi_1) - p_0][p(\theta_2,\varphi_2) - p_0]\, do_1 do_2 f(\theta_{12}), \qquad (4)$$

$$\cos\theta_{12} = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2).$$

We now expand $f(\theta_{12})$ in a series of surface harmonics:

$$f(\theta_{12}) = \sum_l f_l P_l(\cos\theta_{12})$$

$$= \sum_l \sum_{m'}^{l} Y_{lm'}(\theta_1,\varphi_1) Y_{lm'}(\theta_2,\varphi_2) \frac{(l-m')!}{(l+m')!} f_l, \qquad (5)$$

$$f_l = \frac{2l+1}{4\pi} \int f(\theta) P_l(\cos\theta)\, do.$$

Substituting Eq. (5) into (4) and combining with Eq. (3), we get the total energy functional

$$E = \frac{p_0^2 v_0}{(2\pi\hbar)^3} \sum \Phi^2_{lm} \frac{4\pi}{2l+1} \frac{(l+m)!}{(l-m)!}$$

$$+ \frac{2p_0^4}{(2\pi\hbar)^6} \sum_{lm} \Phi^2_{lm} \left(\frac{4\pi}{2l+1}\right)^2 f_l \frac{(l+m)!}{(l-m)!}. \qquad (6)$$

The conditions for stability are written separately for each l, m:

$$1 + \frac{8\pi}{2l+1} \frac{p_0^2 f_l}{v_0(2\pi\hbar)^3} > 0, \quad 1 + \frac{2p_0^2}{v_0(2\pi\hbar)^3} \int f P_l(\cos\theta)\, do > 0. \qquad (7)$$

For $l = 0$ and $l = 1$ these conditions agree with the conditions (11) and (18) of reference 1, where they characterized the positiveness of the square of the speed of sound and of the effective mass.

So far we have supposed that $f(\mathbf{p}, \mathbf{p}')$ does not depend on the mutual orientation of the spins of the excitations. If we take into account this dependence, then instead of f we have the function

$$f(\mathbf{pp}') + (\sigma_1\sigma_2)\, g(\mathbf{p}_1\mathbf{p}_2). \qquad (8)$$

The perturbation of the Fermi surface is now expressed by the formula

$$p = p_0 + \sum_{lm} \sigma \Phi_{lm} Y_{lm}(\theta,\varphi). \qquad (9)$$

instead of Eq. (2). Since E involves the trace with respect to σ and σ', we get instead of Eq. (6):

$$E = \frac{p_0^2 v_0}{2(2\pi\hbar)^3} \sum_{lm} \Phi^2_{lm} \frac{4\pi}{2l+1} \frac{(l+m)!}{(l-m)!}$$

$$+ \frac{p_0^4}{2(2\pi\hbar)^6} \sum_{lm} \Phi^2_{lm} \left(\frac{4\pi}{2l+1}\right)^2 \frac{(l+m)!}{(l-m)!} g_l, \qquad (10)$$

where g_l are the expansion coefficients of the function $g(\mathbf{p}_1\mathbf{p}_2) = g(\cos\theta_{12})$ (all taken from the Fermi surface) in a series of Legendre polynomials:

$$g(\cos\theta_{12}) = \sum_l g_l P_l(\cos\theta_{12}). \qquad (11)$$

The stability conditions following from Eq. (10) have the form

$$1 + \frac{p_0^2}{v_0(2\pi\hbar)^3} \int g(\cos\theta) P_l(\cos\theta)\, do > 0. \qquad (12)$$

For $l = 0$ this condition means the absence of ferromagnetism and is contained in the condition $\chi > 0$ (Eq. (26) of reference 1).

In conclusion I express my gratitude to L. D. Landau for a discussion of this note.

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ON THE MECHANISM OF THE DAMPING OF FREE OSCILLATIONS IN A CYCLIC ACCELERATOR

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REFERENCES 1 and 2 contained a statement about the absence of damping of betatron oscillations, which is connected with the emission of radiation in an electron accelerator. We show here that, in actual fact, both the normal adiabatic damping and the damping connected with emission of radiation occur when the beam is straightened when passing through the accelerating gaps.

Let an arbitrary number of accelerating gaps be distributed along the length of the accelerator ring. When a particle passes through a section with a uniform electric field directed along the ideal unperturbed equilibrium trajectory, the components of its momentum on the transverse axes $p_z = Ec^{-2}\,dz/dt$ and $p_r = Ec^{-2}\,dr/dt$ remain constant, where E is the total energy of the particle. Therefore $\dot{z}$ and $\dot{r}$ experience a jump $\Delta\dot{z} = -e\dot{z}U/E$, $\Delta\dot{r} = -e\dot{r}U/E$, where U is the potential difference passed through by the particle, $eU/E \ll 1$. Assuming the accelerating sections to be sufficiently short, one may assume $\Delta z = \Delta r = 0$. This consideration is not essential, but it corresponds to the usual condition of acceleration in a synchroton. The action of the gap is thus described by the matrix

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 - eU/E \end{pmatrix}, \quad z_1 = \Pi z_0, \quad z = \begin{pmatrix} z \\ dz/dt \end{pmatrix} \quad (1)$$

with the determinant $|\Pi_{ik}| = 1 - eU/E$. Outside the accelerating gaps the transition from the point $x_1 = vt_1$ to the point $x_2 = vt_2$ is described for free oscillations by a matrix M with determinant $|M_{ik}| = 1$. If in a time Δt the particle passes n times through an accelerating gap, then

$$z(t + \Delta t) = M_n\Pi_n \ldots M_2\Pi_2 M_1\Pi_1 z(t).$$

The determinant of the product of matrices is

$$|M_n\Pi_n \ldots M_1\Pi_1| = \left(1 - \frac{eU_1}{E_1}\right)\left(1 - \frac{eU_2}{E_2}\right)\cdots\left(1 - \frac{eU_n}{E_n}\right) \approx 1 - \frac{edU/dt}{E(t)}\Delta t, \quad \frac{eU_k}{E} \ll 1, \quad (2)$$

where dU/dt is the potential difference passed through by the particle in unit time. It is well known that the determinant of the matrix of the transition from t to $t + \Delta t$ is equal to the ratio of the phase volumes. From expression (2) it is clear that the phase volume for the variables $z, \dot{z}$ or $r, \dot{r}$ is decreased for each passage through an accelerating gap. From (2) it follows that the free transverse oscillations are damped according to the relation

$$z \sim \exp\left\{-\frac{1}{2}\int_0^t \frac{edU/dt}{E(t)}\,dt\right\}. \quad (3)$$

In an electron accelerator we have $e\,dU/dt = \dot{\bar{E}} + \bar{P}_\gamma$, where $\dot{\bar{E}}$ is the average energy increment of the particle, $\bar{P}_\gamma$ the average power spent in compensating for the radiation losses. In a proton accelerator $\bar{P}_\gamma = 0$ and Eq. (3) gives (taking the change of frequency into account) for the oscillation amplitude $A(t)/A(0) = (p(0)/p(t))^{1/2}$.

The clearcut considerations given here confirm the correctness of the equations for the vertical and radial oscillations for an electron accelerator obtained by Kolomenskii.[3]

The damping mechanism just described does not pertain to forced radial-phase oscillations. The mechanism of damping of synchrotron oscillations is essentially different. In particular, radiation leads directly to the damping of phase oscillations, thanks to the fact that the radiation intensity is proportional to E^2. If the particle energy is increased by an amount δE, the average radiation power is also increased by an amount $2\bar{P}_\gamma\delta E/E$ so that the phase oscillations are damped according to the rule

$$\varphi \sim \exp\left\{-\int_0^t \frac{\bar{P}_\gamma}{E}\,dt\right\},$$

in agreement with the equations obtained in references 3 and 4.

The presence of a connection between radial and phase oscillations which shows up on the one hand in the fact that $\bar{P}_\gamma$ is proportional to H^2, $H = H(r)$, and on the other hand in the appearance of a term $\delta p/p \sim d\varphi/dt$ in the equation for the radial oscillations, leads to a redistribution of the damping decrements while their sum remains equal to its original value.[3-5] In the usual accelerator with fixed focusing, the radial-phase connection leads to unstable radial oscillations.[3] The most convenient method to damp the radial oscillations is, apparently, the variation of the magnetic field along the orbit.[5]

The fact that the energy loss by radiation does not directly influence the velocity of damping of the free transverse oscillations (as follows, in particular, from reference 6) is connected with the fact that the radiation is directed on the average along the motion of the particle so that the particle receives at the moment of emission a transverse recoil momentum proportional to dz/dx: $c\Delta p_2 = -\epsilon\,dz/dx$, where ϵ is the energy of the quantum. Hence $\Delta(E\,dz/dx) = -\epsilon\,dz/dx$, $x = ct$, which gives $(\Delta dz/dx)_{rad} = 0$, while at the same time we have for the passage through an accelerating gap $\Delta dz/dx = -(eU/E)\,dz/dx$.

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PHENOMENOLOGICAL EQUATIONS OF STATISTICAL DYNAMICS OF AN INCOMPRESSIBLE TURBULENT LIQUID

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A closed system of differential equations, which describe statistically the turbulent flow of a liquid, can be obtained only by limiting the number of variables that characterize the motion of the liquid. The appropriate variables to take here are the mean velocity $U_i = \overline{u}_i$ at a given point, and the mean pressure $P = \overline{p}$, and also the (one-sided) second moments of the fluctuating velocity $R_{ij} = v_i v_j$, and the turbulent viscosity N.

The equations for U_i are obtained from the usual averaging of the equations of hydrodynamics:

$$\frac{dU_i}{dt} + \frac{\partial R_{ik}}{\partial x_k} + \frac{\partial P}{\partial x} = 0, \quad \frac{\partial U_k}{\partial x_k} = 0. \tag{1}$$

In the averaging of the equation for $v_i v_j$, obtained from the Navier-Stokes equation, there appear third moments of the fluctuating velocity $v_i v_j v_k$, and also moments that contain derivatives with respect to the coordinates. All these moments can be expressed phenomenologically, in first approximation, in terms of the variables enumerated above and their derivatives. The requirements of dimensionality, tensor invariance, and parity must be satisfied here. If tensor combinations are chosen and known experimental data[1] are followed, then a quasi-diffusion equation is obtained for R_{ij}:

$$\begin{aligned}\frac{dR_{ij}}{dt} &- \frac{\partial}{\partial x_k}\left[\frac{1}{3} N\left(\delta_{ij}\frac{\partial R}{\partial x_k} + \delta_{jk}\frac{\partial R}{\partial x_i} + \delta_{ki}\frac{\partial R}{\partial x_j}\right) - \alpha R\left(\delta_{ij}\frac{\partial N}{\partial x_k} + \delta_{jk}\frac{\partial N}{\partial x_i} + \delta_{ki}\frac{\partial N}{\partial x_j}\right)\right] \\ &+ \frac{\partial}{\partial x_i}\left(\frac{1}{3} N\frac{\partial R}{\partial x_j} - \frac{5\alpha}{2} R\frac{\partial N}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{3} N\frac{\partial R}{\partial x_i} - \frac{5\alpha}{2} R\frac{\partial N}{\partial x_i}\right) \\ &+ R_{ik}\frac{\partial U_j}{\partial x_k} + R_{jk}\frac{\partial U_i}{\partial x_k} + \frac{\beta}{N}\left(R_{ik}R_{jk} - \frac{\delta_{ij}}{3}R_{kl}^2\right) + \frac{\delta_{ij}}{3}\frac{\gamma R^2}{N} = 0.\end{aligned} \tag{2}$$

Here $R = R_{kk}$. The physical meaning of the separate terms is as follows: the second term is connected with the transfer of turbulence energy ($v_i v_j v_k$ is contained in the square brackets); this transfer is directed, as is well known, from the walls to the middle of the flow. The maximum of R on the boundary between the turbulent boundary layer and the laminar underlayer serves as the source. The third and fourth terms take into account the transfer of energy by the fluctuating pressure; it proceeds in the opposite direction and partially cancels the second term. In summation, all these terms give:

$$-\frac{\delta_{ij}}{3}\frac{\partial}{\partial x_k}\left(N\frac{\partial R}{\partial x_k}\right) + \alpha\delta_{ij}\frac{\partial}{\partial x_k}\left(R\frac{\partial N}{\partial x_k}\right) - \frac{3\alpha}{2}\left[\frac{\partial}{\partial x_i}\left(R\frac{\partial N}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(R\frac{\partial N}{\partial x_i}\right)\right].$$

Further, the increase of the turbulent energy derives from the gradient of the mean velocity; it is obtained directly from the Navier-Stokes equation. The next-to-last term of the equation describes the shift of the anisotropy of the turbulence due to the scattering of the fluctuating velocity on the fluctuating pressure; finally, the last term represents the viscous dissipation of energy in fine-grained turbulence. The dimensional universal constants α, β, and γ must be determined by experiment.

With regard to the turbulent viscosity, in first approximation, for each given flow, we must set N = const everywhere, with the exception of the boundary layer. In the turbulent boundary layer, N falls off with approach to the wall. In this case, however, there is an added condition: the divergence of the total flow of turbulent energy is practically equal to zero:

$$\frac{\partial}{\partial x_k}\left(N\frac{\partial R}{\partial x_k}\right) = 0. \tag{3}$$

This also gives the lacking equation for the determination of the value of N in the boundary layer.

As a condition for joining the solution of the equation for the two regions [the middle part of the flow (N = const) and the boundary layer — condition (3)] we use the requirement of continuity of all functions and their first derivatives. On the