being described appears in processes which take place in the presence of vortices in which high gradients of ρ_s exist. Under these conditions the quantum terms in the equations turn out to be important. The equations which have been derived make it possible to compute the coefficients B and B^1 of Hall and Vinen[9] because of friction between the superfluid part and the normal part in the presence of a vortex; it is proposed to carry out this calculation in a subsequent paper.

In conclusion the authors wish to express their gratitude to L. D. Landau and V. L. Ginzburg for help and advice in the course of this work.

[1] V. L. Ginzburg and L. P. Pitaevskii, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 1240 (1958), Soviet Phys. JETP **7**, 858 (1958).

[2] L. D. Landau and E. M. Lifshitz, Статистическая физика (Statistical Physics) Gostekhizdat (1951) Chapter XIV.

[3] L. D. Landau and E. M. Lifshitz, Механика сплошных сред (Mechanics of Continuous Media) 2nd edition, Gostekhizdat (1954) Chapter XVI.

[4] I. M. Khalatnikov, Usp. Fiz. Nau, **60**, 69 (1956).

[5] I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **23**, 265 (1952).

[6] L. D. Landau and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR, **96**, 469 (1954).

[7] I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **23**, 8 (1952).

[8] C. E. Chase, Proc. Roy. Soc. (London) **220A**, 116 (1953).

[9] H. E. Hall and W. F. Vinen, Proc. Roy. Soc. (London) **238A**, 204 (1956).

Translated by H. Lashinsky
73

SOVIET PHYSICS JETP VOLUME 35(8), NUMBER 2 FEBRUARY, 1959

THE SPECTRAL REPRESENTATION OF THE TWO-MESON GREEN'S FUNCTION

V. N. GRIBOV

Leningrad Physico-Technical Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor March 13, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 416-427 (August, 1958)

A spectral representation has been found for the two-meson Green's function which is analogous to that found earlier[1-3] for the vertex part. The analytic properties of this function are examined. It follows from the representations that dispersion relations exist for the scattering amplitude for a fixed scattering angle in the center-of-mass system. These relations are obtained and discussed.

1. INTRODUCTION

IN a previous paper by the writer[3] the causality conditions were used to obtain the spectral representation of the vacuum expectation value of the T-product of three Heisenberg operators. In the present paper this same method is used to obtain the spectral representation of the vacuum expectation value of the T-product of four operators, i.e., for the two-particle Green's function. In order not to encumber the discussion with elaborate computations associated with the spinor character of the meson-nucleon Green's function and its possession of poles, we shall confine ourselves in this paper to the derivation of the spectral representation for the two-meson Green's function and the study of the analytical properties of the meson-meson scattering amplitude.

From the spectral representation it follows that the scattering amplitude is an analytic function in the upper half-plane and has no essential singularity at infinity, not only for a fixed preassigned momentum, but also for a fixed scattering angle in the center-of-mass system. Owing to this, it is possible to write dispersion relations for a fixed value of the scattering angle. These relations are written out, but they contain the scattering amplitude in the region of imaginary angles. Analytic continuation into this region by means of expansion in Legendre polynomials is evidently impossible.

2. THE SPECTRAL REPRESENTATION FOR $\langle| Tj_i(x_1) j_j(x_2) j_k(x_3) j_l(x_4) |\rangle$

Let us consider the symmetrical pseudoscalar meson theory. In this theory one ordinarily takes the two-meson Green's function to mean the expression

$$\langle| T\varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3)\varphi_l(x_4)|\rangle,$$

with terms corresponding to unconnected diagrams removed. The ψ_i are meson field operators, which can be taken to be Hermitian. This quantity, however, does not occur directly in the scattering amplitude, and it contains singularities not related to the interaction of the mesons. A more interesting quantity, with its Fourier components proportional to the matrix elements of the S matrix is (cf. reference 4)

$$K_1 K_2 K_3 K_4 \langle| T\varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3)\varphi_l(x_4)|\rangle = \tau_{ijkl}(x_1, x_2, x_3, x_4), \qquad K_1 = \Box_1 - \mu^2. \tag{1}$$

The differentiations in this expression are easily performed if we use the formula

$$\begin{aligned} K_1 \langle| T\varphi_i(x_1) A(x_2) B(x_3) C(x_4)|\rangle &= \langle| Tj_i(x_1) A(x_2) B(x_3) C(x_4)|\rangle \\ &+ \delta(t_1 - t_2)\langle| T[\dot{\varphi}_i(x_1) A(x_2)] B(x_3) C(x_4)\rangle + (2\to 3) + (2\to 4), \qquad j_i(x_1) = K_1\varphi_i(x_1), \end{aligned} \tag{2}$$

for which the proof is simple if we use the fact that the operators commute on a space-like surface. Here A, B, C are arbitrary operators.

Using Eq. (2) and dropping terms that are of the form $f_1(x_1 - x_2)\cdot f_2(x_3 - x_4)$ (these correspond to unconnected diagrams and make no contribution to the scattering amplitude), we get after simple calculations:

$$K_1 K_2 K_3 K_4 \langle| T\varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3)\varphi_l(x_4)|\rangle = \langle| Tj_i(x_1) j_j(x_2) j_k(x_3) j_l(x_4)|\rangle + \ldots \tag{3}$$

The terms not written out here are given in detail in Appendix I and correspond to diagrams in which two, three, or four external meson lines emerge from a single point. All these terms can be expressed in terms of vacuum expectation values of two or three operators for which the spectral representations are known. We note that they are all equal to zero if there is no contact interaction of the mesons described by a term $\lambda\varphi_i\varphi_k\varphi_k$ in the current j_i.

We assume that terms corresponding to the mass renormalization and a term $\lambda\varphi^4$ are introduced into the Hamiltonian, but the multiplicative renormalization is carried out only in the final result.

We note that the essential part of all the following does not depend on the explicit form of the interaction.

To establish the spectral representation for $\langle| Tj_i(x_1) j_j(x_2)\cdot j_k(x_3) j_l(x_4)|\rangle$ we first consider, as in reference 3, the simple product:

$$\begin{aligned} \langle| j_k(x_1) j_j(x_2) j_k(x_3) j_l(x_4)|\rangle &= \int d^4p_1\, d^4p_4\, d^4p \exp\{ip_1(x_1 - x_2) + ip(x_2 - x_3) \\ &+ ip_4(x_3 - x_4)\}\, \vartheta(p_{10})\,\vartheta(p_{40})\,\vartheta(p_0) \sum (j_i)_{0p_1}(j_j)_{p_1p}(j_k)_{pp_4}(j_l)_{p_40}, \qquad (j_j)_{p_1p} = \langle p_1 | j_j(0) | p\rangle. \end{aligned} \tag{4}$$

The summation is taken over all states with given p_1, p, p_4. In virtue of the properties of the operators $j_i(x)$ the smallest possible values of $-p_1^2$ and $-p_4^2$ correspond to three-meson states, and the four-dimensional momentum p can correspond either to the vacuum state or to states beginning with the two meson states. To the vacuum state there corresponds the term in Eq. (4)

$$\delta(p)\left[\sum (j_i)_{0p_1}(j_j)_{p_10}\right]\left[\sum (j_k)_{0p_4}(j_l)_{p_40}\right].$$

For what follows it is convenient to split off this term and consider it separately. Individually, however, this term does not satisfy the causality condition (that the operators commute on a space-like surface). Therefore we also separate from Eq. (4) a number of further terms, which will satisfy the causality condition together with the above term. If we imagine the operators $j_i(x_1)$, $j_j(x_2)$, $j_k(x_3)$, and $j_l(x_4)$ expanded in terms of creation and annihilation operators, then the term that has been written out corresponds to those terms of Eq. (4) in which the creation and annihilation operators of $j_i(x_1)$ are contracted completely with corresponding operators of $j_j(x_2)$ and are not contracted with any others. There also exist other terms in which the operators of $j_i(x_1)$ are contracted only with operators of $j_k(x_3)$, or only with those of $j_l(x_4)$. They contain the respective factors $\delta(p-p_1-p_4)$ and $\delta(p_1-p_4)$. Separating out these terms also, we can write:

$$\sum (j_i)_{0p_1}(j_j)_{p_1p}(j_k)_{pp_4}(j_l)_{p_40} = \frac{\delta(p)}{(2\pi)^6}\rho_{ij}(-p_1^2)\rho_{kl}(-p_4^2) + \frac{\delta(p-p_1-p_4)}{(2\pi)^6}\rho_{ik}(-p_1^2)\rho_{jl}(-p_4^2) + \frac{\delta(p_1-p_4)}{(2\pi)^6}\rho_{il}(-p_1^2)\rho_{jk}(-p^2)$$
$$+\frac{1}{(2\pi)^{18}}\rho_{ijlk}(p_1^2, p^2, p_4^2, (p_1-p)^2, (p_1-p_4)^2, (p-p_4)^2); \qquad \rho_{ij}(-k^2) = (2\pi)^{-3}\sum (j_i)_{0k}(j_j)_{k0}. \tag{5}$$

After the vacuum state has been separated out, the smallest value of $-p^2$ belongs to a two-meson state, and consequently

$$\rho_{ijlk}(p_1^2, p^2, p_4^2, (p-p_1)^2, (p_1-p_4)^2, (p-p_4)^2) = 0,$$

if one of the inequalities

$$-p_1^2 < 9\mu^2, \qquad -p_4^2 < 9\mu^2, \qquad -p^2 < 4\mu^2. \tag{6}$$

is satisfied. Substituting Eq. (5) into Eq. (4), and Eq. (4) into the causality condition, written in the form (for example for $x_{12}^2 > 0$)

$$\langle | j_i(x_1) j_j(x_2) j_k(x_3) j_l(x_4) |\rangle = \langle | j_j(x_2) j_i(x_1) j_k(x_3) j_l(x_4) |\rangle, \tag{7}$$

one can easily show that the terms that have been separated out satisfy this condition automatically. In addition they make no contribution to the scattering amplitude (they correspond to unconnected diagrams), and therefore they can be omitted from the further treatment.

In order to satisfy the causality condition, in analogy with reference 3 we shall write $\rho_{ijlk}(p_1^2, p^2, p_4^2, (p_1-p)^2, (p_1-p_4)^2, (p-p_4)^2)$ in the form:

$$\rho_{ijlk}(p_1, p, p_4) = \int \vartheta(k_{12})\vartheta(k_{13})\vartheta(k_{14})\vartheta(k_{23})\vartheta(k_{24})\vartheta(k_{34})\; f_{ijlk}(-k_{12}^2, -k_{13}^2, \ldots -k_{34}^2)\,\delta(p_1-k_{12}-k_{13}-k_{14})$$
$$\times\,\delta(p-k_{13}-k_{14}-k_{23}-k_{24})\,\delta(p_4-k_{14}-k_{24}-k_{34})\,(dk); \qquad \vartheta(k) = \begin{cases}1 & k_0>0\\ 0 & k_0<0,\end{cases} \tag{8}$$

regarding this relation as an equation with respect to $f_{ijlk}(-k_{12}^2, -k_{13}^2, \ldots, -k_{34}^2)$.
From the invariance of ρ_{ijlk} it follows that $f_{ijlk} = 0$ if any one of its arguments is smaller than zero (if $k_{12}^2 > 0$, then $\vartheta(k_{12})$ is a noninvariant quantity).

We shall not carry out a detailed analysis of the equation (8), as was done at the corresponding point in the argument of reference 3, but we shall assume that it has a solution. We remark only that if we write Eq. (8) in the form

$$\rho_{ijlk}(p_1, p, p_4) = \int_0^\infty\!\!\cdot\!\cdot\!\int_0^\infty d\varkappa_{12}^2\ldots d\varkappa_{34}^2 K(p_1, p, p_4; \varkappa_{12}^2\ldots\varkappa_{34}^2)\, f_{ijlk}(\varkappa_{12}^2, \varkappa_{13}^2\ldots\varkappa_{34}^2);$$
$$K(p_1, p, p_4; \varkappa_{12}^2, \varkappa_{13}^2\ldots\varkappa_{34}^2) = \int \vartheta(k_{12})\delta(k_{12}^2+\varkappa_{12}^2)\,\vartheta(k_{13})\,\delta(k_{13}^2+\varkappa_{13}^2)\ldots$$
$$\ldots\vartheta(k_{34})\,\delta(k_{34}^2+\varkappa_{34}^2)\,\delta(p_1-k_{12}-k_{13}-k_{14})\,\delta(p-k_{13}-k_{14}-k_{23}-k_{24})\,\delta(p_4-k_{14}-k_{24}-k_{34})\,(dk), \tag{9}$$

then it is easy to see that $K(p_1, p, p_4, \varkappa_{12}^2\ldots\varkappa_{34}^2) \neq 0$ only if

$$-p_1^2 \geqslant (\varkappa_{12}+\varkappa_{13}+\varkappa_{14})^2; \; -p_4^2 \geqslant (\varkappa_{14}+\varkappa_{24}+\varkappa_{34})^2, \qquad -p_1^2 \geqslant (\varkappa_{13}+\varkappa_{14}+\varkappa_{23}+\varkappa_{24})^2. \tag{10}$$

In order that the function $\rho_{ijlk}(p_1, p, p_4)$ automatically satisfy the conditions (6), we set, as in reference 3,

$$f_{ijlk}(\varkappa_{12}^2, \varkappa_{13}^2, \ldots \varkappa_{34}^2) = 0,$$

whenever even a single one of the conditions

$$\varkappa_{12}+\varkappa_{13}+\varkappa_{14} \geqslant 3\mu; \; \varkappa_{13}+\varkappa_{14}+\varkappa_{23}+\varkappa_{24} \geqslant 2\mu; \; \varkappa_{14}+\varkappa_{24}+\varkappa_{34} \geqslant 3\mu; \tag{11}$$

is not satisfied.

Substituting Eq. (9) into Eq. (5) and then into Eq. (4), we get:

$$\langle | j_i(x_1) j_j(x_2) j_l(x_3) j_k(x_4) |\rangle = \int_0^\infty\!\!\cdot\!\cdot\!\int_0^\infty d\varkappa_{12}^2\ldots d\varkappa_{34}^2\,\Delta^+(x_{12}, \varkappa_{12})\ldots\Delta^+(x_{34}, \varkappa_{34})\; f_{ijlk}(\varkappa_{12}^2, \varkappa_{13}^2\ldots\varkappa_{34}^2);$$
$$\Delta^+(x_{12}, \varkappa_{12}) = (2\pi)^{-3}\int d^4k e^{ikx_{12}}\delta(k^2+\varkappa_{12}^2)\,\vartheta(k). \tag{12}$$

Substituting Eq. (12) into the causality condition (7) and using the fact that $\Delta^+(x, \kappa) = \Delta^+(-x, \kappa)$ for $x^2 > 0$, we get

$$\int_0^\infty \cdots \int_0^\infty dx_{12}^2 \dots dx_{34}^2 \Delta^+(x_{12}, \varkappa_{12}) \dots \Delta^+(x_{34}, \varkappa_{34}) [f_{ijlk}(\varkappa_{12}^2, \varkappa_{13}^2, \varkappa_{14}^2, \varkappa_{23}^2, \varkappa_{24}^2, \varkappa_{34}^2) - f_{jilk}(\varkappa_{12}^2, \varkappa_{23}^2, \varkappa_{24}^2, \varkappa_{13}^2, \varkappa_{14}^2, \varkappa_{34}^2)] = 0 \quad (13)$$

and other analogous conditions from which if follows that $f_{ijlk}(\kappa_{12}^2, \kappa_{13}^2, \dots k_{34}^2)$ is a symmetric function with respect to simultaneous interchange of any two of the indices $ijlk$ and the corresponding indices on the κ_{ik}.

From the symmetry of $f_{ijlk}(\kappa_{12}^2, \dots \kappa_{34}^2)$ and the conditions (11) it follows that $f_{ijlk}(\kappa_{12}^2, \kappa_{13}^2, \dots \kappa_{34}^2) = 0$ if any one of the conditions

$$\begin{gathered}\varkappa_{12} + \varkappa_{13} + \varkappa_{14} \geqslant 3\mu, \quad \varkappa_{12} + \varkappa_{23} + \varkappa_{24} \geqslant 3\mu, \quad \varkappa_{13} + \varkappa_{23} + \varkappa_{34} \geqslant 3\mu, \quad \varkappa_{14} + \varkappa_{24} + \varkappa_{34} \geqslant 3\mu, \\ \varkappa_{13} + \varkappa_{14} + \varkappa_{23} + \varkappa_{24} \geqslant 2\mu, \quad \varkappa_{12} + \varkappa_{13} + \varkappa_{24} + \varkappa_{34} \geqslant 2\mu, \quad \varkappa_{23} + \varkappa_{34} + \varkappa_{12} + \varkappa_{14} \geqslant 2\mu.\end{gathered} \quad (14)$$

is not satisfied.

We obtain further information about $f_{ijlk}(\kappa_{12}^2, \kappa_{13}^2, \dots \kappa_{34}^2)$ if we use the fact that the operators $\varphi_i(x)$, and consequently also $j_i(x)$, can be chosen to be Hermitian. With such a choice it follows from Eqs. (4) and (5) that

$$\rho_{ijlk}(p_1, p, p_4) = \rho^*_{klji}(p_4, p, p_1), \quad (15)$$

but from symmetry of $f_{ijlk}(\kappa_{11}^2, \dots \kappa_{34}^2)$ it follows that

$$\rho_{ijlk}(p_1, p, p_4) = \rho_{klji}(p_4, p, p_1). \quad (16)$$

Consequently the $\rho_{ijlk}(p_1, p, p_4)$ are real, and therefore the $f_{ijlk}(\kappa_{12}^2 \dots \kappa_{34}^2)$ can also be taken to be real.

Using the symmetry of the function $f_{ijlk}(\kappa_{12}^2, \kappa_{13}^2, \dots \kappa_{34}^2)$, we can go over to the representation of the vacuum expectation value of the T-product of the operators $j_i(x)$. To do this we have only to replace the functions $\Delta^+(x, \kappa)$ in Eq. (12) by functions

$$\Delta_F(x) = \vartheta(x)\Delta^+(x) + \vartheta(-x)\Delta^+(-x). \quad (17)$$

Making this replacement, we obtain the desired spectral representation in the coordinate space

$$\begin{gathered}\langle | T j_i(x_1) j_j(x_2) j_k(x_3) j_l(x_4) | \rangle = \int_0^\infty \cdots \int_0^\infty d\varkappa_{12}^2 \dots d\varkappa_{34}^2 f_{ijlk}(\varkappa_{12}^2, \varkappa_{13}^2 \dots \varkappa_{34}^2) \\ \times \Delta_F(x_{12}, \varkappa_{12}) \Delta_F(x_{13}, \varkappa_{13}) \Delta_F(x_{14}, \varkappa_{14}) \Delta_F(x_{23}, \varkappa_{23}) \Delta_F(x_{24}, \varkappa_{24}) \Delta_F(x_{34}, \varkappa_{34}).\end{gathered} \quad (18)$$

Equation (18) follows directly from Eqs. (12), (17) and the definition of the T-product when account is taken of the symmetry properties of $f_{ijlk}(\kappa_{12}^2, \dots \kappa_{34}^2)$.

To obtain the spectral representation in the momentum space it is necessary to calculate the following integral

$$(2\pi)^{-12} \int dk_{12} \dots dk_{34} \frac{\delta(l_1 - k_{12} - k_{13} - k_{14})\,\delta(l_2 + k_{12} - k_{13} - k_{14})\,\delta(l_3 + k_{13} + k_{23} - k_{34})\,\delta(l_4 + k_{14} + k_{24} + k_{34})}{(k_{12}^2 + \varkappa_{12}^2)(k_{13}^2 + \varkappa_{13}^2)(k_{14}^2 + \varkappa_{14}^2)(k_{23}^2 + \varkappa_{23}^2)(k_{24}^2 + \varkappa_{24}^2)(k_{34}^2 + \varkappa_{34}^2)} \quad (19)$$

which diverges logarithmically. Since, however, we assume that the meson-meson scattering amplitude exists, this divergence must not appear in the result. If to remove the divergence we introduce a cut-off radius Λ, then the term that goes to infinity for $\Lambda \to \infty$ will be proportional to $\ln \Lambda$ and will not depend on κ_{ik}. When substituted into the integral with respect to κ_{ik}^2 it gives a contribution proportional to

$$\ln\Lambda \int d\varkappa_{12}^2\, d\varkappa_{13}^2 \dots d\varkappa_{34}^2 f_{ijkl}(\varkappa_{12}^2, \varkappa_{13}^2, \dots \varkappa_{34}^2).$$

In order for this term to make no contribution to the scattering amplitude, it must either be equal to zero or else cancel with the infinite terms written out in Appendix 1.

As is shown in Appendix 2, the integral (19) can be brought into a simpler form. We thus get the following spectral representation in momentum space:

$$\tau_{ijkl}(l_1, l_2, l_3, l_4) = \frac{-i}{3(4\pi)^6}\int_0^\infty \cdot\cdot \int_0^\infty d\varkappa_{12}^2 \ldots d\varkappa_{34}^2 f_{ijkl}(\varkappa_{12}^2 \varkappa_{13}^2 \ldots \varkappa_{34}^2)$$
$$\times \int_0^1 \cdot\cdot \int_0^\infty d\alpha_{12} \ldots d\alpha_{34}\delta(\alpha_{12} + \alpha_{13} + \ldots + \alpha_{34} - 1)\, \varphi^{-2} \ln(\square - i\varepsilon)\,\delta(l_1 + l_2 + l_3 + l_4). \tag{20}$$

(The factor $\delta(l_1 + l_2 + l_3 + l_4)$ will be omitted hereafter). Here

$$\varphi(\alpha) = \alpha_{12}\alpha_{13}\alpha_{23} + \alpha_{12}\alpha_{13}\alpha_{24} + \alpha_{12}\alpha_{13}\alpha_{34} + \alpha_{12}\alpha_{14}\alpha_{23} + \alpha_{12}\alpha_{14}\alpha_{23} + \alpha_{12}\alpha_{14}\alpha_{34} + \alpha_{12}\alpha_{23}\alpha_{34} + \alpha_{12}\alpha_{24}\alpha_{34}$$
$$+ \alpha_{13}\alpha_{14}\alpha_{23} + \alpha_{13}\alpha_{14}\alpha_{24} + \alpha_{13}\alpha_{14}\alpha_{34} + \alpha_{13}\alpha_{23}\alpha_{24} + \alpha_{13}\alpha_{24}\alpha_{34} + \alpha_{14}\alpha_{23}\alpha_{24} + \alpha_{14}\alpha_{23}\alpha_{34} + \alpha_{23}\alpha_{24}\alpha_{34} \tag{21}$$

(the sum of all products of three different variables α_{ik} under the condition that no index can occur more than twice in a given term);

$$\square = l_1^2\alpha_{12}\alpha_{13}\alpha_{14}(\alpha_{23} + \alpha_{24} + \alpha_{34}) + l_2^2\alpha_{12}\alpha_{23}\alpha_{24}(\alpha_{13} + \alpha_{14} + \alpha_{34}) + l_3^2\alpha_{13}\alpha_{23}\alpha_{34}(\alpha_{12} + \alpha_{14} + \alpha_{24}) + l_4^2\alpha_{14}\alpha_{24}\alpha_{34}(\alpha_{12} + \alpha_{13} + \alpha_{23})$$
$$+ \alpha_{13}\alpha_{14}\alpha_{23}\alpha_{24}(l_1 + l_2)^2 + \alpha_{12}\alpha_{14}\alpha_{23}\alpha_{34}(l_1 + l_3)^2 + \alpha_{12}\alpha_{13}\alpha_{24}\alpha_{34}(l_2 + l_3)^2 + \varphi(\alpha)\varkappa^2; \quad \varkappa^2 = \varkappa_{12}^2\alpha_{12} + \varkappa_{13}^2\alpha_{13} + \ldots + \varkappa_{34}^2\alpha_{34}. \tag{22}$$

Formulas (20) to (22) provide a convenient means for studying the analytical properties of $\tau(l_1, l_2, l_3, l_4)$ as affected by the variation of any one of the independent invariants. Since, however, we shall be interested in the scattering amplitude, we set $l_1^2 = l_2^2 = l_3^2 = l_4^2 = -\mu^2$ and consider τ as a function of $(l_1 + l_2)^2$, $(l_1 + l_3)^2$, and $(l_2 + l_3)^2$, regarding them at first as independent.

The behavior of $\tau((l_1 + l_2)^2, (l_1 + l_3)^2, (l_2 + l_3)^2)$ depends essentially on whether, for given values of $(l_1 + l_2)^2$, $(l_1 + l_3)^2$, and $(l_2 + l_3)^2$, the quantity $\square$ can vanish for any particular values of the variables α_{ik} and κ_{ik}^2 We shall show that $\square > 0$ if

$$-(l_1 + l_2)^2 < 4\mu^2; \quad -(l_1 + l_3)^2 < 4\mu^2; \quad -(l_2 + l_3)^2 < 4\mu^2. \tag{23}$$

The proof reduces to a matter of regrouping the terms in Eq. (22) and using the conditions (14).

Let us first consider those terms $\varphi(\alpha)$ which, when multiplied by κ^2, can give terms containing $\alpha_{13}\alpha_{14}\alpha_{23}\alpha_{24}$. These are the following terms:

$$\alpha_{13}\alpha_{14}\alpha_{23} + \alpha_{13}\alpha_{14}\alpha_{24} + \alpha_{13}\alpha_{23}\alpha_{24} + \alpha_{14}\alpha_{23}\alpha_{24}.$$

Multiplying them by κ^2 and adding and subtracting terms containing the products $\kappa_{13}\kappa_{14}$, $\kappa_{13}\kappa_{23}$, and so on, we get

$$\alpha_{13}\alpha_{14}\alpha_{23}\alpha_{24}(\varkappa_{13} + \varkappa_{14} + \varkappa_{23} + \varkappa_{24})^2 + \alpha_{13}\alpha_{14}(\varkappa_{23}\alpha_{23} - \varkappa_{24}\alpha_{24})^2 + \alpha_{13}\alpha_{23}(\varkappa_{14}\alpha_{14} - \varkappa_{24}\alpha_{24}) + \alpha_{13}\alpha_{24}(\varkappa_{14}\alpha_{14} - \varkappa_{23}\alpha_{23})^2$$
$$+ \alpha_{14}\alpha_{23}(\varkappa_{13}\alpha_{13} - \varkappa_{24}\alpha_{24})^2 + \alpha_{14}\alpha_{24}(\varkappa_{13}\alpha_{13} - \varkappa_{23}\alpha_{23})^2 + \alpha_{23}\alpha_{24}(\varkappa_{13}\alpha_{13} - \varkappa_{14}\alpha_{14})^2 + \varkappa_{12}^2(\alpha_{12}\alpha_{13}\alpha_{14}(\alpha_{23} + \alpha_{24})$$
$$+ \alpha_{13}\alpha_{23}\alpha_{24}(\alpha_{13} + \alpha_{14})] + \varkappa_{34}^2[\alpha_{13}\alpha_{23}\alpha_{34}(\alpha_{14} + \alpha_{24}) + \alpha_{14}\alpha_{24}\alpha_{34}(\alpha_{13} + \alpha_{23})].$$

Selecting in the same way the terms of $\varphi(\alpha)$ which when multiplied by κ^2 will contain $\alpha_{12}\alpha_{14}\alpha_{23}\alpha_{34}$ and $\alpha_{12}\alpha_{13}\alpha_{24}\alpha_{34}$, and grouping the remaining terms not containing squares of differences, we get

$$\square = \alpha_{13}\alpha_{14}\alpha_{23}\alpha_{24}[(l_1 + l_2)^2 + (\varkappa_{13} + \varkappa_{14} + \varkappa_{23} + \varkappa_{24})^2] + \alpha_{12}\alpha_{14}\alpha_{23}\alpha_{24}[(l_1 + l_2)^2 + (\varkappa_{12} + \varkappa_{14} + \varkappa_{23} + \varkappa_{24})^2] + \alpha_{12}\alpha_{13}\alpha_{24}\alpha_{34}[(l_2 + l_3)^2$$
$$+ (\varkappa_{12} + \varkappa_{13} + \varkappa_{24} + \varkappa_{34})^2] + \alpha_{12}\alpha_{13}\alpha_{14}(\alpha_{23} + \alpha_{24} + \alpha_{34})(l_1^2 + \varkappa_{12}^2 + \varkappa_{13}^2 + \varkappa_{14}^2) + (1 \to 2) + (1 \to 3) + (1 \to 4)$$
$$+ \sum_{ikl}(\alpha_{ik}\varkappa_{ik})^2\alpha_{il}\alpha_{kl} + \sum_{\substack{i \neq m \\ k \neq l}} \alpha_{ik}\alpha_{il}(\varkappa_{mk}\alpha_{mk} - \varkappa_{ml}\alpha_{ml})^2. \tag{24}$$

From the condition $\kappa_{12} + \kappa_{13} + \kappa_{14} \geq 3\mu$ it follows that $\kappa_{12}^2 + \kappa_{13}^2 + \kappa_{14}^2 \geq 3\mu^2$. Since $l_1^2 = -\mu^2$, all the terms in Eq. (24) except the first three are always positive. The condition that the first three terms be positive, together with the second group of conditions (14), reduces to Eq. (23).

If we regard τ as a function of one of the variables [for example $(l_1 + l_2)^2$] for fixed values of the others, then starting from Eq. (20) we can easily show (for example, by means of Titchmarsh's theorem) that on the hypothesis of uniform convergence of the integrals with respect to κ_{ik}^2 for real $(l_1 + l_2)^2$ this formula defines a function analytic in the entire plane with the exception of those points of the real axis for which $\square$ can be equal to zero.

If these singularities are removed by the introduction of a term $-i\epsilon$, then at the corresponding points $\mathrm{Re}\,\tau \neq 0$. It is easy to show, by using the results of reference 3, that terms depending on a contact interaction $\lambda\varphi^4$ do not change this result.

In concluding this section we point out the following important fact. The expression (20) has a direct meaning provided that

$$\int \ln(\varkappa^2\varphi)\, f(\varkappa_{12}^2 \ldots \varkappa_{34}^2)\, d\varkappa_{12}^2 \ldots d\varkappa_{34}^2 < \infty.$$

In cases in which $f(\kappa_{12}^2, \ldots \kappa_{34}^2)$ does not decrease rapidly enough, the integral must be regularized, and its divergent terms must be cancelled with the infinite terms written out in Appendix 1. The degree of divergence of the integral is closely connected with the behavior of τ_{ijkl} at large energies. Therefore in obtaining spectral representations, just as in the derivation of dispersion relations, it is necessary to make an assumption about the behavior of the quantity in question at large energies.

3. THE DISPERSION RELATIONS FOR THE SCATTERING AMPLITUDE FOR CONSTANT SCATTERING ANGLE IN THE CENTER-OF-MASS SYSTEM

In order to go over directly to the relations for the scattering amplitude, we take l_1 and l_2 as the four-momenta of the mesons in the initial state. Then

$$\begin{aligned}(l_1 + l_2)^2 &= -4\omega^2;\\ (l_1 + l_3)^2 &= 2p^2(1 - \cos\vartheta) = 2(\omega^2 - \mu^2)(1 - \cos\vartheta);\\ (l_2 + l_3)^2 &= 2p^2(1 + \cos\vartheta) = 2(\omega^2 - \mu^2)(1 + \cos\vartheta);\end{aligned} \tag{25}$$

ω is the energy of one of the mesons in the center-of-mass system, and ϑ is the scattering angle. The scattering amplitude is

$$T_{kl,\, ij}(\omega, \cos\vartheta) = \frac{-i}{4\pi\omega}\tau_{ijkl}[(l_1 + l_2)^2, (l_1 + l_3)^2, (l_2 + l_3)^2]. \tag{26}$$

From the formulas of the preceding section it follows that the function $\tau((l_1 + l_2)^2, (l_1 + l_3)^2, (l_2 + l_3)^2)$ is analytic not only as a function of one of the invariants for fixed values of the others, but also as a function of ω^2 for fixed $\cos\vartheta$, if we draw the cuts along the real axis from $\omega^2 = \mu^2$ to $\omega^2 = \infty$ and, depending on whether $\vartheta < \pi/2$ or $\vartheta > \pi/2$, from $\omega^2 = -\mu^2 \tan^2\vartheta/2$ or $\omega^2 = -\mu^2 \times \cot^2\vartheta/2$ to $-\infty$. The position of the cuts is determined by means of the conditions (23) and the formulas (25).

Using these analytic properties or, with a procedure like that of Goldberger,[5] changing the order of integration in Eq. (20), we can write the following dispersion relations for $\tau_{ijkl}(\omega, \cos\vartheta)$ for $\vartheta < \pi/2$:

$$\begin{aligned}&-\frac{\operatorname{Im}\tau_{ijkl}(\omega, \cos\vartheta)}{(\omega^2 - \omega_{10}^2)(\omega^2 - \omega_{20}^2)} + \frac{\operatorname{Im}\tau_{ijkl}(\omega_{10}, \cos\vartheta)}{(\omega^2 - \omega_{10}^2)(\omega_{10}^2 - \omega_{20}^2)}\\ &+ \frac{\operatorname{Im}\tau_{ijkl}(\omega_{20}, \cos\vartheta)}{(\omega^2 - \omega_{20})^2(\omega_{20}^2 - \omega_{10}^2)} = \frac{P}{\pi}\int_{\mu^2}^{\infty} \frac{\operatorname{Re}\tau_{ijkl}(\omega', \cos\vartheta)\, d\omega'^2}{(\omega'^2 - \omega^2)(\omega'^2 - \omega_{10}^2)(\omega'^2 - \omega_{20}^2)}\\ &- \frac{P}{\pi}\int_{-\infty}^{-\mu^2\tan^2(\vartheta/2)} \frac{\operatorname{Re}\tau_{ijkl}(\omega', \cos\vartheta)\, d\omega'^2}{(\omega'^2 - \omega^2)(\omega'^2 - \omega_{10}^2)(\omega'^2 - \omega_{20}^2)}.\end{aligned} \tag{27}$$

In obtaining this formula we have made the same assumption about the behavior of τ for large ω^2 as is made (cf. reference 5) for the scattering amplitude of π mesons on nucleons. ω_{10}^2, ω_{20}^2 are so far arbitrary.

The second term of the right member appears with a minus sign, since in the derivation of the dispersion relations we integrate along the top edge of the cut, whereas for $(l_1 + l_3)^2 < -4\mu^2$ or $(l_2 + l_3)^2 < -4\mu^2$ the function $\tau((l_1 + l_2)^2, (l_1 + l_3)^2, (l_2 + l_3)^2)$ is defined by Eq. (20) as the value on the bottom side of the cut.

The first integral in the right member contains the scattering amplitude in the physical region. The second contains the scattering amplitude for imaginary energies. Using the symmetry of the function $\tau_{ijkl}(l_1, l_2, l_3, l_4)$, we transform it into an integral over real energies, but containing the scattering angle for an imaginary angle. For this purpose we change the variable in the second integral of Eq. (27) in the following way:

$$2(\omega'^2 - \mu^2)(1 + \cos\vartheta) = -4\omega_1^2, \tag{28}$$

and define a quantity $\cos\vartheta_1$ by the relation:

$$-4\omega'^2 = 2(\omega_1^2 - \mu^2)(1 + \cos\vartheta_1). \tag{29}$$

In the new variables

$$\begin{aligned}(l_1 + l_2)^2 &= 2(\omega_1^2 - \mu^2)(1 + \cos\vartheta_1),\ (l_1 + l_3)^2\\ &= 2(\omega_1 - \mu^2)(1 - \cos\vartheta_1);\\ (l_2 + l_3)^2 &= -4\omega_1^2.\end{aligned} \tag{30}$$

Since

$$\begin{aligned}&\tau_{ijkl}[(l_1 + l_2)^2, (l_1 + l_3)^2, (l_2 + l_3)^2]\\ &= \tau_{kjil}[(l_2 + l_3)^2, (l_1 + l_3)^2, (l_1 + l_2)^2],\end{aligned} \tag{31}$$

we get

$$\tau_{ijkl}(\omega', \cos\vartheta) = \tau_{kjil}(\omega_1, \cos\vartheta_1). \tag{32}$$

In particular

$$\tau_{ijkl}\left(-\mu\tan\frac{\vartheta}{2}, \cos\vartheta\right) = \tau_{kjil}(\mu, \infty), \tag{33}$$

where use has been made of the fact that

$$\cos\vartheta_1 = 1 + \frac{2\omega_1^2}{\omega_1^2 - \mu^2}\tan^2\frac{\vartheta}{2}. \tag{34}$$

Since, however, the scattering amplitude at zero energy does not depend on the angle, we have

$$\tau_{ijkl}(\mu,\ \infty)=\tau_{ijkl}(\mu,\ 0)\equiv\tau(\mu). \tag{35}$$

Using these equations, we take $\omega_{10}^2=\mu^2$, $\omega_{20}^2=-\mu^2\tan^2\vartheta/2$. Then, using also Eq. (26), we can write instead of Eq. (27)

$$\begin{aligned}&\frac{\omega\,\mathrm{Re}\,T_{kl,ij}(\omega,\ \cos\vartheta)}{(\omega^2-\mu^2)(\omega^2+\mu^2\tan^2(\vartheta/2))}-\frac{\cos^2(\vartheta/2)}{\mu}\left[\frac{\mathrm{Re}\,T_{kl,ij}(\mu)}{\omega^2-\mu^2}-\frac{\mathrm{Re}\,T_{il,kj}(\mu)}{\omega^2+\mu^2\tan^2(\vartheta/2)}\right]\\&=\frac{2}{\pi}P\int_\mu^\infty\frac{\omega'^2d\omega'\,\mathrm{Im}\,T_{kl,ij}(\omega',\ \cos\vartheta)}{(\omega'^2-\omega^2)(\omega'^2-\mu^2)(\omega'^2+\mu^2\tan^2(\vartheta/2))}\\&-\frac{2}{\pi}\cos^4\frac{\vartheta}{2}\int_\mu^\infty\frac{\mathrm{Im}\,T_{il,kj}(\omega_1,\ \cos\vartheta_1)\,d\omega_1}{(\omega^2+\omega_1^2-\mu^2\cos^2(\vartheta/2))(\omega_1^2-\mu^2)}.\end{aligned} \tag{36}$$

For $\pi/2<\vartheta<\pi$ one gets analogous relations differing only in the replacement of ϑ by $\pi-\vartheta$.

By taking advantage of isotopic invariance, we can express the amplitudes $T_{kl,ij}$ in terms of the scattering amplitudes in the states with total isotopic spins T_0, T_1, and T_2 and reduce the system of equations (36) to three equations for T_0, T_1, and T_2. It is easy to show that

$$\begin{aligned}T_{kl,ij}&=\frac{2}{3}T_0\delta_{ij}\delta_{kl}+T_1(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})\\&+T_2\left\{\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2}{3}\delta_{ij}\delta_{kl}\right\}.\end{aligned} \tag{37}$$

Substituting this expression into Eq. (36), we get the following system of equations:

$$\begin{aligned}&\frac{\omega\,\mathrm{Re}\,T_i(\omega,\ \cos\vartheta)}{(\omega^2-\mu^2)\left(\omega^2+\mu^2\tan^2\frac{\vartheta}{2}\right)}-\frac{\cos^2(\vartheta/2)}{\mu}\left[\frac{\mathrm{Re}\,T_i(\mu)}{\omega^2-\mu^2}-\frac{\sum_k A_{ik}\,\mathrm{Re}\,T_k(\mu)}{\omega^2+\mu^2\tan^2\frac{\vartheta}{2}}\right]\\&=\frac{2}{\pi}P\int_\mu^\infty\frac{\omega'^2d\omega'\,\mathrm{Im}\,T_i(\omega',\ \cos\vartheta)}{(\omega'^2-\omega^2)(\omega'^2-\mu^2)(\omega'^2+\mu^2\tan^2(\vartheta/2))}\\&-\frac{2}{\pi}\cos^4\frac{\vartheta}{2}\sum_k A_{ik}\int_\mu^\infty\frac{\mathrm{Im}\,T_k(\omega,\ \cos\vartheta_1)\,d\omega_1}{(\omega^2+\omega_1^2-\mu^2\cos^2(\vartheta/2))(\omega_1^2-\mu^2)};\end{aligned} \tag{38}$$

$$(A_{ik})=\begin{pmatrix}2/3 & -1/2 & 5/6\\ -2/3 & 1 & 5/3\\ 2/3 & 1 & 1/3\end{pmatrix}.$$

The equations that have been obtained would be of considerably greater interest if one could indicate a simple method for analytic continuation of the scattering amplitude into the region of imaginary angles. It seems natural to use for this purpose an expansion of the scattering amplitude in series of Legendre polynomials

$$T_i(\omega,\ \cos\vartheta)=\sum_l T_i^l(\omega)\,P_l(\cos\vartheta). \tag{39}$$

Such an expansion is especially attractive because the conditions for the S matrix to be unitary have a simple form for the quantities T_i^l. Such an expansion, however, can scarcely allow us to continue the scattering amplitude into a region of values of $\cos\vartheta$ that are much larger than 1. Namely, it follows from Eqs. (23) and (24) that the scattering amplitude has singularities of the type of branch points for those values of the variables at which one of the invariants $(l_1+l_2)^2$, $(l_1+l_3)^2$, $(l_2+l_3)^2$ becomes equal to $-4\mu^2$. According to Eq. (25) this means that as a function of $\cos\vartheta$ the scattering amplitude has singularities at

$$\cos\vartheta=\pm(\omega^2+\mu^2)/(\omega^2-\mu^2). \tag{40}$$

Therefore the radius of convergence of the series (39) cannot exceed this value. This means that we can use the expansion (39) in the last integral in the right member of Eq. (38) in the region of values of ω_1 for which

$$\cos\vartheta_1<(\omega_1^2+\mu^2)/(\omega_1^2-\mu^2).$$

Recalling the expression (34) for $\cos\vartheta_1$, we get the condition

$$\omega_1^2<\mu^2\cot^2\frac{\vartheta}{2}, \tag{41}$$

which shows that the expansion can be used only for the forward scattering amplitude, since the contribution to the integral in question from the region $\omega_1>\mu\cot(\vartheta/2)$ is of the order

$$\left(\mu\cot\frac{\vartheta}{2}\right)^{-2}\approx\mu^{-2}(1-\cos\vartheta),$$

i.e., of the order of the first term of the expansion of $T_i(\omega,\ \cos\theta)$ in powers of $1-\cos\vartheta$. Therefore in order to give meaning to the dispersion relations for the scattering amplitude one must find some other method of analytic continuation.

APPENDIX 1

Let us write out the formula (3) in detail using the fact that when we introduce a contact interaction

$$[\dot{\varphi}_i(x),\ j_k(x')]\,|_{t=t'}=\lambda\delta(\mathbf{r}-\mathbf{r}')\,\{\varphi_\lambda^2\delta_{ik}+2\varphi_i\varphi_k\};$$

this gives

$$K_1K_2K_3K_4 \langle|T\varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3)\varphi_l(x_4)|\rangle = \langle|Tj_i(x_1)j_j(x_2)j_k(x_3)j_l(x_4)|\rangle$$
$$+\lambda\{\delta(x_1-x_2)\langle|T\{\varphi_\lambda^2(x_1)\delta_{ij}+2\varphi_i(x_1)\varphi_j(x_1)\}j_k(x_3)j_l(x_4)|\rangle+\ldots\}_{\rm sym}$$
$$+2\lambda\{\delta(x_1-x_2)\delta(x_2-x_3)[\delta_{kj}\langle|T\varphi_i(x_1)j_l(x_4)|\rangle+\delta_{ik}\langle|T\varphi_j(x_2)j_l(x_4)|\rangle$$
$$+\delta_{ij}\langle|T\varphi_k(x_3)j_l(x_4)|\rangle]+\ldots\}_{\rm sym}+\lambda^2\{\delta(x_1-x_2)\delta(x_3-x_4)\langle|T\{\varphi_\lambda^2(x_1)\delta_{ij}+2\varphi_i(x_1)\varphi_j(x_1)\}\{\varphi_\nu^2(x_3)\delta_{kl}$$
$$+2\varphi_k(x_3)\varphi_l(x_3)\}|\rangle+\ldots\}_{\rm sym}+2\lambda\delta(x_1-x_2)\delta(x_2-x_3)\delta(x_3-x_4)[\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{kj}\delta_{il}].$$

Here $\{\ \}_{\rm sym}$ means that we are to take the sum of terms corresponding to all permutations of x_1, x_2, x_3, x_4 together with the associated isotopic indices.

APPENDIX 2

The integral (19) can be easily reduced to integrations over parameters if we use the formula

$$\frac{1}{a_1a_2\ldots a_n}=(n-1)!\int_0^1\!\!\int_0^1 dz_1dz_2\ldots dz_n\frac{\delta(z_1+z_2+z_3+\ldots+z_{n-1})}{(a_1z_1+a_2z_2+\ldots+a_nz_n)^n}. \tag{A.1}$$

Denoting the integral (19) by I and dropping the factor $\delta(l_1+l_2+l_3+l_4)$, we can write

$$I=-\frac{5!}{(2\pi)^{12}}\int d\alpha_{12}d\alpha_{12}\ldots d\alpha_{34}\int\frac{dk_{12}dk_{13}dk_{23}\delta(\alpha_{12}+\alpha_{13}+\ldots+\alpha_{34}-1)}{[k_{12}^2\alpha_{12}+\ldots+k_{34}^2\alpha_{34}+\varkappa_{12}^2\alpha_{12}+\ldots+\varkappa_{34}^2\alpha_{34}]^6},$$
$$k_{14}=l_1-k_{12}-k_{13},\ k_{24}=l_2+k_{12}-k_{23};\ k_{34}=l_3+k_{13}+k_{23}. \tag{A.2}$$

Instead of the variables $z_1, z_2, \ldots, z_6$ we have here introduced the variables α_{12}, α_{13}, α_{14}, α_{23}, α_{24}, α_{34}.

If in the denominator of the expression (A.2) we insert instead of k_{14}, k_{24}, and k_{34} their expressions in terms of l_1, l_2, l_3 and k_{12}, k_{13}, k_{23}, then besides quadratic terms the denominator will contain linear terms in k_{12}, k_{13}, and k_{23}. To get rid of these terms we make the change of variables

$$k_{12}=q_{12}+\Delta_{12};\ k_{13}=q_{13}+\Delta_{13};\ k_{23}=q_{23}+\Delta_{23}. \tag{A.3}$$

The conditions for the determination of Δ_{ik} have the form:

$$\Delta_{12}(\alpha_{12}+\alpha_{14}+\alpha_{24})+\alpha_{14}\Delta_{13}-\alpha_{24}\Delta_{23}+l_2\alpha_{24}-l_1\alpha_{14}=0;\quad \alpha_{14}\Delta_{12}+(\alpha_{13}+\alpha_{14}+\alpha_{34})\Delta_{13}+\alpha_{34}\Delta_{23}+l_3\alpha_{34}-l_1\alpha_{14}=0;$$
$$-\alpha_{24}\Delta_{12}+\alpha_{34}\Delta_{13}+(\alpha_{23}+\alpha_{24}+\alpha_{34})\Delta_{23}+l_3\alpha_{34}-l_2\alpha_{24}=0. \tag{A.4}$$

The determinant $\varphi(\alpha)$ of this system is the sum (21). Then

$$\Delta_{12}=\{l_1\alpha_{14}\varphi_{24}^1-l_2\alpha_{24}\varphi_{14}^2+l_3\alpha_{34}(\alpha_{23}\alpha_{14}-\alpha_{13}-\alpha_{24})\}/\varphi;\qquad \Delta_{13}=\{l_1\alpha_{14}\varphi_{34}^1-l_3\alpha_{34}\varphi_{14}^3+l_2\alpha_{24}(\alpha_{14}\alpha_{23}-\alpha_{12}\alpha_{34})\}/\varphi; \tag{A.5}$$

$$\Delta_{23}=\{l_2\alpha_{24}\varphi_{34}^2-l_3\alpha_{34}\varphi_{24}^3+l_1\alpha_{14}(\alpha_{13}\alpha_{24}-\alpha_{12}\alpha_{34})\}/\varphi;\qquad \varphi_{24}^1=\alpha_{13}\alpha_{23}+\alpha_{13}\alpha_{24}+\alpha_{13}\alpha_{34}+\alpha_{23}\alpha_{34}. \tag{A.6}$$

φ_{24}^1 consists of the products by pairs of all the variables except α_{12} and α_{14}, taken so that the indices 2 and 4 do not occur twice in any product. Substituting Eqs. (A.5) and (A.6) into the denominator of (A.2), we get for the latter the expression

$$(\alpha_{12}+\alpha_{14}+\alpha_{24})q_{12}^2+(\alpha_{13}+\alpha_{14}+\alpha_{34})q_{13}^2+(\alpha_{23}+\alpha_{24}+\alpha_{34})q_{23}^2+2\alpha_{14}q_{12}q_{13}+2\alpha_{13}q_{13}q_{23}-2\alpha_{24}q_{12}q_{23}+\Box/\varphi;$$
$$\Box=\varphi\Delta_{12}(l_2\alpha_{24}-l_1\alpha_{14})+\varphi\Delta_{13}(-l_1\alpha_{14}+l_3\alpha_{34})+\varphi\Delta_{23}(l_3\alpha_{34}-l_2\alpha_{24}) \tag{A.7}$$
$$+\varphi(\alpha_{14}l_1^2+\alpha_{24}l_2^2+\alpha_{34}l_3^2)+\varphi\varkappa^2=l_1^2\alpha_{14}\varphi_{14}+l_2^2\alpha_{24}\varphi_{24}+l_3^2\alpha_{34}\varphi_{34}+2l_1l_2\alpha_{14}\alpha_{24}\varphi_{12}^4+2l_1l_3\alpha_{14}\alpha_{34}\varphi_{13}^4+2l_2l_3\alpha_{24}\alpha_{34}\varphi_{23}^4+\varphi\varkappa^2;$$

φ_{14} is the sum of those terms of φ which do not contain α_{14}; and

$$\varkappa^2=\varkappa_{12}^2\alpha_{12}+\varkappa_{13}^2\alpha_{13}+\varkappa_{14}^2\alpha_{14}+\varkappa_{23}^2\alpha_{23}+\varkappa_{24}^2\alpha_{24}+\varkappa_{34}^2\alpha_{34}.$$

[If in Eq. (A.7) we express l_1l_2, l_1l_3 and l_2l_3 in terms of $(l_1+l_2)^2$, $(l_1+l_3)^2$, and $(l_2+l_3)^2$ and note that

$$(l_1+l_2)^2+(l_1+l_3)^2+(l_2+l_3)^2=l_1^2+l_2^2+l_3^2+l_4^2,$$

then we can easily get for $\Box$ the expression (22) given in the text.]

It is now easy to carry out the integrations with respect to q_{12}, q_{13}, q_{23} by introducing new variables q'_{12}, q'_{13}, q'_{23} in such a way that the quadratic terms reduce to a sum of squares. We thus get

$$\int\frac{d^4q'_{12}d^4q'_{13}d^4q'_{23}}{[\lambda_{12}q'^2_{12}+\lambda_{13}q'^2_{13}+\lambda_{23}q'^2_{23}+\Box/\varphi]^6}=\frac{(i\pi^2)}{5!\lambda_3^2\lambda_{23}^2}\int d^4q'_{12}\frac{1}{[\lambda_{12}q'^2_{12}+\Box/\varphi]^2}.$$

To perform the last integration we introduce the cut-off parameter Λ, and then

$$I = \frac{i}{(4\pi)^6} \frac{i}{\lambda_{12}^2\lambda_{13}^2\lambda_{23}^2} [\ln \varphi\Lambda + 1 - \ln(\Box - i\varepsilon)]. \tag{A.8}$$

Noting that $\lambda_{12}\lambda_{13}\lambda_{23} = -\varphi$ and that the terms that do not depend on κ_{ik} make no contribution, we get the result (20) given in the text.

[1] J. Schwinger, Proceedings of the Seventh Rochester Conference.
[2] Y. Nambu, Nuovo cimento 6, 1064 (1957).
[3] V. N. Gribov, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 1310 (1958), Soviet Phys. JETP 7, 903 (1958).
[4] Lehmann, Symanzik, and Zimmermann, Nuovo cimento 1, 205 (1955).
[5] M. Goldberger, Phys. Rev. 99, 979 (1955).

Translated by W. H. Furry
74

SOVIET PHYSICS JETP VOLUME 35(8), NUMBER 2 FEBRUARY, 1959

PROPAGATION OF A NON-SELF-SIMILAR THERMAL WAVE

E. I. ANDRIANKIN

Institute of Chemical Physics, Academy of Sciences U.S.S.R.

Submitted to JETP editor March 15, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) 35, 428-432 (August, 1958)

The propagation of a thermal wave from an instantaneous point source in a gas is investigated with account of the temperature dependence of the internal energy of the gas. The case when the internal energy is associated not only with the matter but also with radiation is considered. The range of the radiation is assumed to depend on the temperature in accordance with a power law. An approximate method can also be used in the case of an arbitrary dependence of the internal energy and of the heat flux on the temperature.

LET a quantity of heat Q_0 be liberated at a given initial instant of time within a small volume (at a point). Then, if the density of the medium is constant, and the thermal conductivity and the specific heat are each proportional to the temperature raised to a certain power, the problem is a self-similar one and its solution can be obtained in closed form. Such a problem was investigated by Zel'dovich and Kompaneets.[1]

If a thermal wave propagates in a gas then, because of the high temperature, the molecules of the gas break up into atoms and the latter are ionized, and this leads to a temperature dependence of the internal energy of the gas. Calculations[2,3] show that the internal energy of a gas may be approximated over a wide range of temperatures by a power of the temperature ($\sim aT^\lambda$). However, at very high temperatures (on the order of several millions of degrees for air of normal density) it is necessary to take into account, in addition to the energy of the matter, also the radiation energy, which is proportional to the fourth power of the temperature $\sim bT^4$. Such a problem is no longer self-similar even if the radiation range is expressed by a power of the temperature. Another non-self-similar problem will be one in which the internal energy is given by a power of the temperature, but the range of radiation is given not by a single power, but involves two or more terms.

We shall discuss the problem of the propagation of a non-self-similar thermal wave by considering a special case when the internal energy is expressed by the following two term formula

$$E = aT^\lambda + bT^4 \tag{1}$$

(here $b = 4\sigma/c$, c is the velocity of light, σ =